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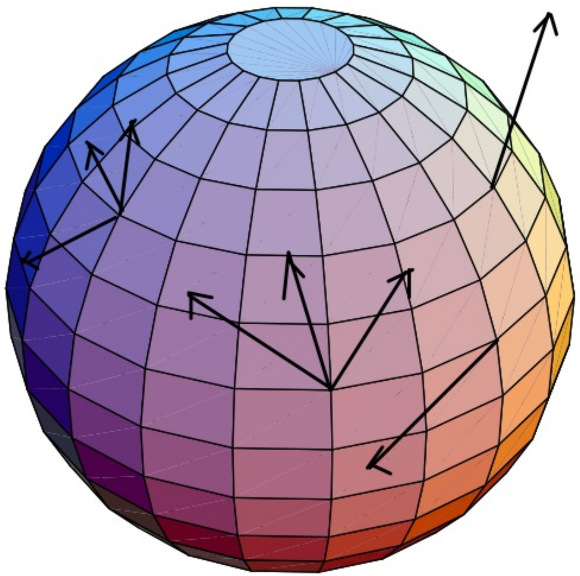

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  - no reference to embeddings
  - not arrows that can be moved around as in flat space
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  - attached to a fixed point
  - no natural way to move them around: a different vector space at each point

Vectors

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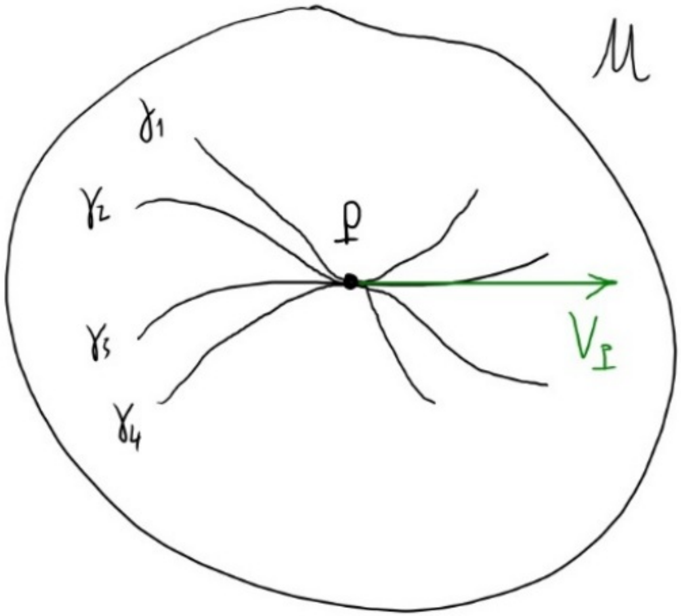

$$\text{vector} \equiv \left(\begin{array}{c} \text{equivalence class} \\ \gamma_i \sim \gamma_j \end{array} \right)$$

Vectors

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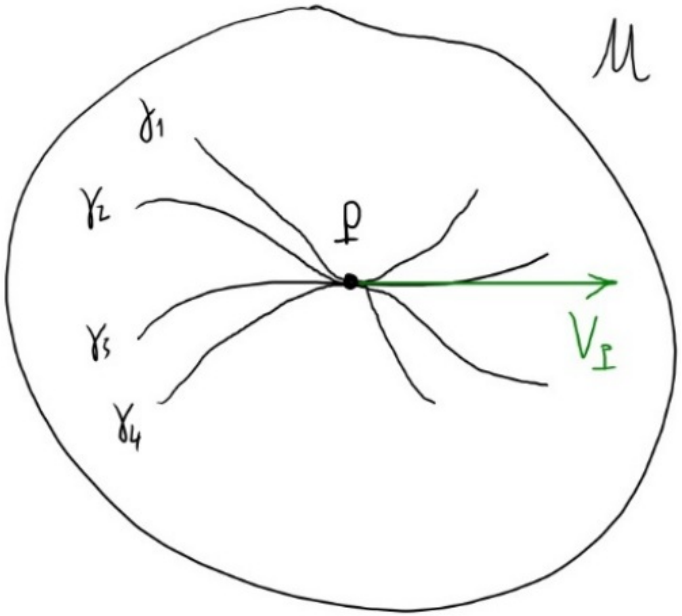
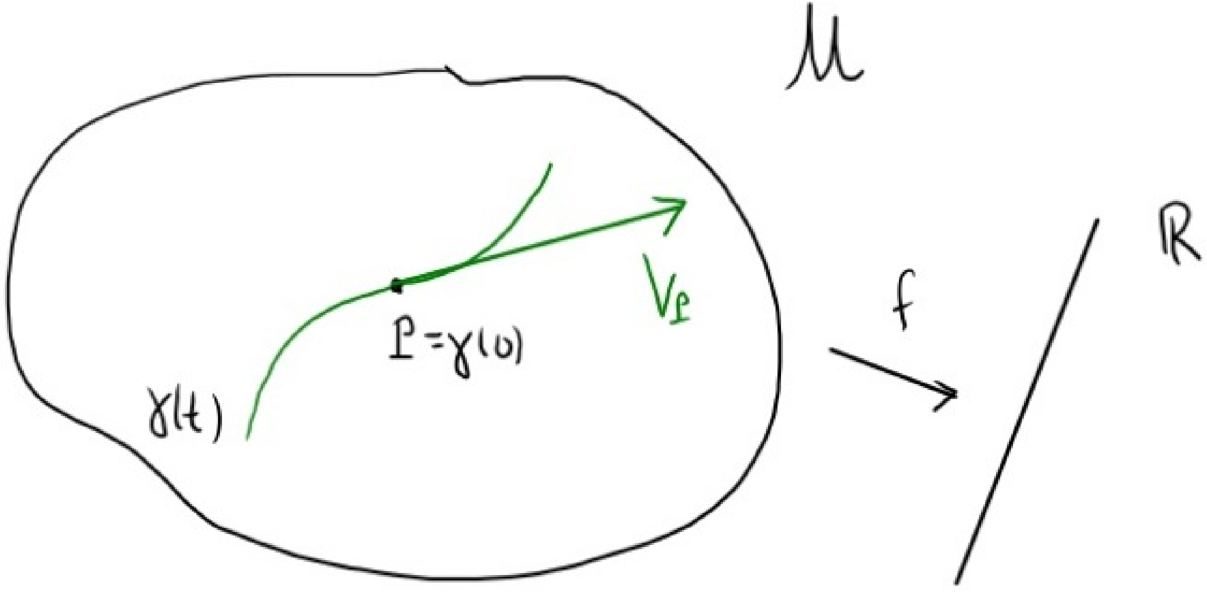


- A vector gives the rate of change of "things" as we move on a curve

- "Things" on which all observers agree on are all the real functions on $\mathcal{M}$

$$f : \mathcal{M} \longrightarrow \mathbb{R}$$



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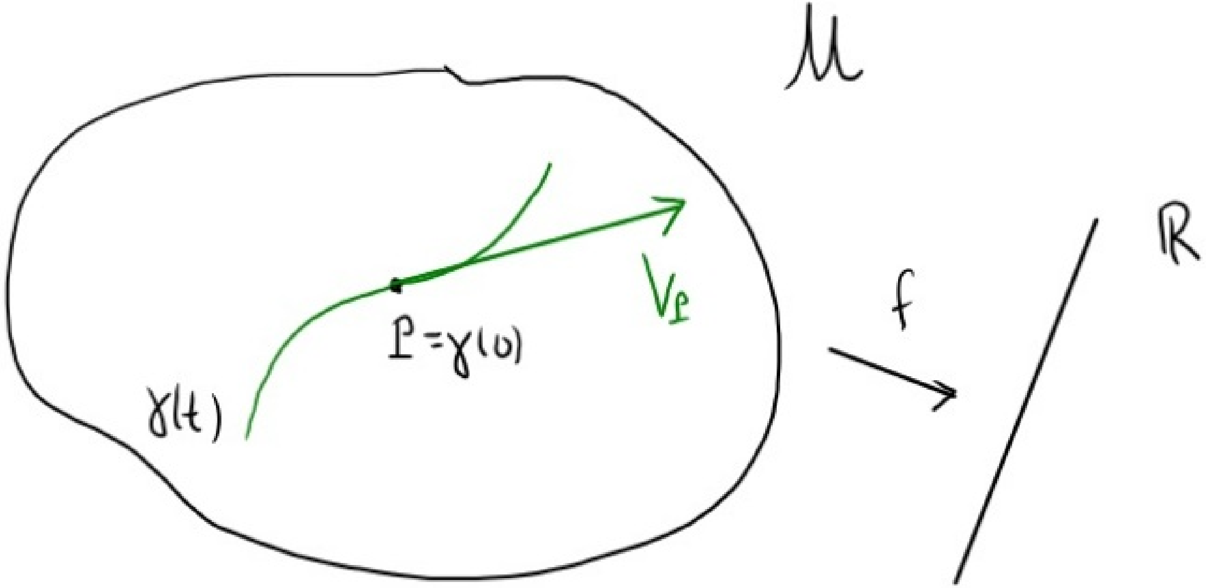
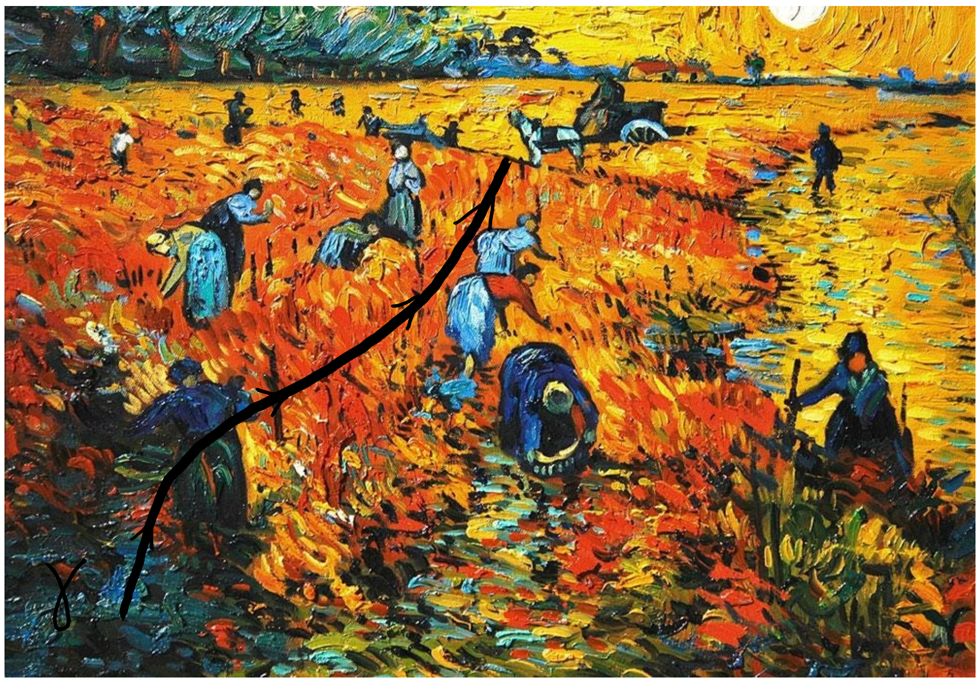


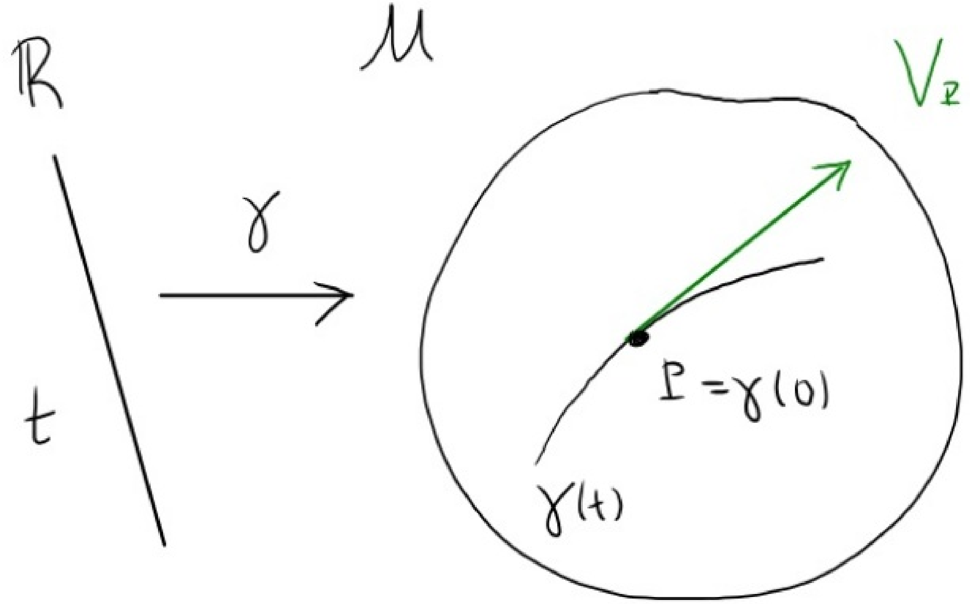
e.g. by measuring the rate of change of color, we obtain information on direction of motion



- we should be able to do this for all functions!

- A curve is a map

$$\gamma : \mathbb{R} \to M$$
$$t \mapsto \gamma(t)$$



here: $\gamma(0) = P$

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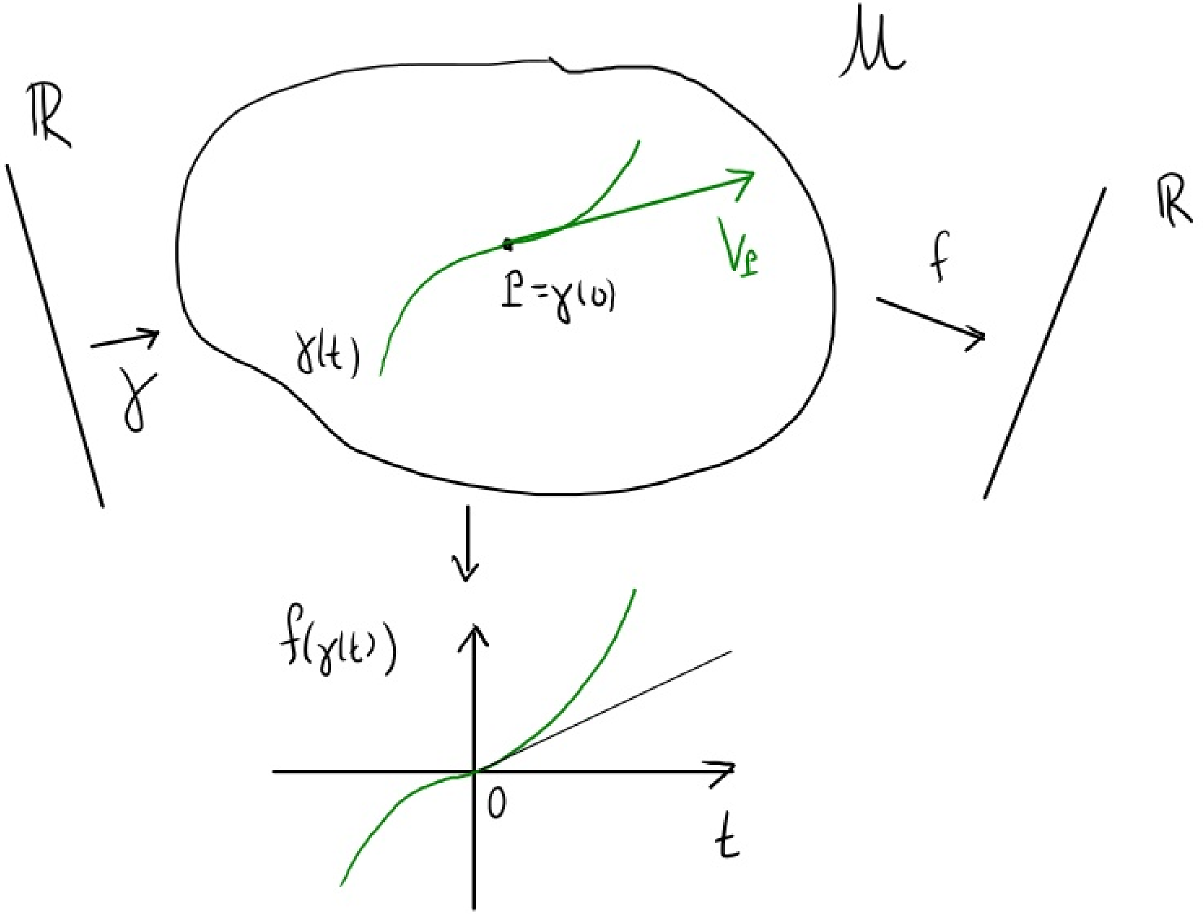


- given a function

$$f : M \to \mathbb{R}$$

we can map

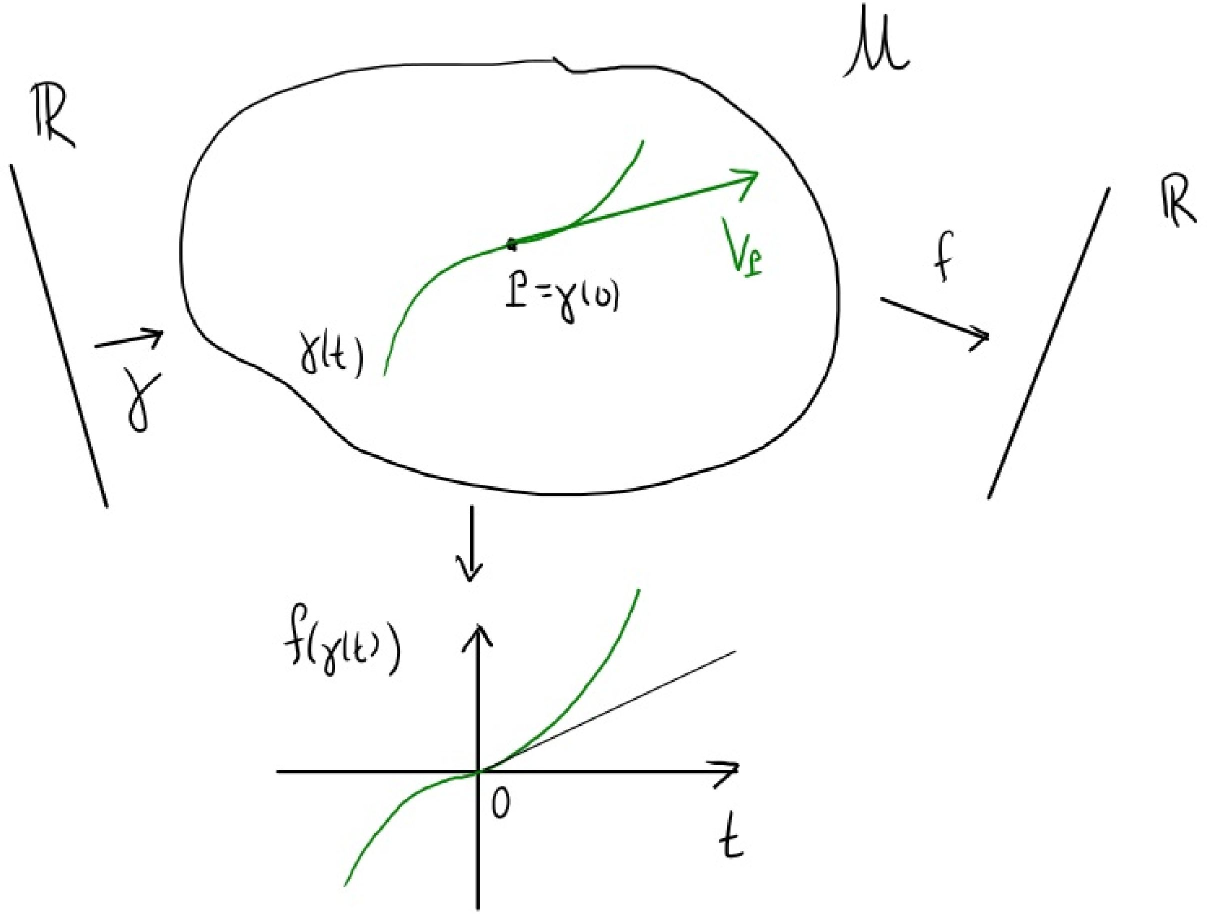
$$f \circ \gamma : \mathbb{R} \to \mathbb{R}, \text{ s.t } t \mapsto f(\gamma(t))$$

$f \circ \gamma : \mathbb{R} \to \mathbb{R}$

a real function.

Compute:

$$\left. \frac{d\, f\circ\gamma}{dt} \right|_0 \equiv \frac{d\, f(\gamma(0))}{dt}$$



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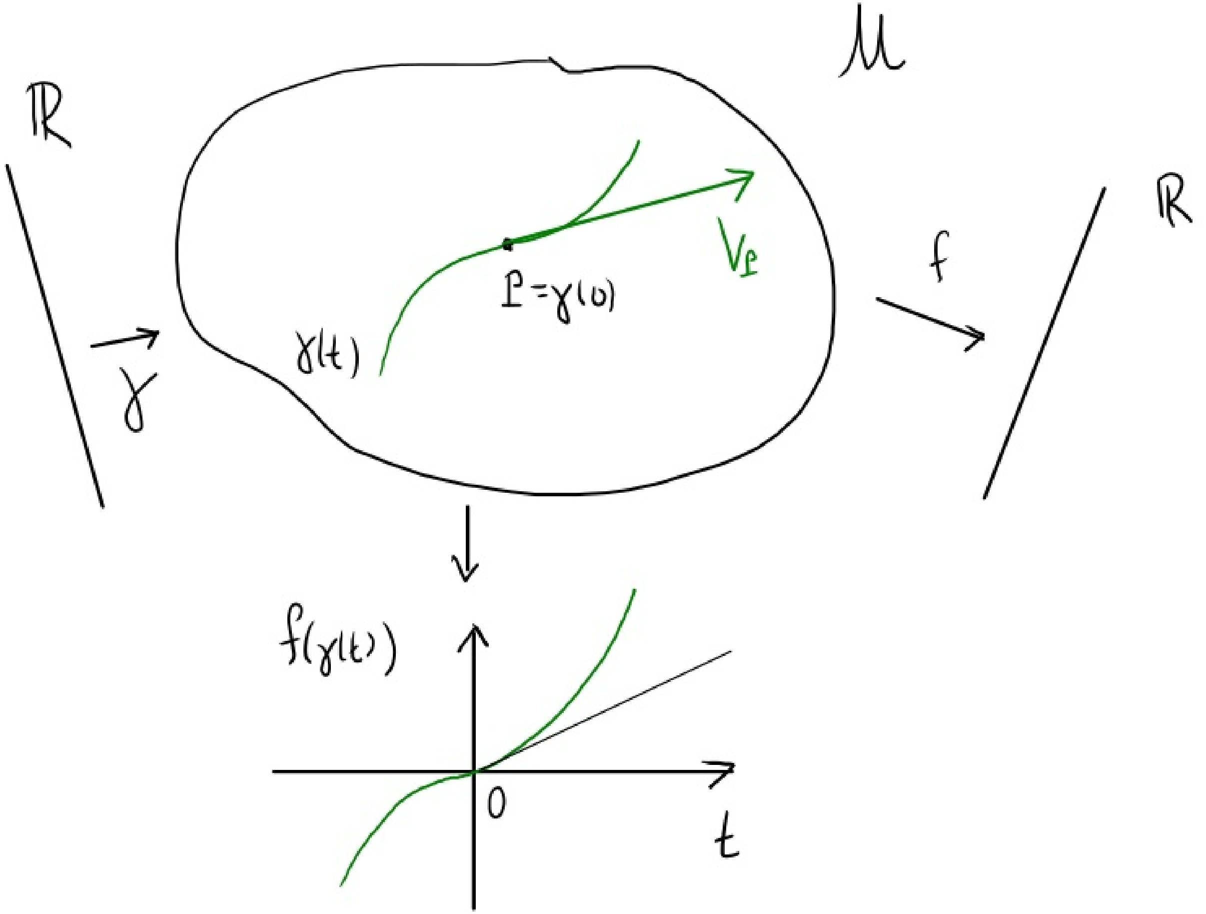
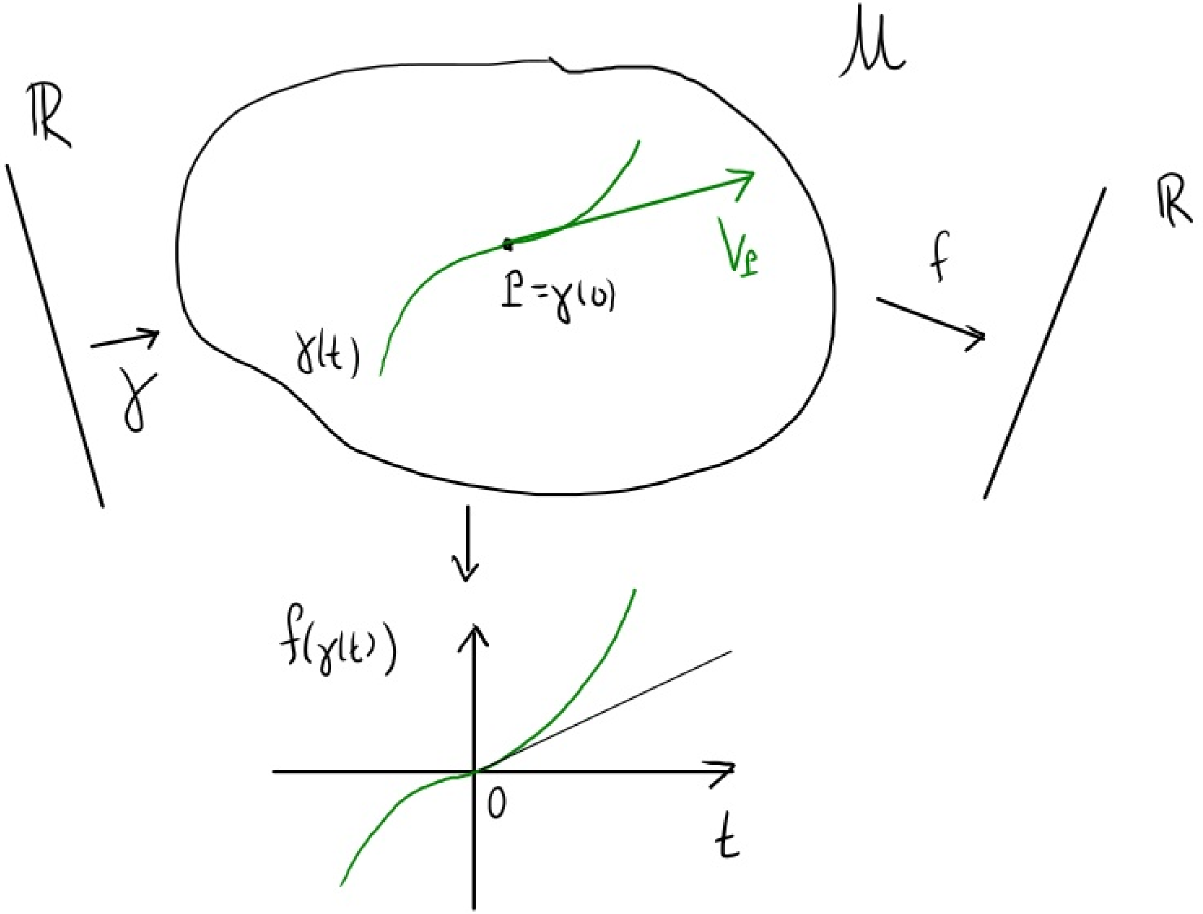


We write: $\dfrac{df}{dt}\bigg|_p \equiv \dfrac{d\, f\circ\gamma}{dt}(0)$ = How fast f changes as we move on γ

- The derivatives $\left.\frac{df}{dt}\right|_P$ for any f are a measure of how things change along γ at P



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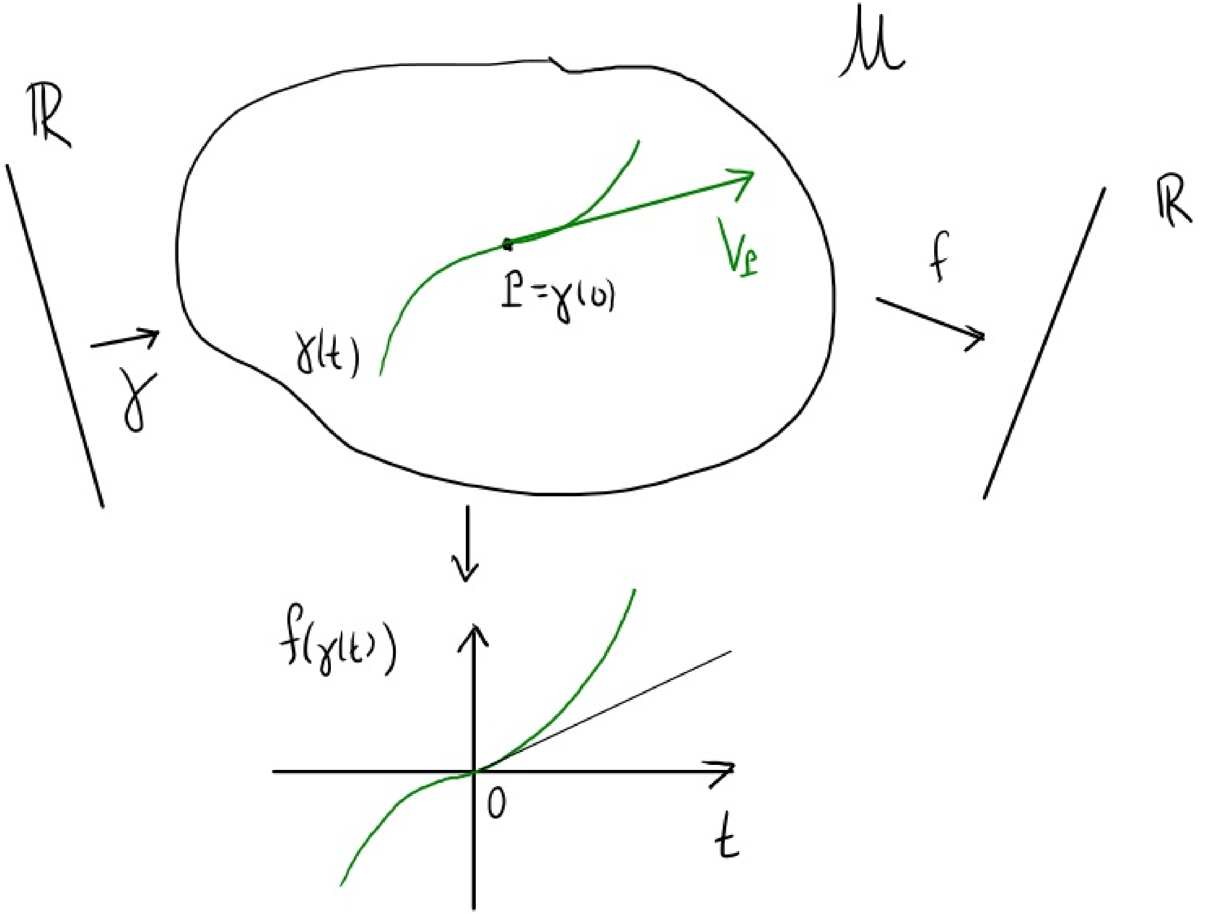

This rate depends on the choice of "time".

A curve is not just a set of points on M, but goes together with the choice of parameter t

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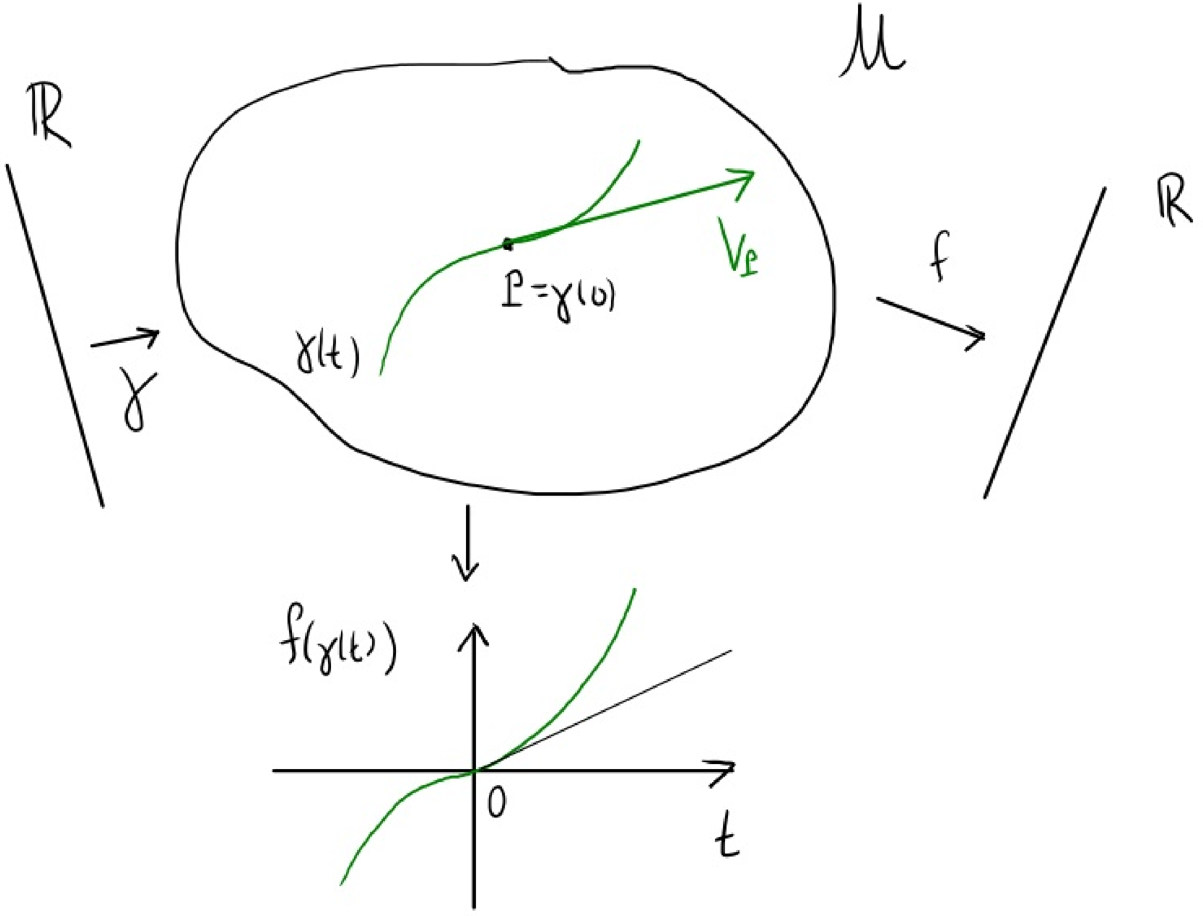
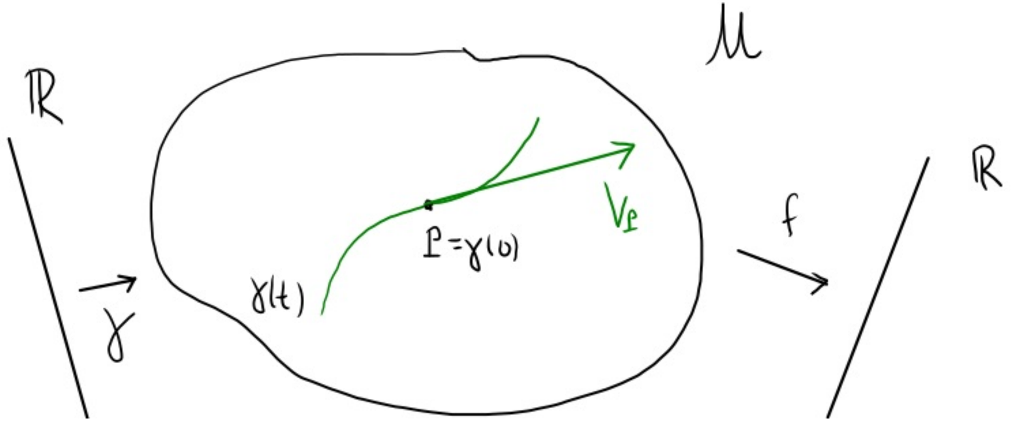


- We define the vector V_P at P to be the operator $\left.\frac{d}{dt}\right|_0$ acting on any f: $V_P(f) = \left.\frac{df}{dt}\right|_P$

- $V_p(f)$ is the directional derivative of f along γ at p



$$V_p = \frac{d}{dt}\Big|_p$$

$$V_p(f) = \frac{df}{dt}\Big|_p = \frac{d\, f\circ\gamma}{dt}(0)$$

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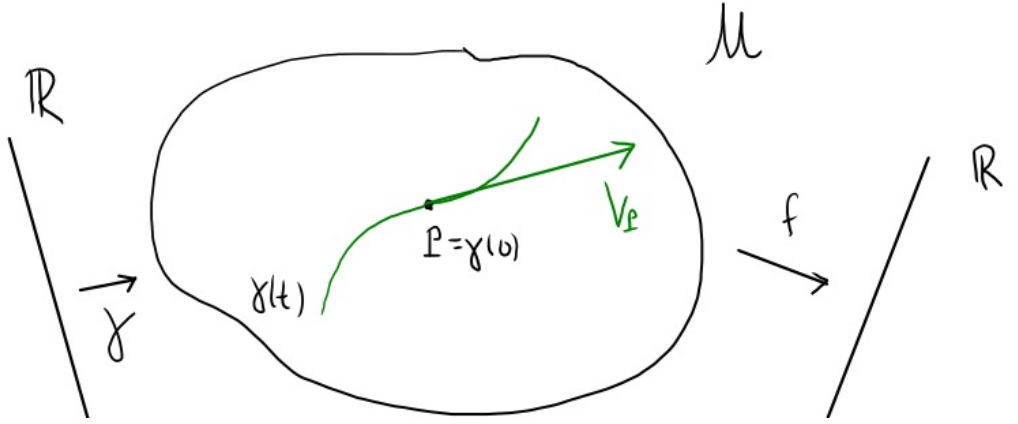

- V_p acts linearly on functions

$$V_p(\alpha f + \beta g) = \alpha V_p(f) + \beta V_p(g) \qquad \alpha, \beta \in \mathbb{R}$$

obvious: $\frac{d}{dt}(\alpha f + \beta g) = \alpha \frac{df}{dt} + \beta \frac{dg}{dt}$

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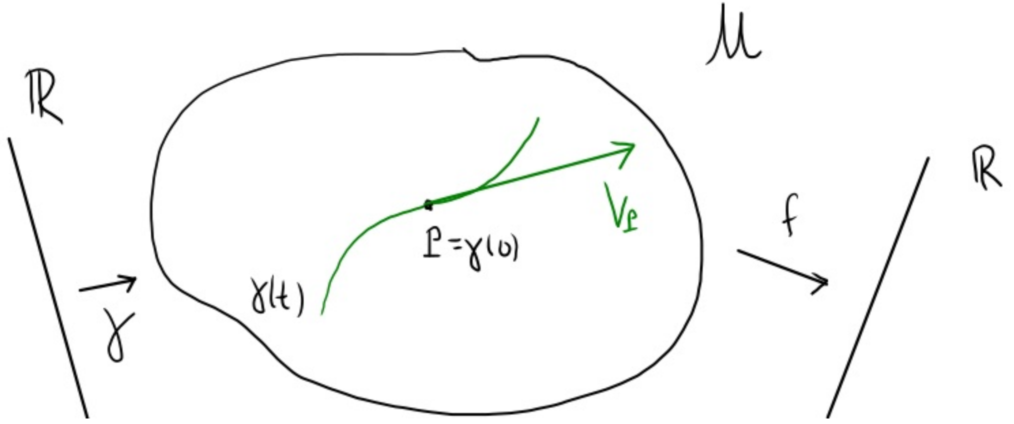

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$$V_p(f \cdot g) = V_p(f) \cdot g + f\, V_p(g)$$

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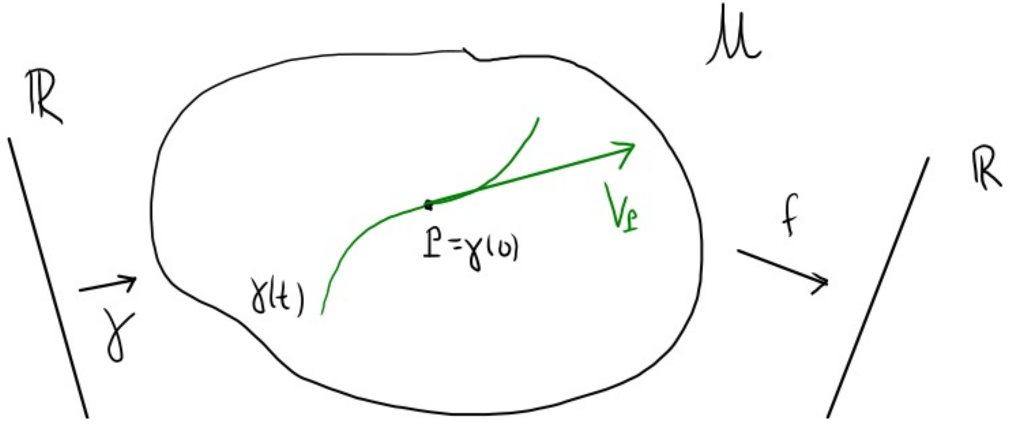

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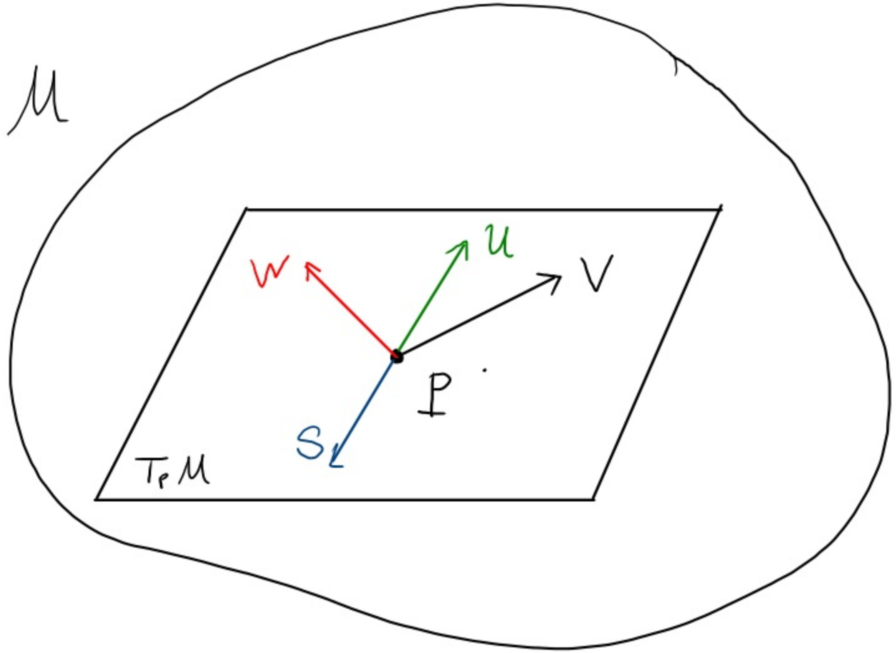
- V_p is a derivation on $F(M)$

smooth functions on M

Vectors at p are all possible derivations on $\mathcal{F}(\mathcal{M})$



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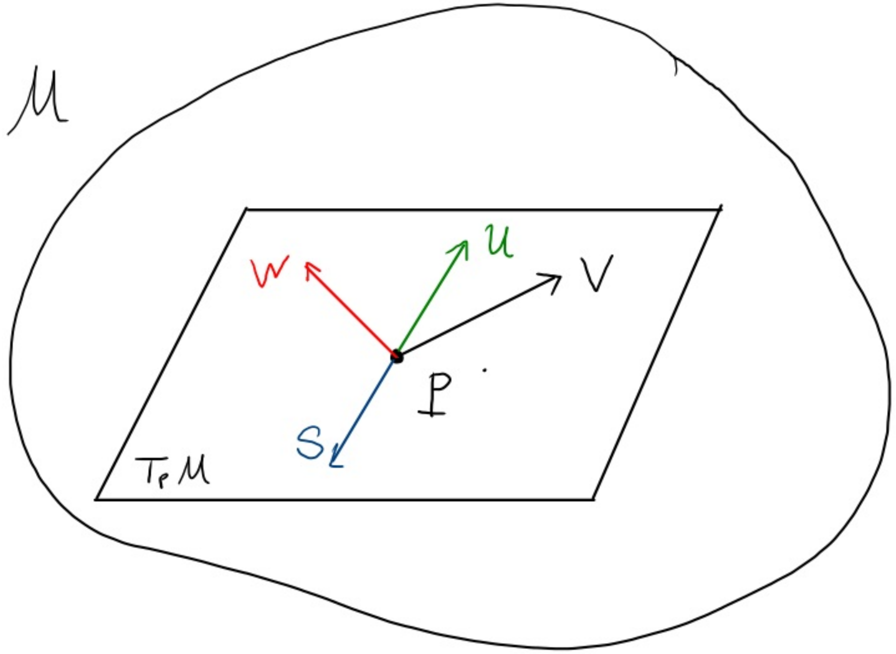


They form a vector space

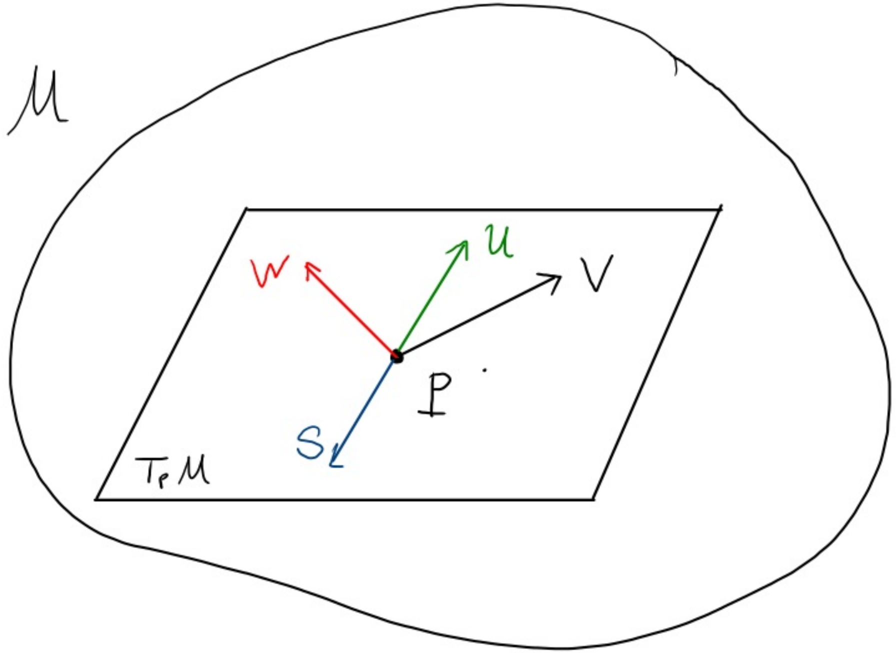
$$T_P M$$

at P

$$U, V \in T_P M \Rightarrow \alpha V + \beta U \in T_P M \qquad \alpha, \beta \in \mathbb{R}$$

Indeed: $W = \alpha V + \beta U$

is a derivation!



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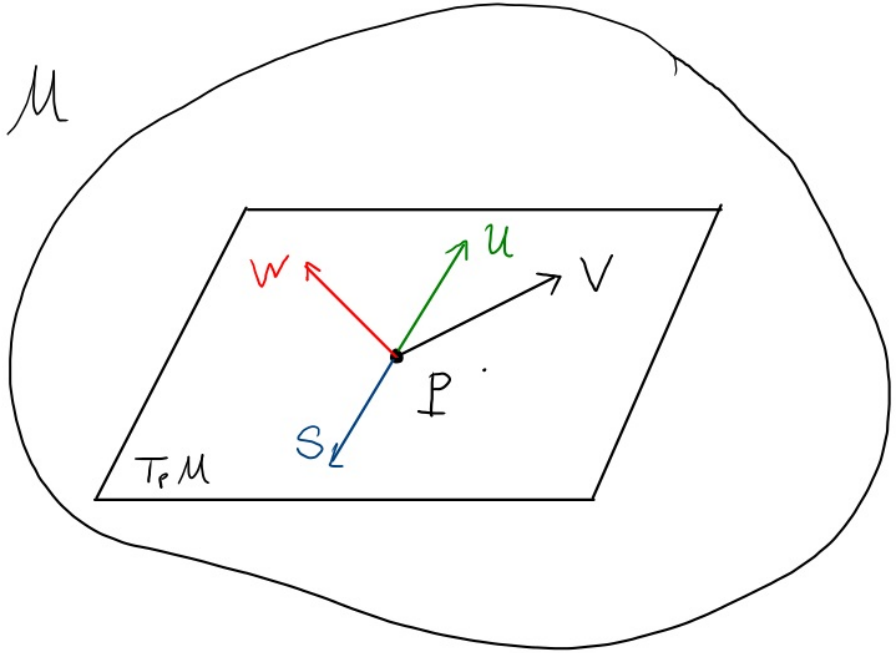
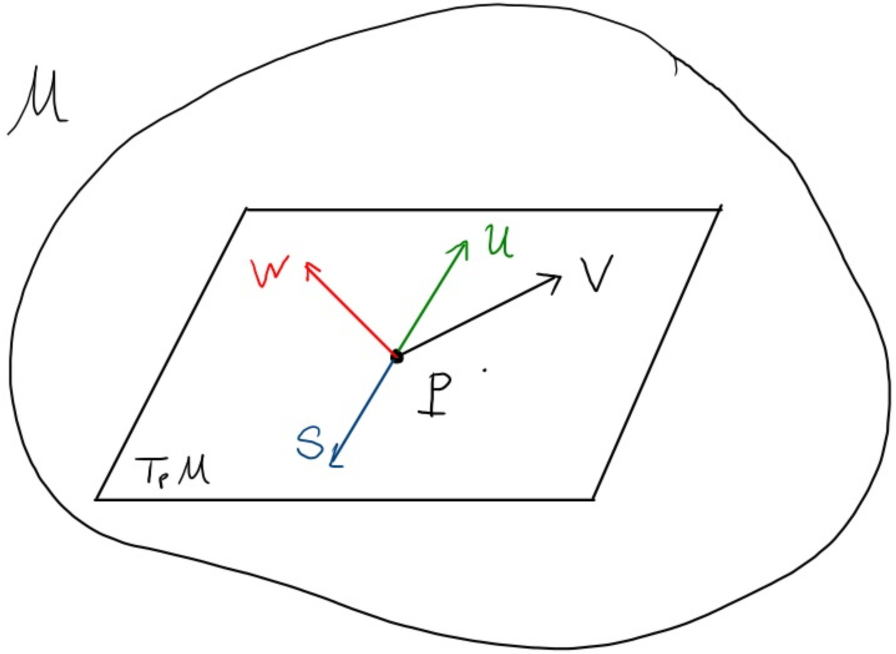


$\forall f, g \in \mathcal{F}(M)$

$$W(c_1 f + c_2 g) = c_1 W(f) + c_2 W(g)$$

$$W(f \cdot g) = W(f) \cdot g + f\, W(g)$$

$$W(c_1 f + c_2 g) = (\alpha V + \beta U)(c_1 f + c_2 g)$$



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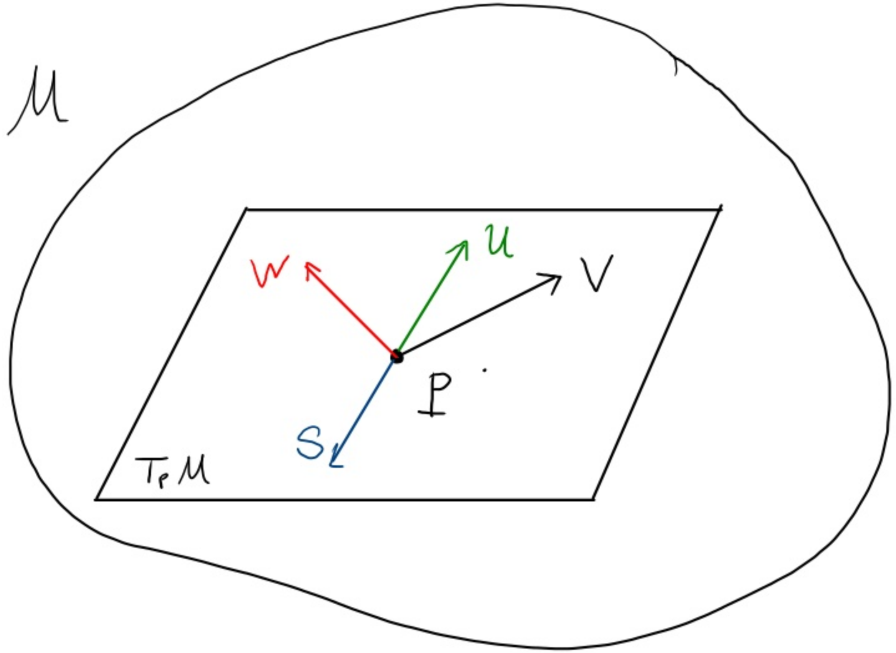

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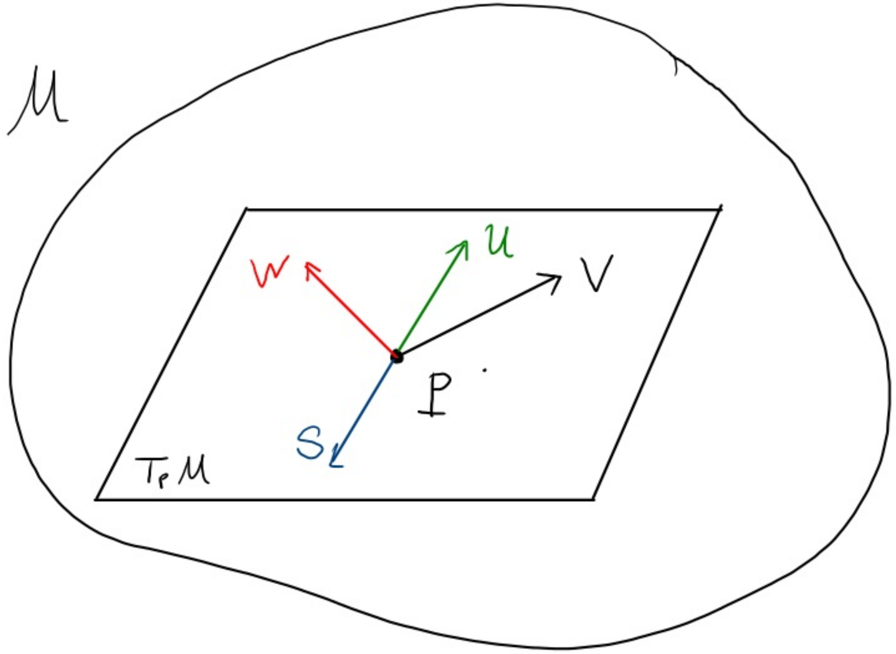

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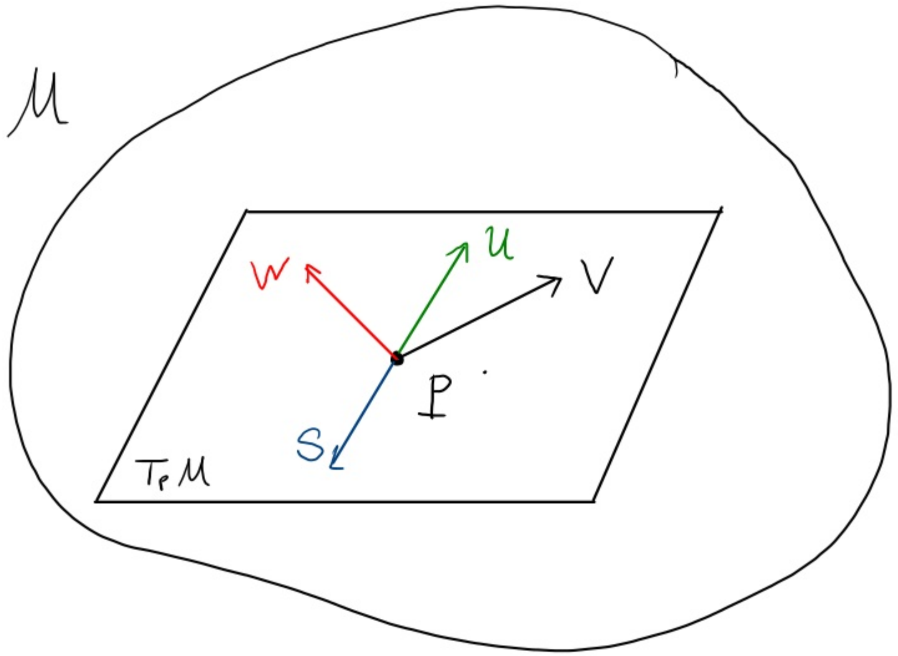

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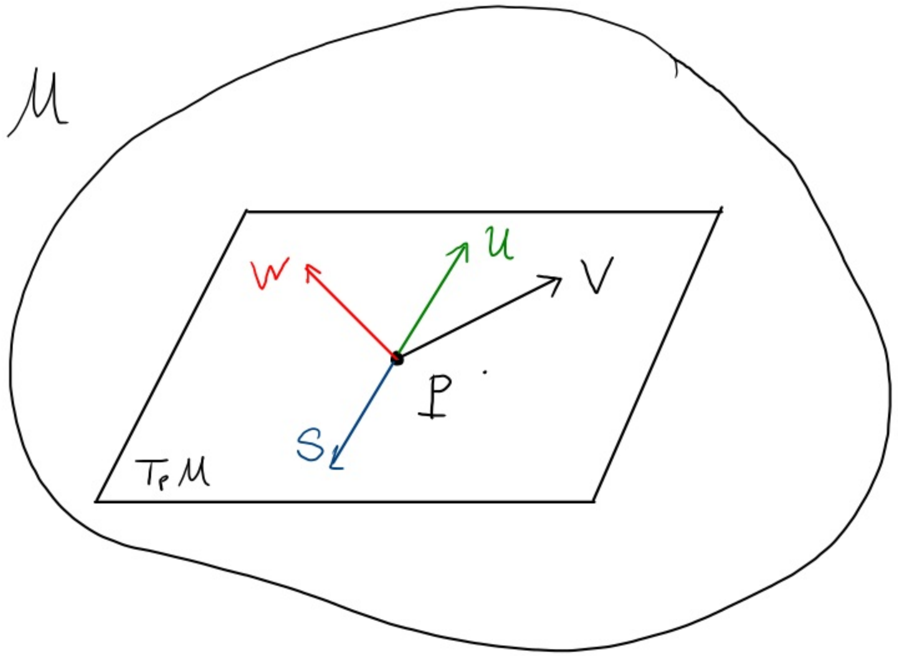

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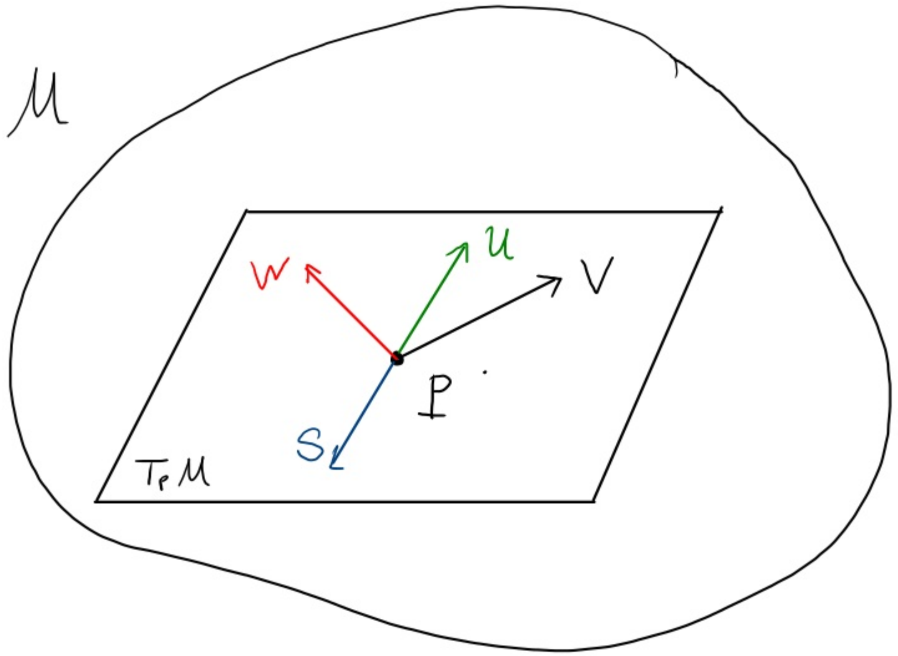

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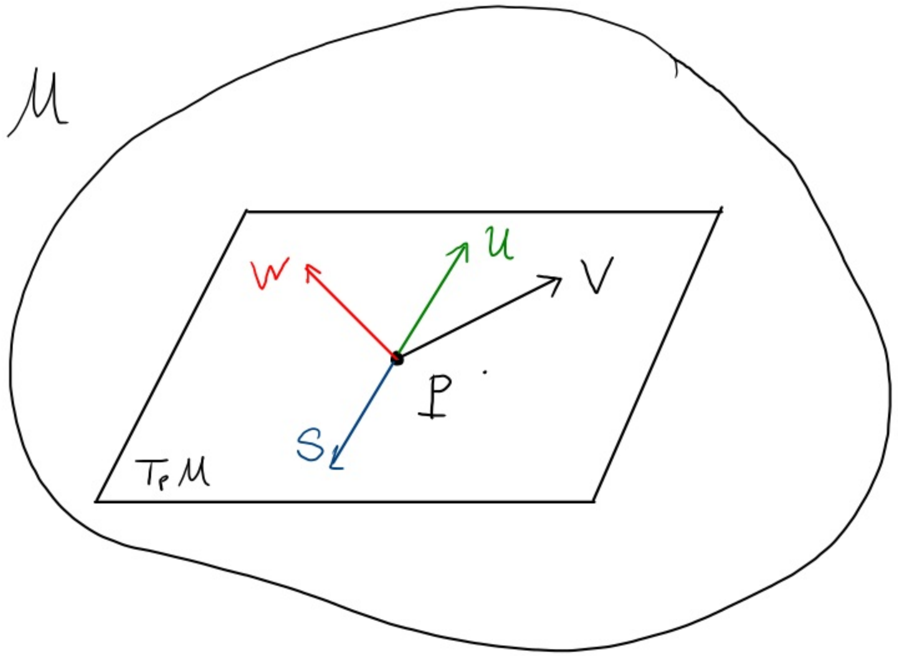

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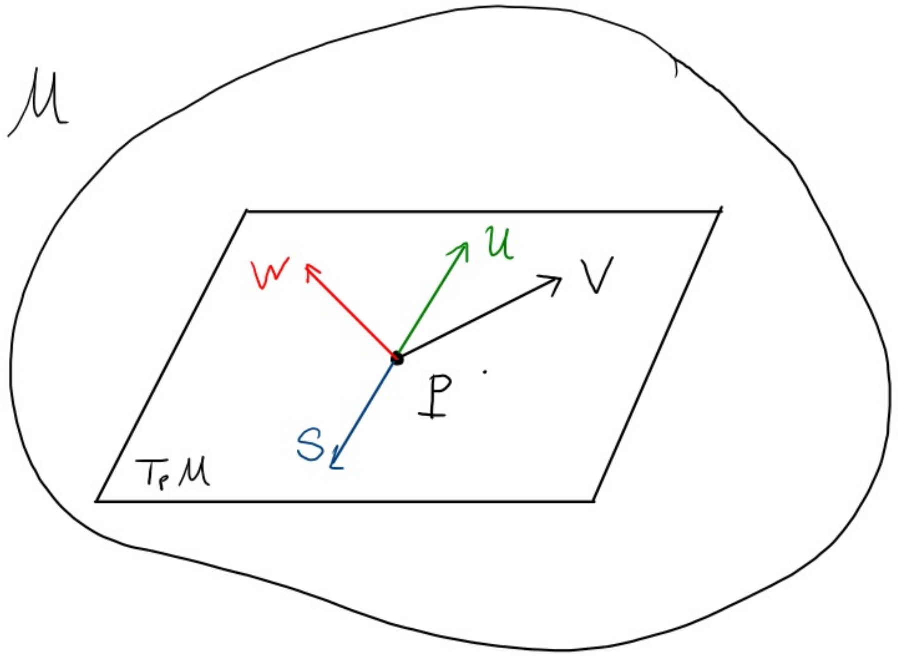
$$+ \beta\left[U(f)\cdot g + f\cdot U(g) \right]$$



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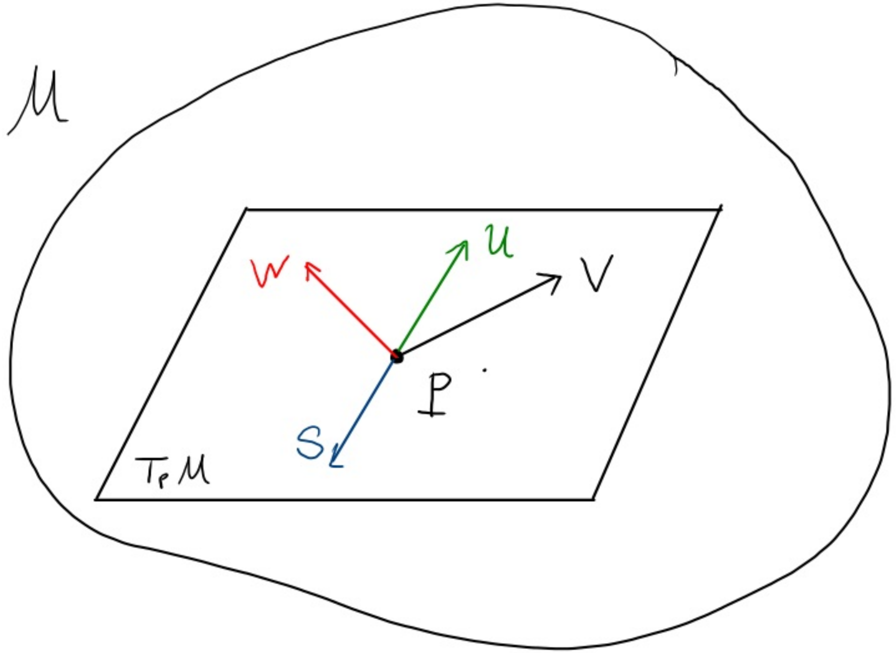

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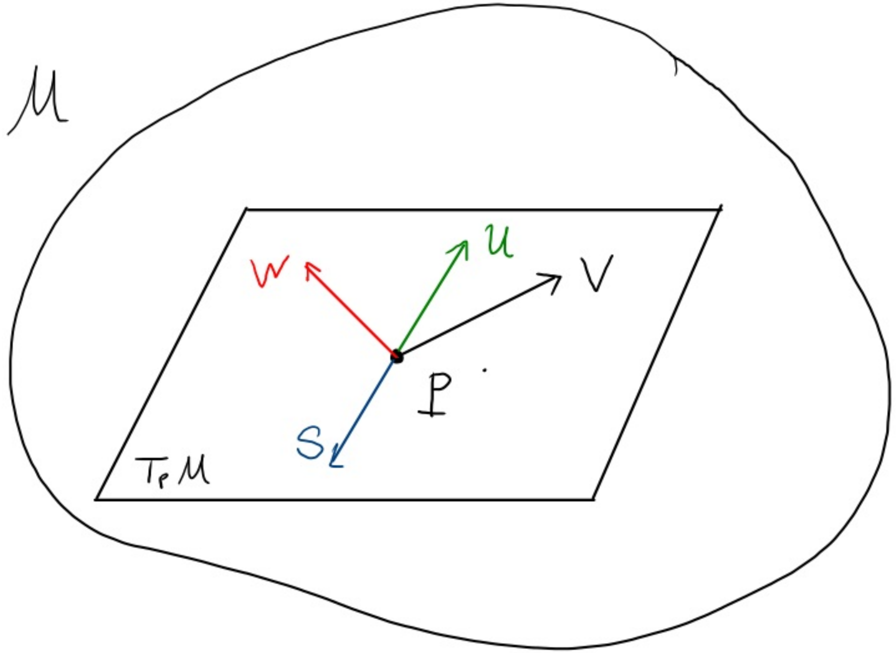
$$= \left[\alpha V(f) + \beta U(f)\right]\cdot g + f\cdot\left[\alpha V(g) + \beta U(g)\right]$$

$$= W(f) \cdot g + f\cdot W(g)$$



$$W = \alpha V + \beta W$$

a vector since $\forall\, f, g \in \mathcal{F}(M)$

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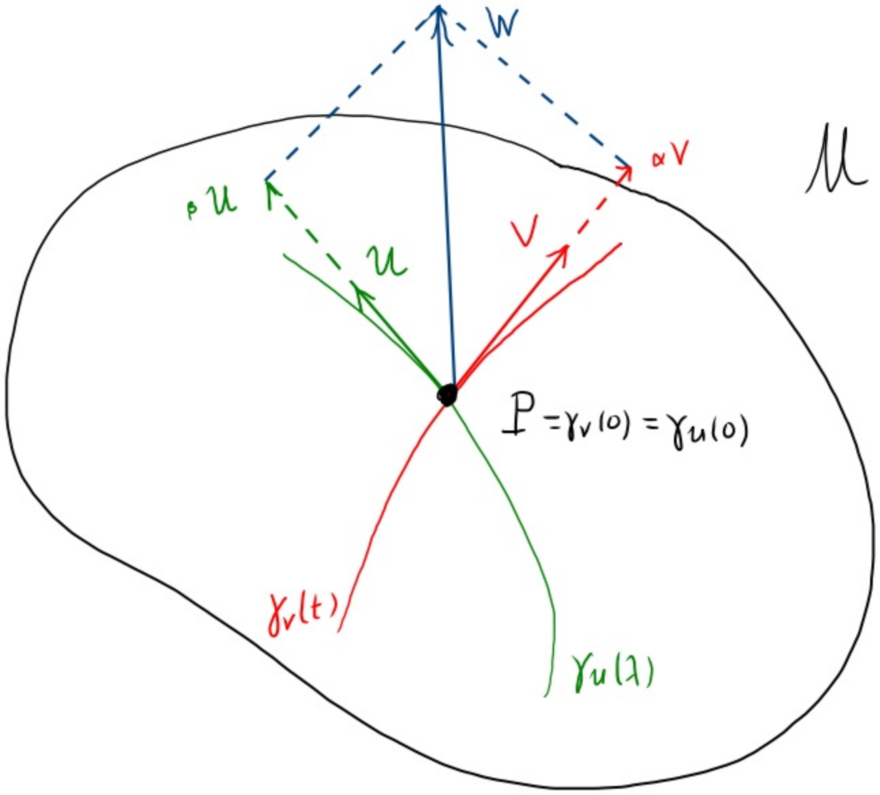
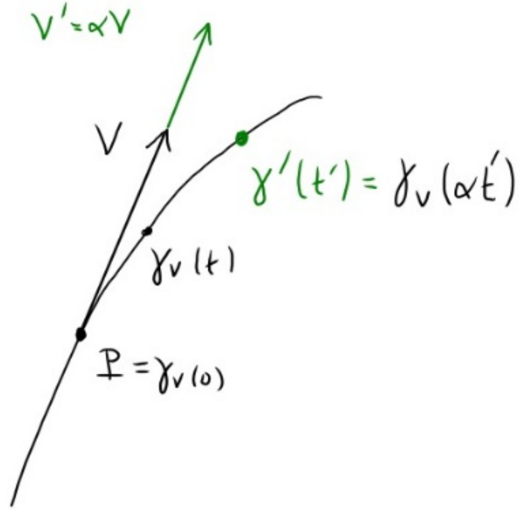
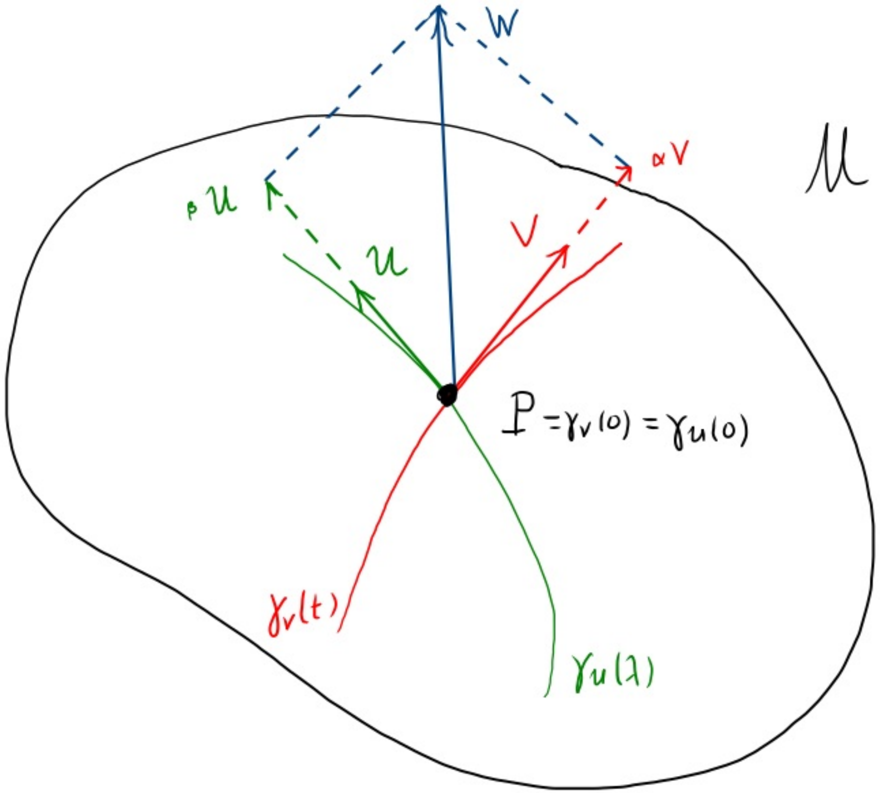
$$W(f \cdot g) = f \cdot W(g) + W(f) \cdot g$$



How can we understand it geometrically?

The vector αV is easy to understand:





It corresponds to a reparametrization of the curve

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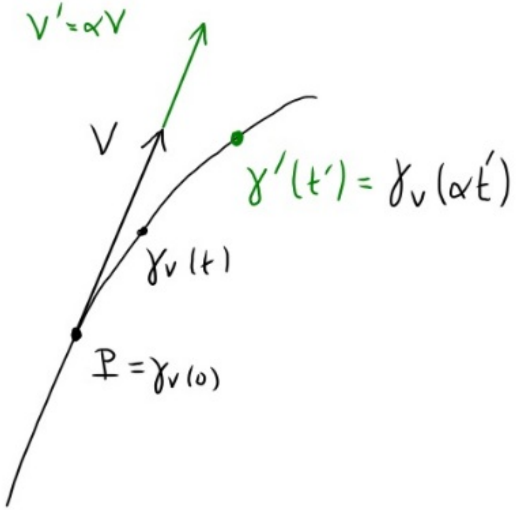

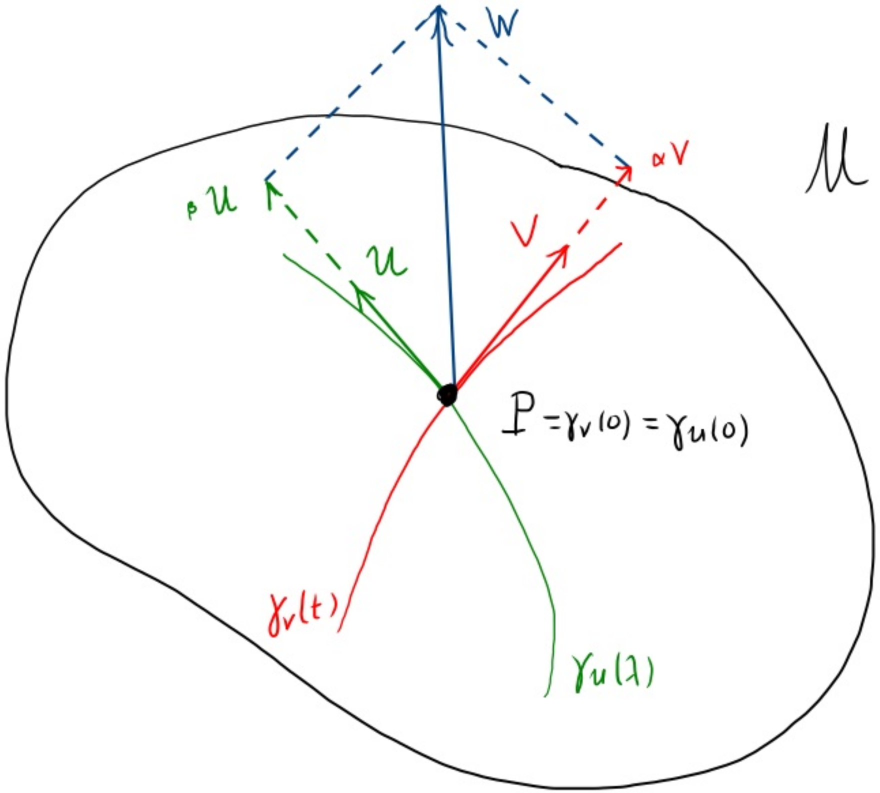

Define a new curve $\gamma'(t') = \gamma_v(\underbrace{\alpha t'}_{t = \alpha t'})$

same points, different parameter

$\Rightarrow$ different curve

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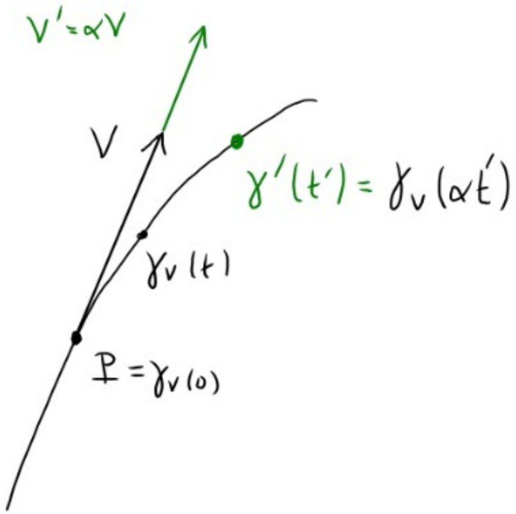

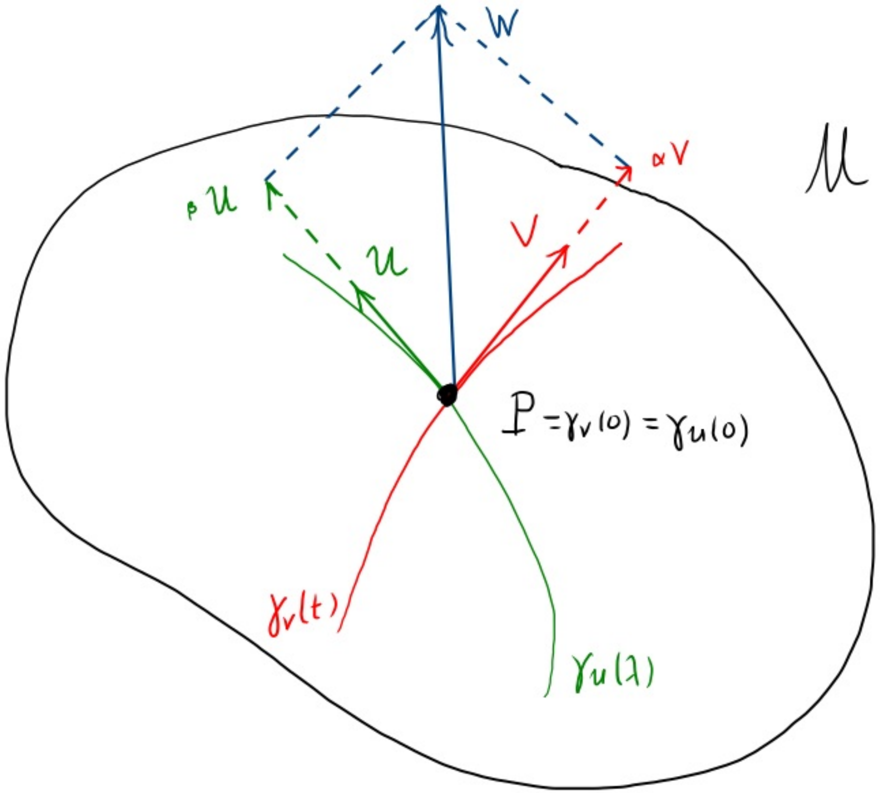

Define a new curve $\gamma'(t') = \gamma_v(\alpha t')$

$$V'_P(f) = \frac{d}{dt'} f \circ \gamma'(t')\Big|_0$$

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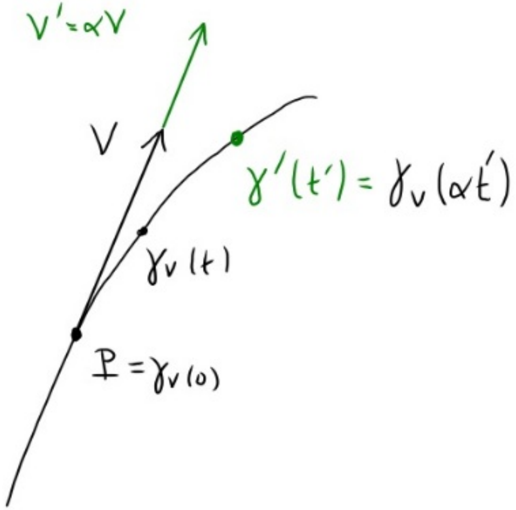

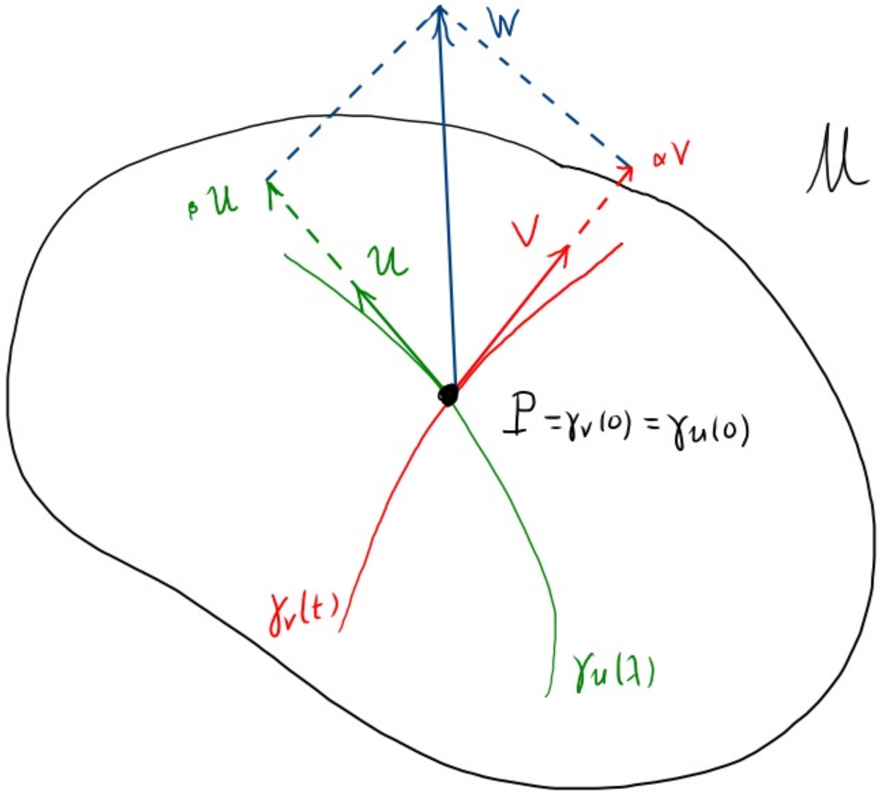

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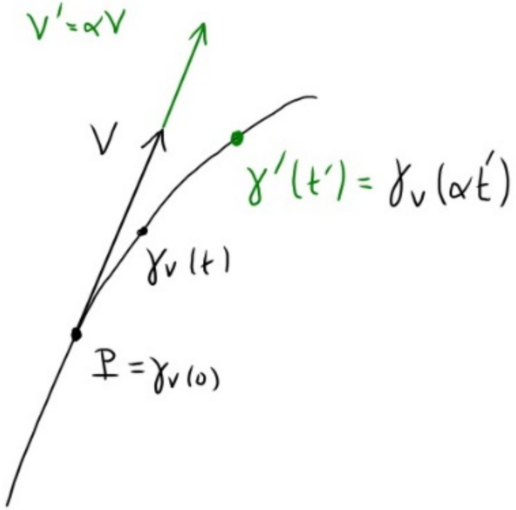
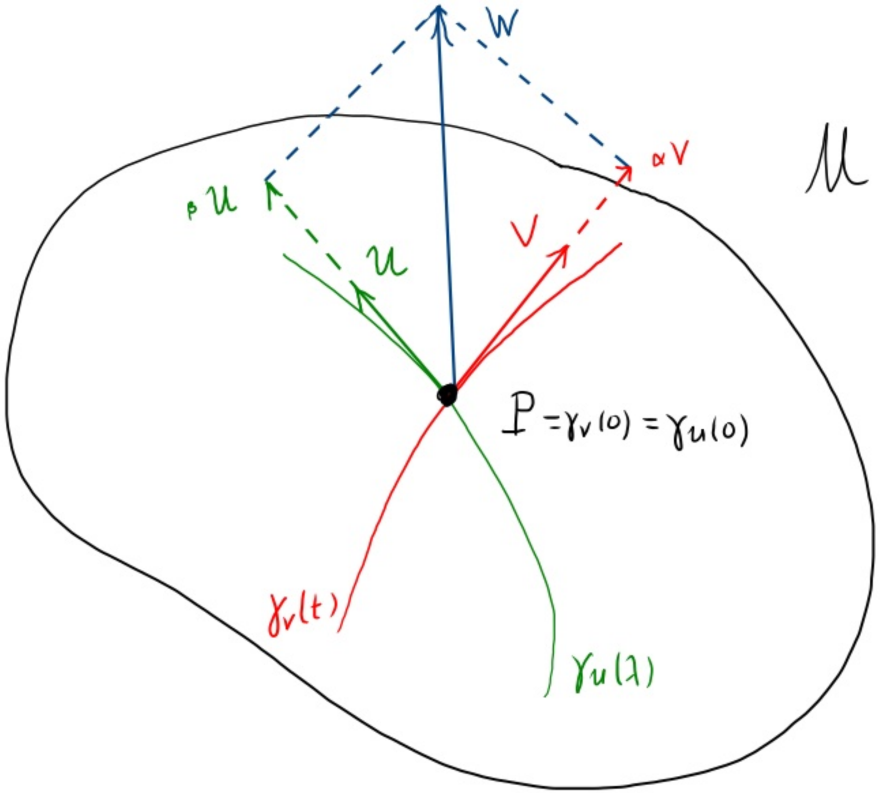
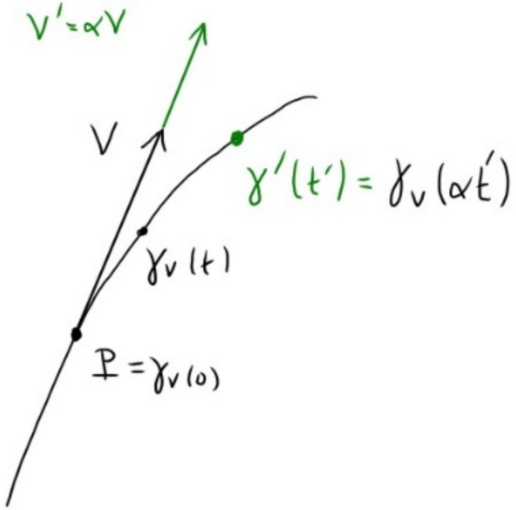


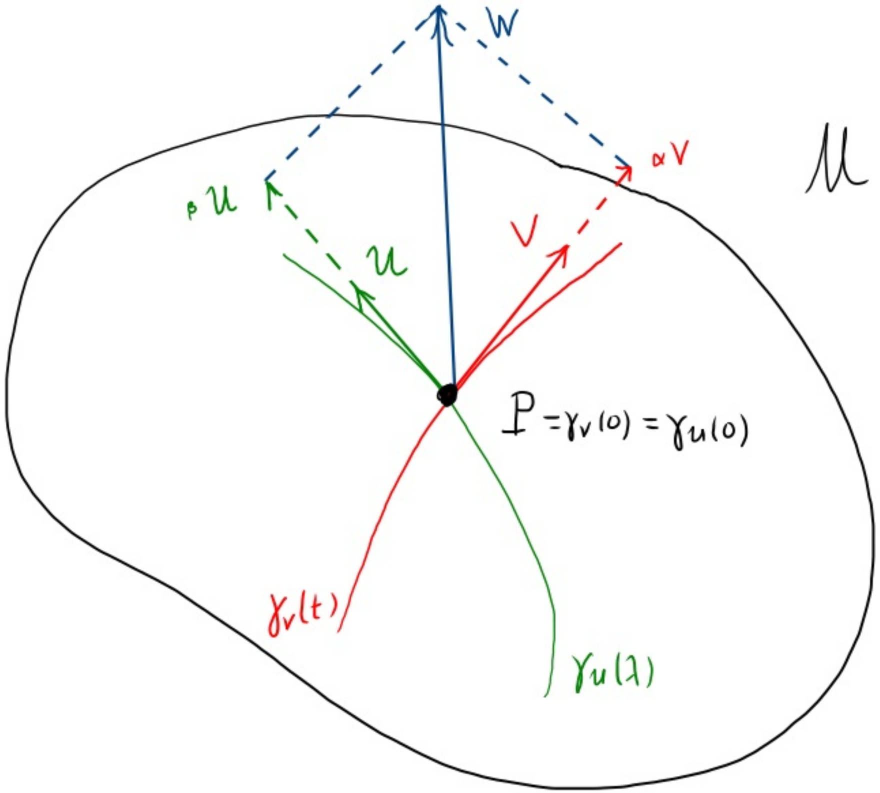


Define a new curve $\gamma'(t') = \gamma_v(\alpha t')$

$$V'_P(f) = \frac{d}{dt'} f \circ \gamma'(t')\Big|_0 = \frac{d}{dt'} f \circ \gamma_v(\alpha t')\Big|_0 = \alpha \frac{d}{dt} f \circ \gamma_v(0)$$

The vector αV is easy to understand:



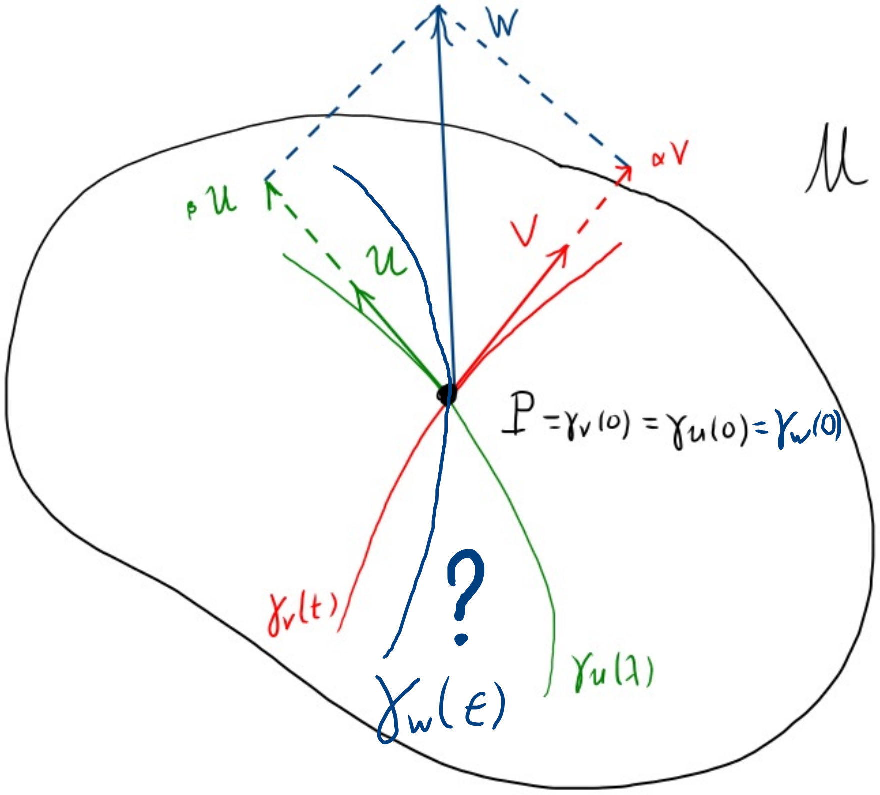


Define a new curve $\gamma'(t') = \gamma_v(\alpha t')$

$$V'_P(f) = \alpha V_P(f)$$

But is there a class of curves like, e.g. $\gamma_w(\epsilon)$ such that

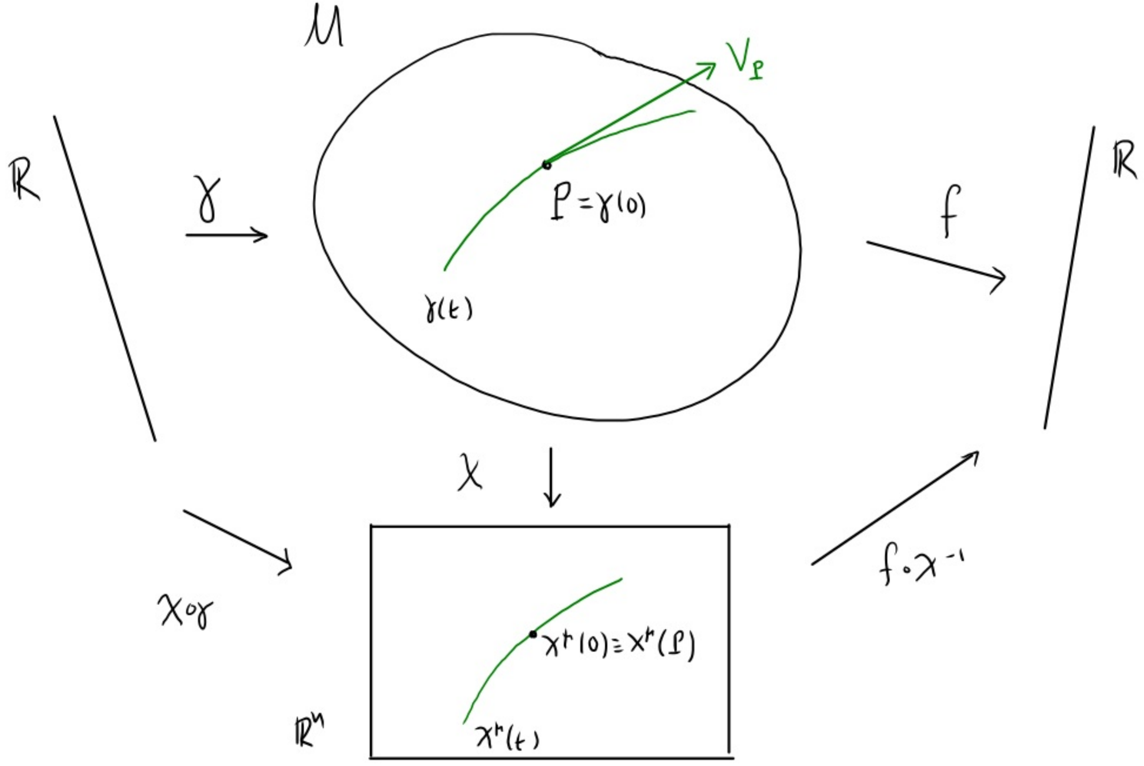
$$\alpha \left.\frac{df}{dt}\right|_p + \beta \left.\frac{df}{d\lambda}\right|_p = \left.\frac{df}{d\varepsilon}\right|_p \quad \forall f?$$



yes ... let's see why:

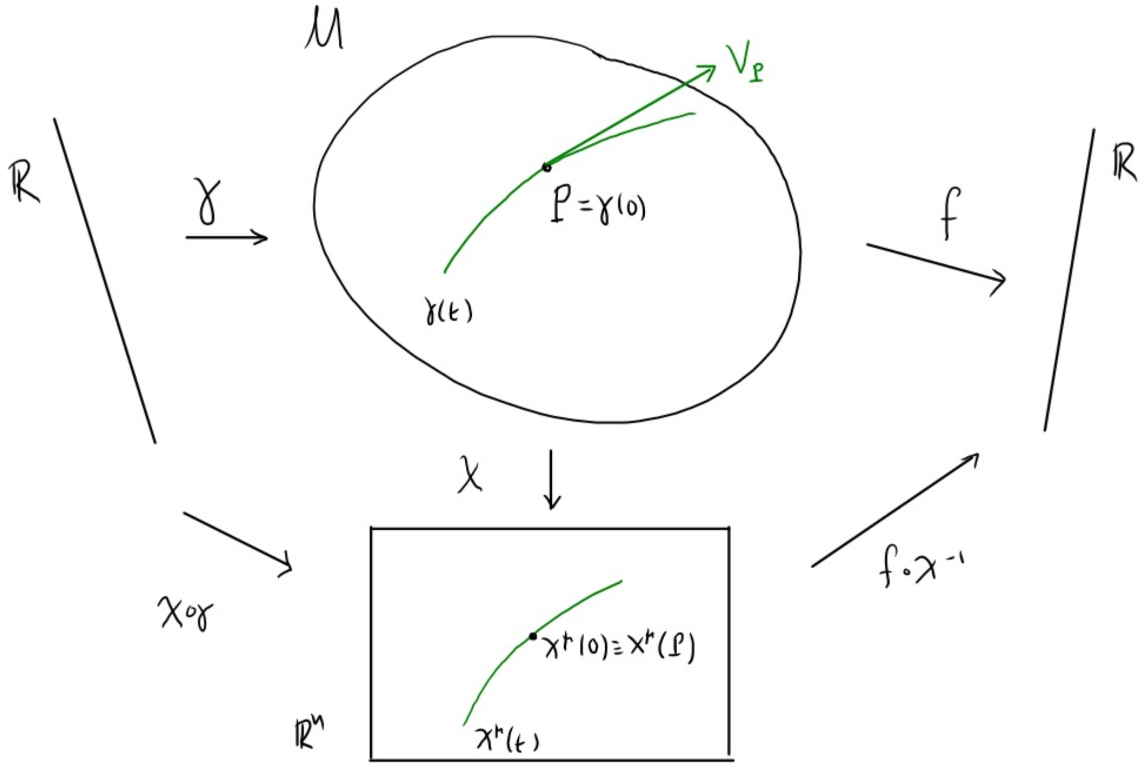
Consider $V_p(f)$:

$$V_p(f) = \left.\frac{df}{dt}\right|_p$$



Consider $V_p(f)$:

$$V_p(f) = \frac{df}{dt}\Big|_p = \frac{d}{dt} f\circ\gamma(t)\Big|_0$$



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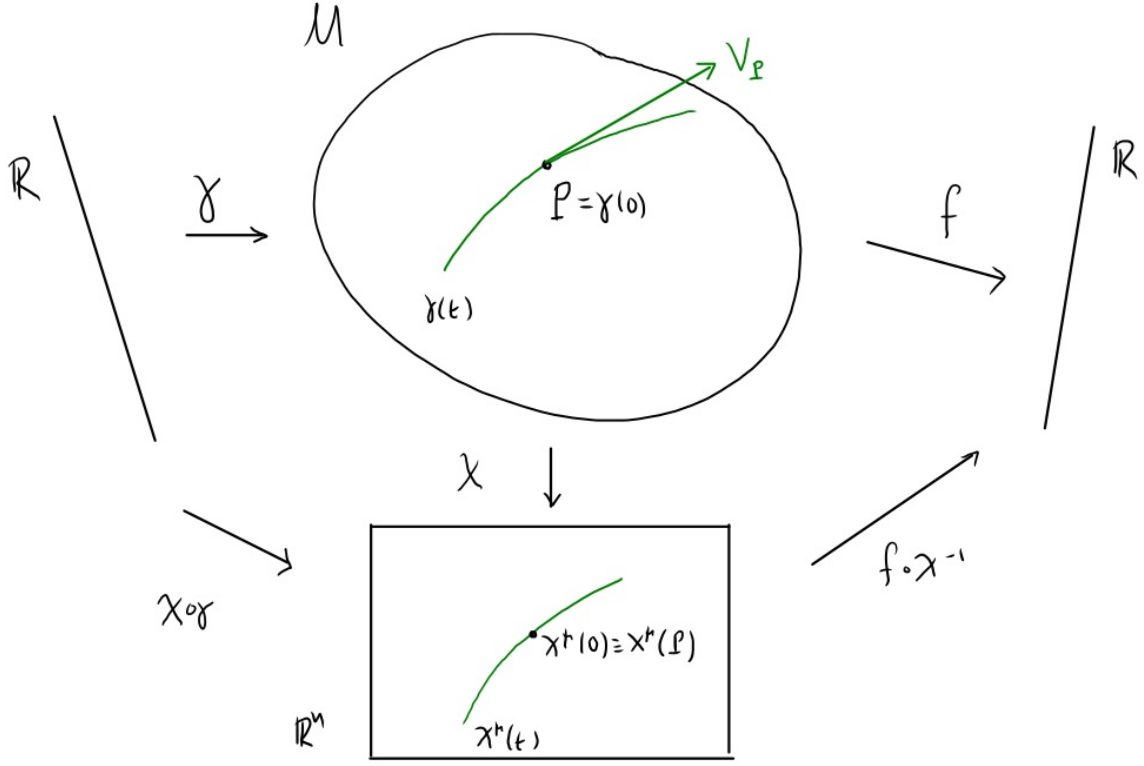

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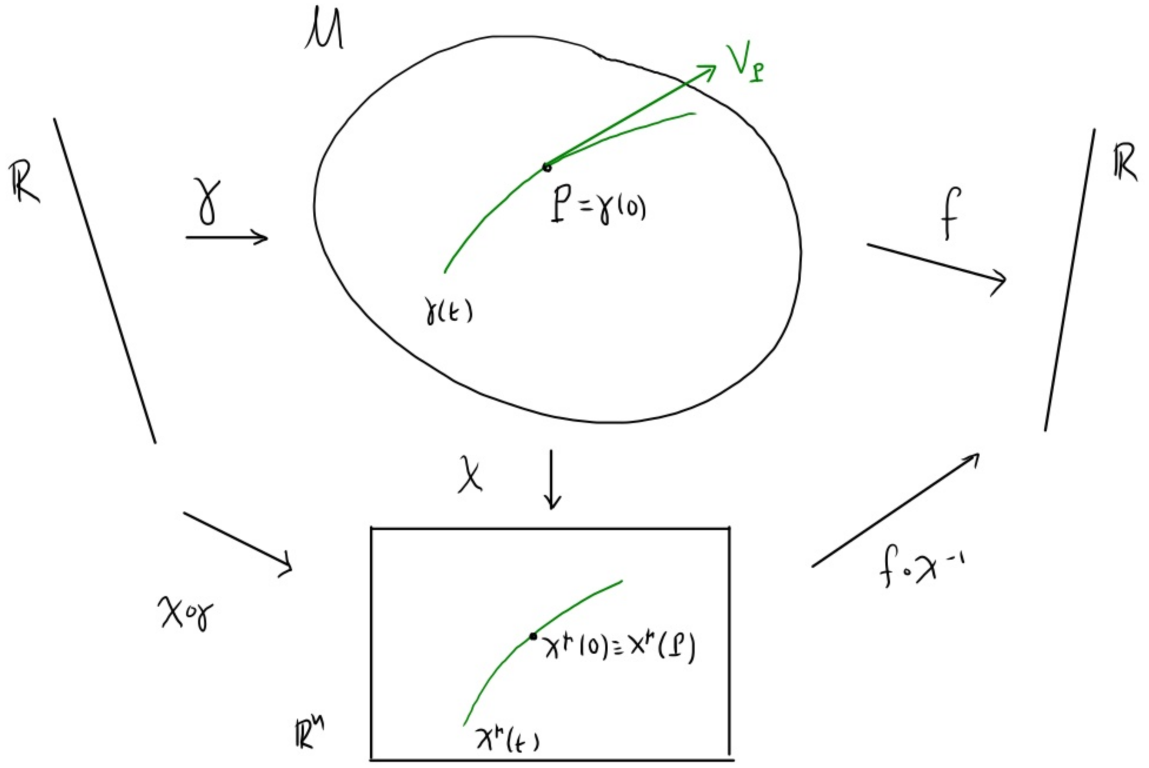
$$= \frac{d}{dt} \underbrace{f\circ\chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi\circ\gamma(t)}_{x^\mu(t)}\Big|_0$$

function of coordinates

Image of curve in $\mathbb{R}^n$



$f(x^\nu) \equiv f(x^0, x^1, \dots, x^{n-1})$

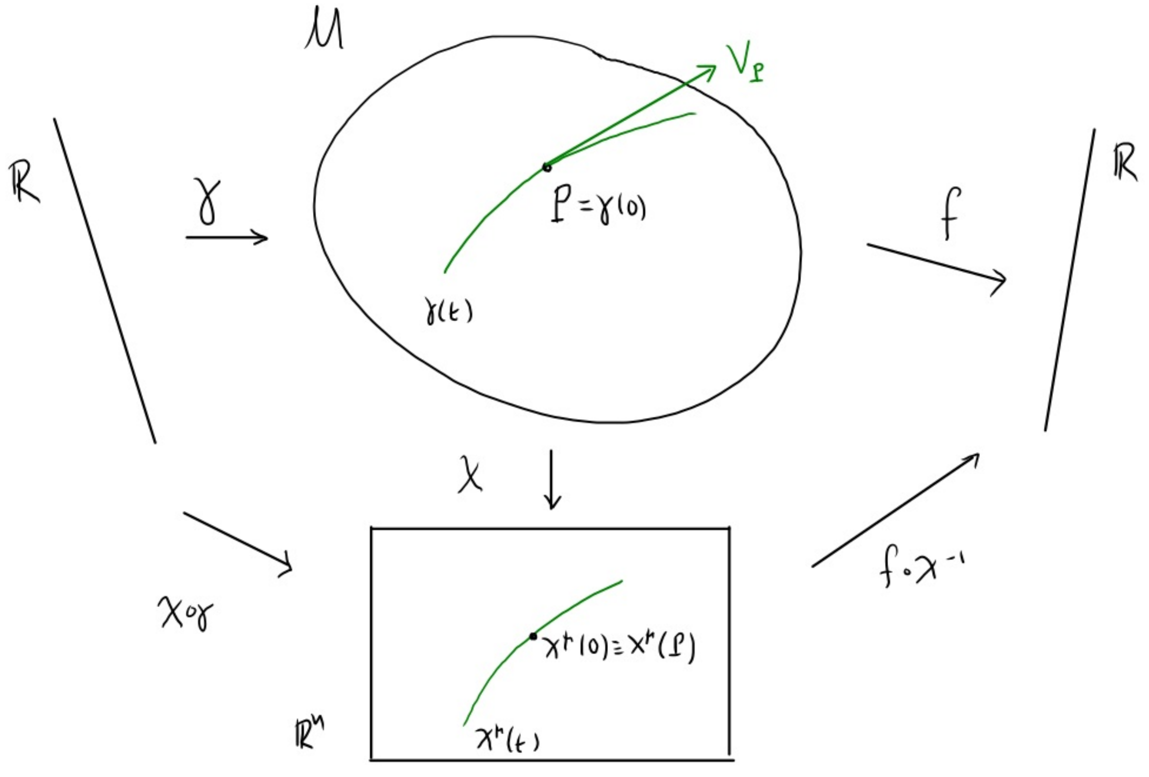
$x^\mu(t) = \left(x^0(t), x^1(t), \dots, x^{n-1}(t)\right)$

Consider $V_p(f)$:

$$V_p(f) = \frac{df}{dt}\Big|_p = \frac{d}{dt} f\circ\gamma(t)\Big|_0$$

$$= \frac{d}{dt}\Big(\underbrace{(f\circ\chi^{-1})}_{f(x^\nu)}\circ\underbrace{(\chi\circ\gamma(t))}_{x^\mu(t)}\Big)\Big|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu} \circ \frac{dx^\mu(t)}{dt}\Big|_0$$



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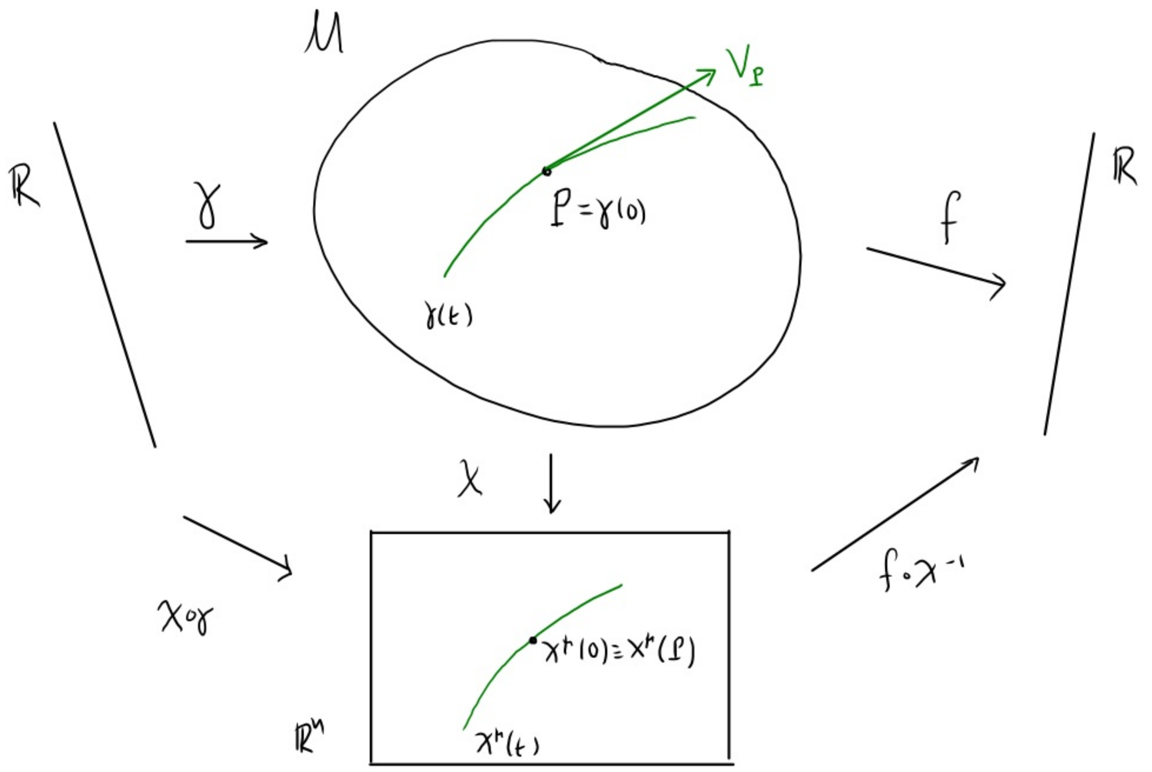


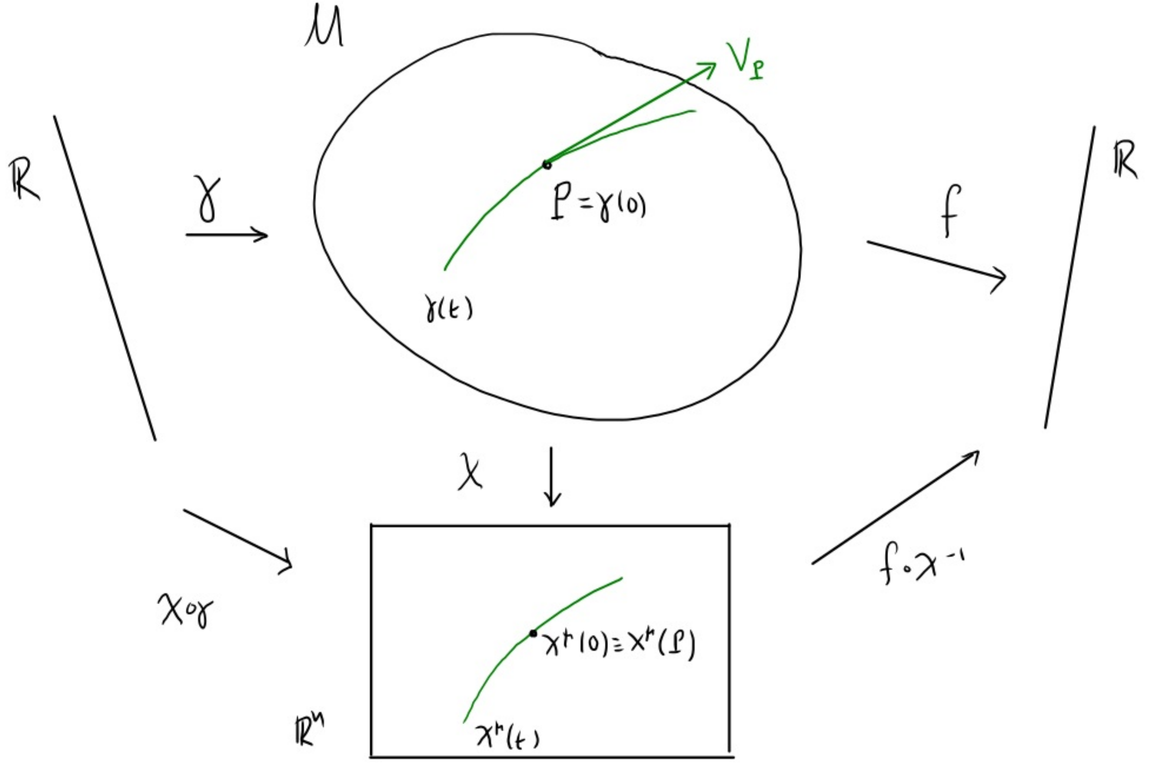
Einstein convention: sum over repeated indices

$$= \sum_{\mu=0}^{n-1} \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}$$

Consider $V_p(f)$:

$$V_p(f) = \frac{\partial f(x^\nu)}{\partial x^\mu} \frac{dx^\mu}{dt}\Big|_0$$

$$\equiv \frac{dx^\mu}{dt} \cdot \underbrace{\partial_\mu f}_{\text{Notation for } \frac{\partial f}{\partial x^\mu}} \Big|_0$$



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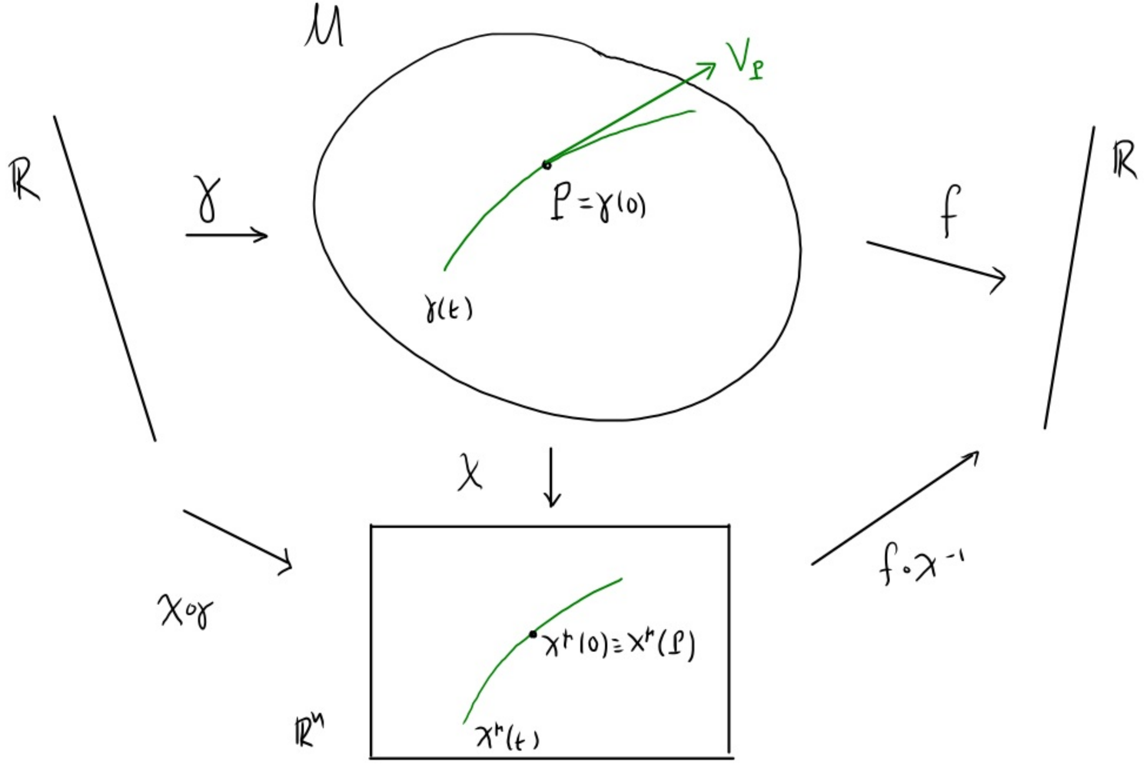
$$\equiv \frac{dx^\mu}{dt} \cdot \partial_\mu f \Big|_0$$

We write:

$$V_p = \frac{dx^\mu}{dt} \cdot \partial_\mu$$



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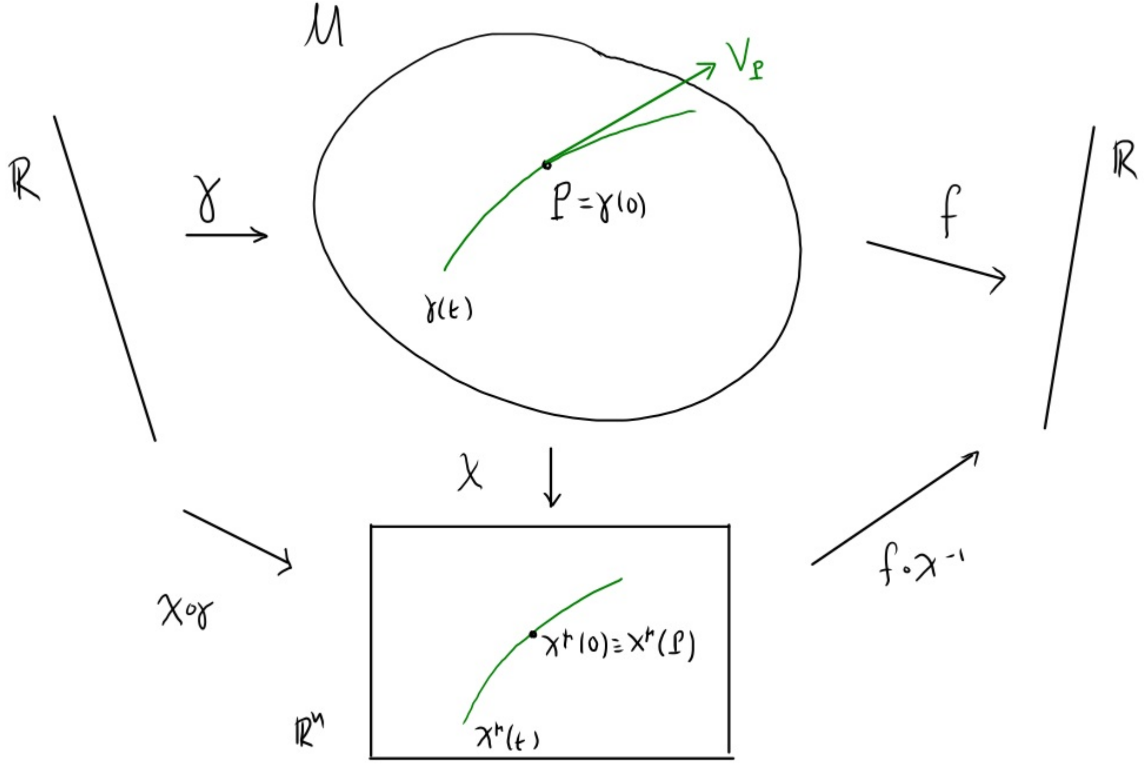


We write:

$$V_p = \underbrace{\frac{dx^\mu}{dt} \cdot \partial_\mu}$$

a linear combination of vectors ∂_μ

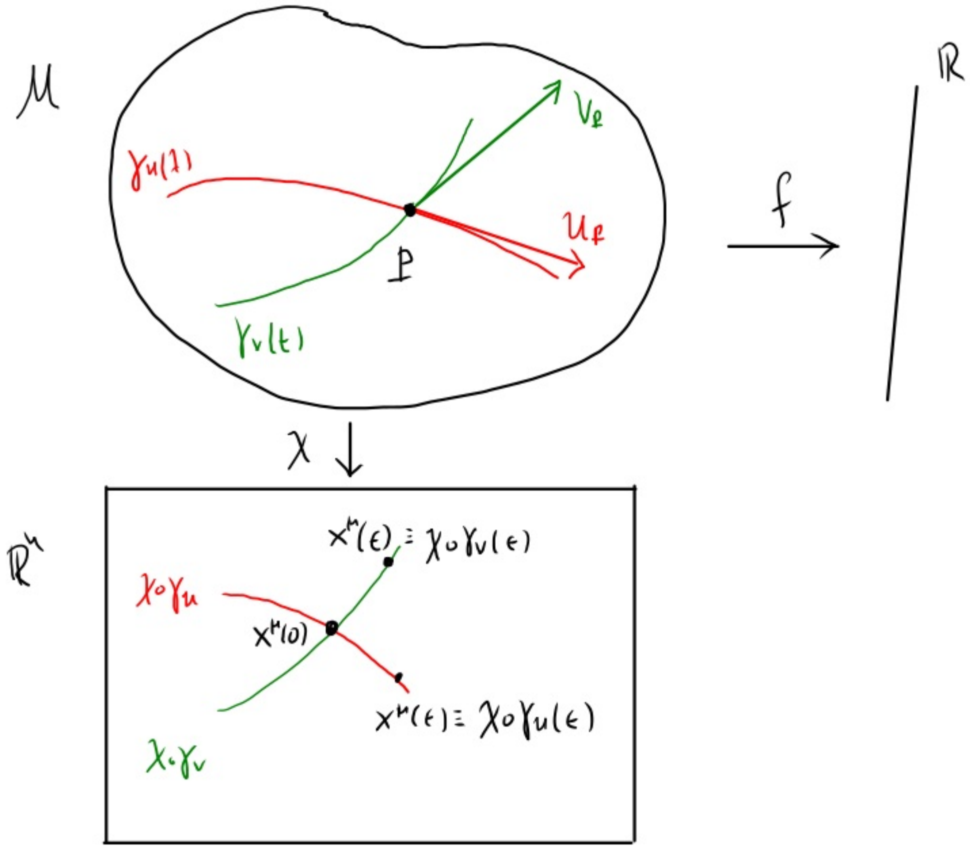
$\{\partial_\mu\}$ a basis in $T_p M$

Back to the proof that $\alpha V + \beta U$ a vector:

Consider the curves

$\gamma_v(t)$ such that $\gamma_v(0) = P$

$\gamma_u(t)$ $\gamma_u(0) = P$



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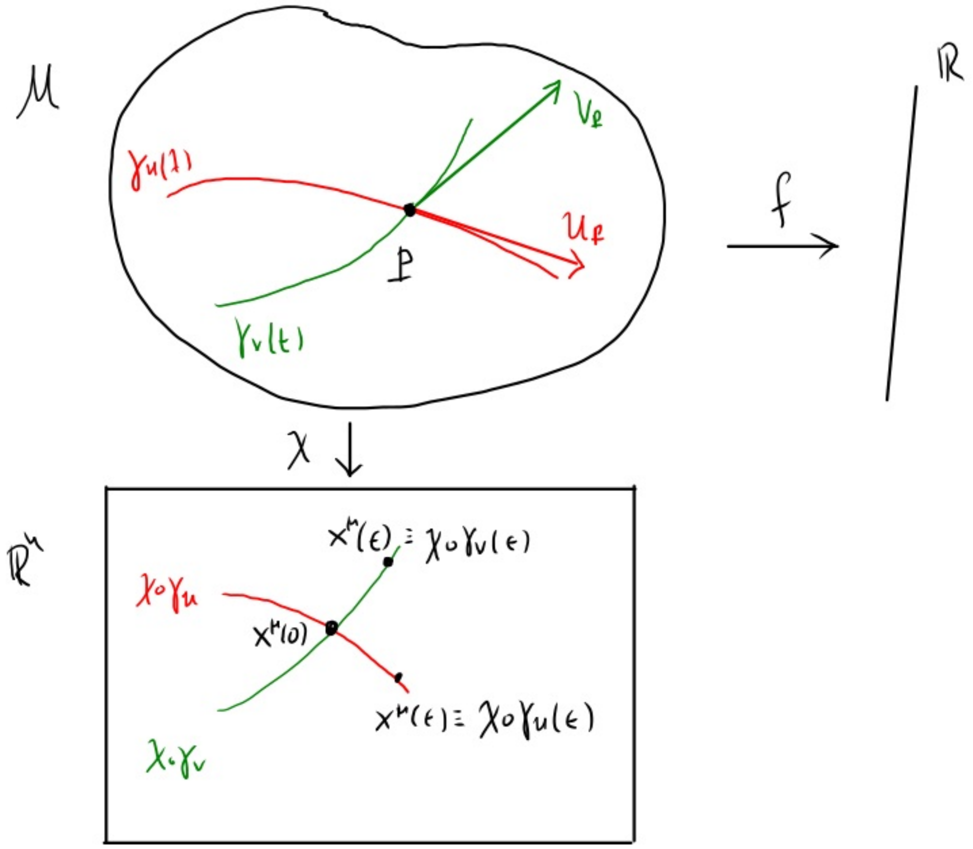


Consider the curves

$\gamma_v(t)$ such that $\gamma_v(0) = p$

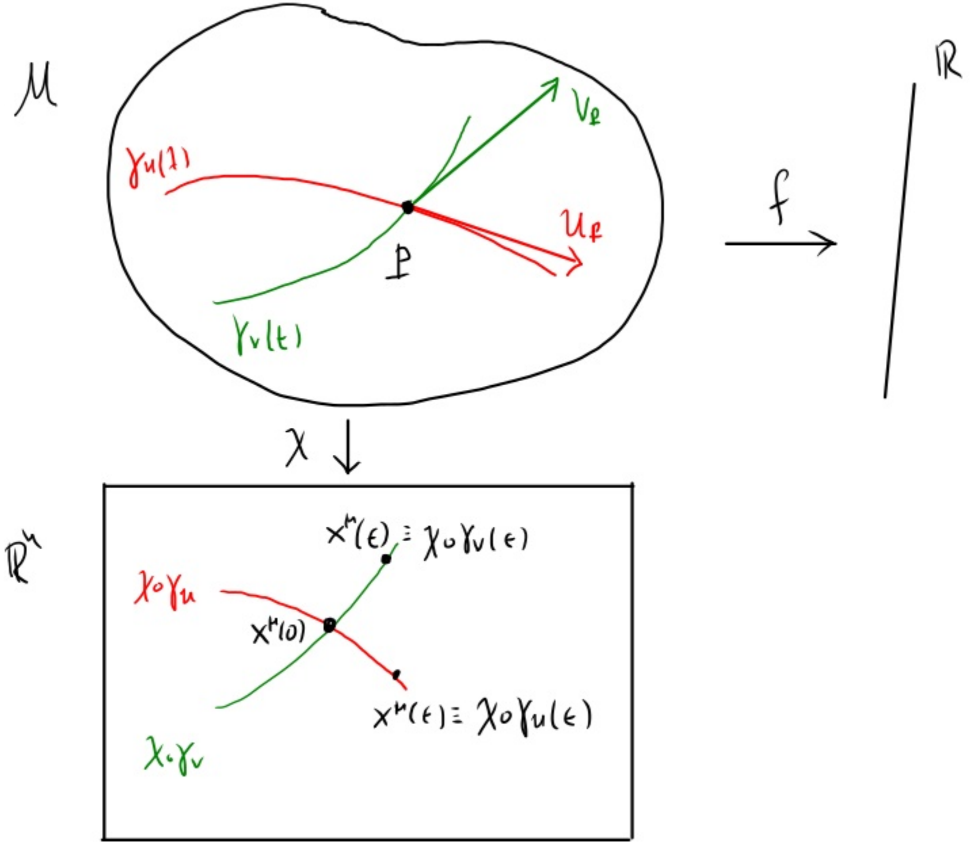
$\gamma_u(t)$ $\qquad\qquad\qquad\qquad$ $\gamma_u(0) = p$

Then

$$V_p(f) = \frac{df}{dt}\bigg|_p = \frac{dx^\mu}{dt} \cdot \frac{\partial f}{\partial x^\mu}\bigg|_p$$

$$U_p(f) = \frac{df}{d\lambda}\bigg|_p = \frac{dx^\mu}{d\lambda} \cdot \frac{\partial f}{\partial x^\mu}\bigg|_p$$

Back to the proof that $\alpha V + \beta U$ a vector:



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rate of change on $\chi\circ\gamma_v(t)$

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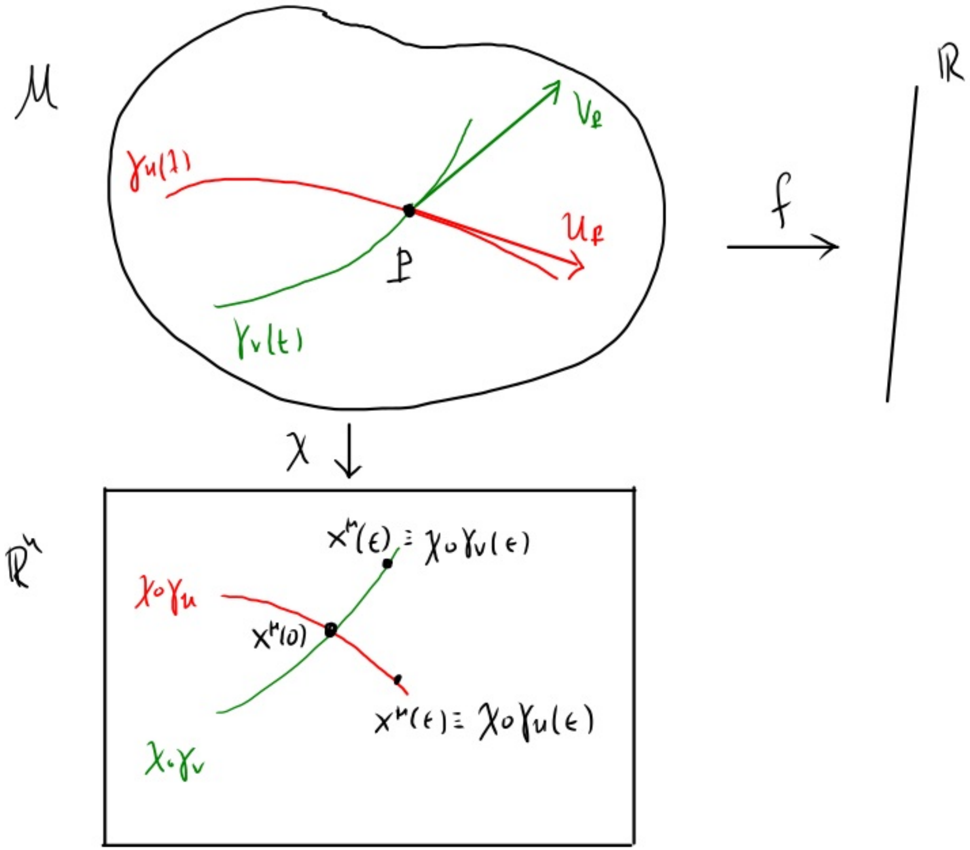
rate of change on $\chi\circ\gamma_u(\lambda)$

For a coordinate system with

$\chi : \underline{P} \mapsto x^{\mu}(\underline{P})$

choose a specific μ



For a coordinate system with

$\chi : P \mapsto x^\mu(P)$

choose a specific μ

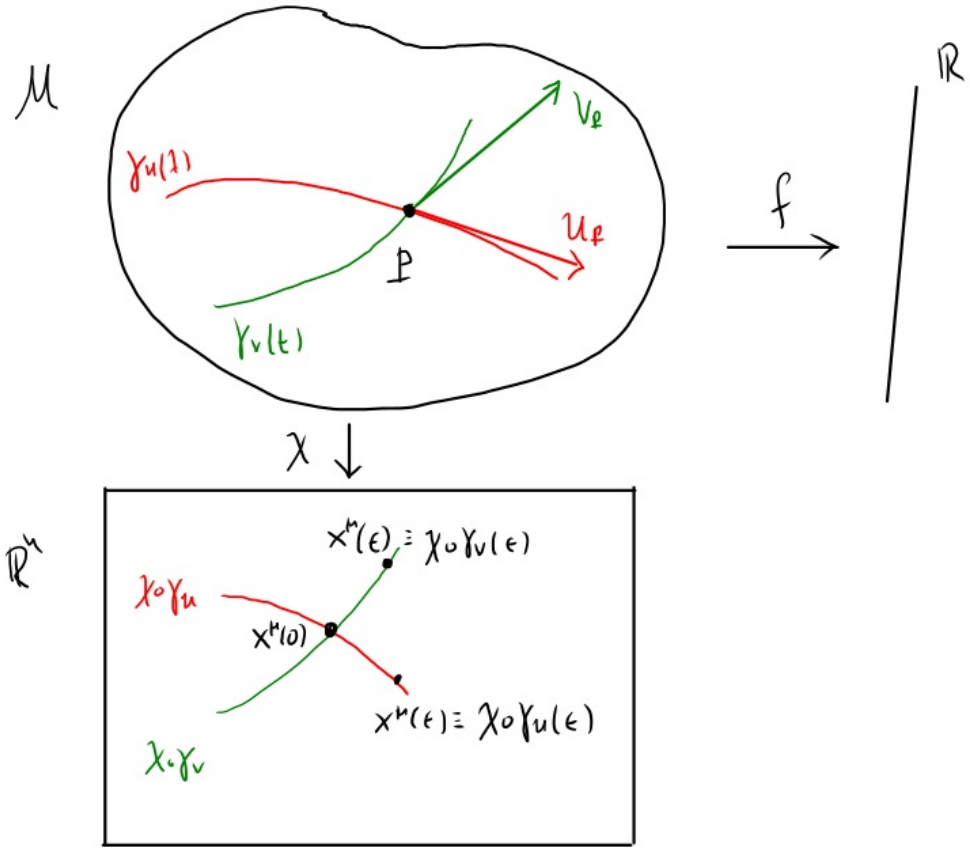
Then

$x^\mu : P \mapsto x^\mu(P)$

a real function on $\mathcal{M}$

such that:

$$V_P(x^\mu) = \frac{dx^\nu}{dt} \cdot \frac{\partial x^\mu}{\partial x^\nu}\Big|_P$$



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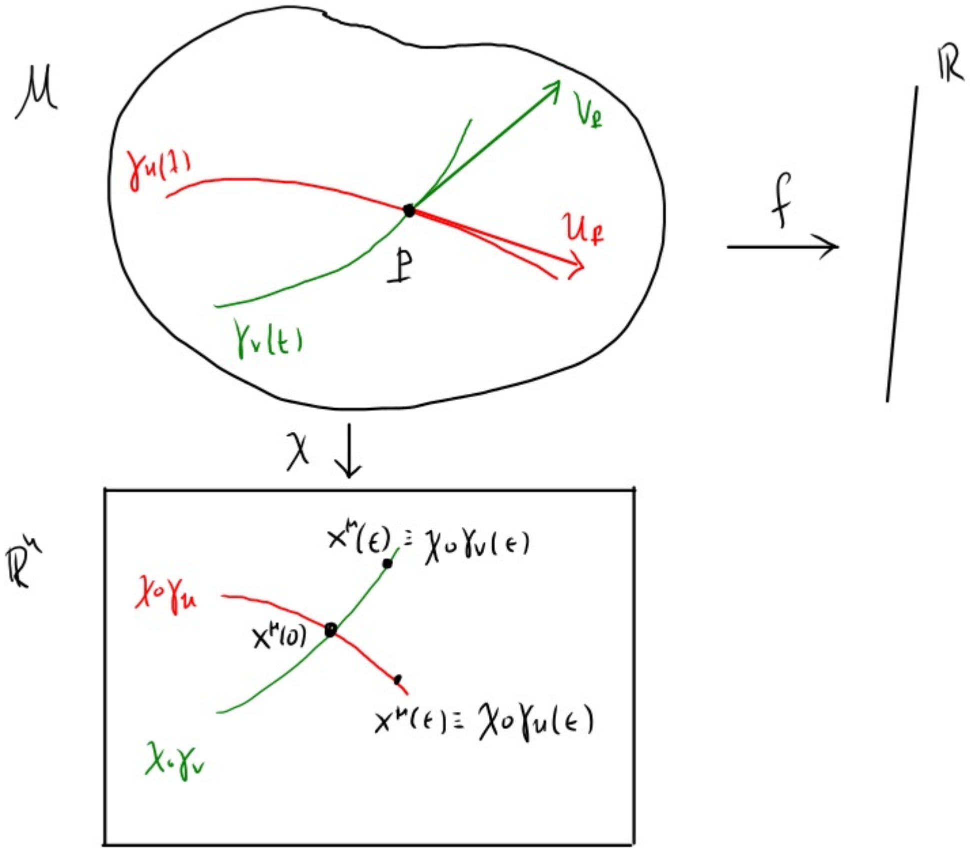

$$V_{\underline{P}}(x^\mu) = \frac{dx^\nu}{dt} \cdot \frac{\partial x^\mu}{\partial x^\nu}\bigg|_{\underline{P}} = \frac{dx^\nu}{dt}\,\delta^\mu{}_\nu\bigg|_{\underline{P}} = \frac{dx^\mu}{dt}\bigg|_{\underline{P}}$$

For a coordinate system with

$\chi : P \mapsto x^\mu(P)$

choose a specific μ

Then

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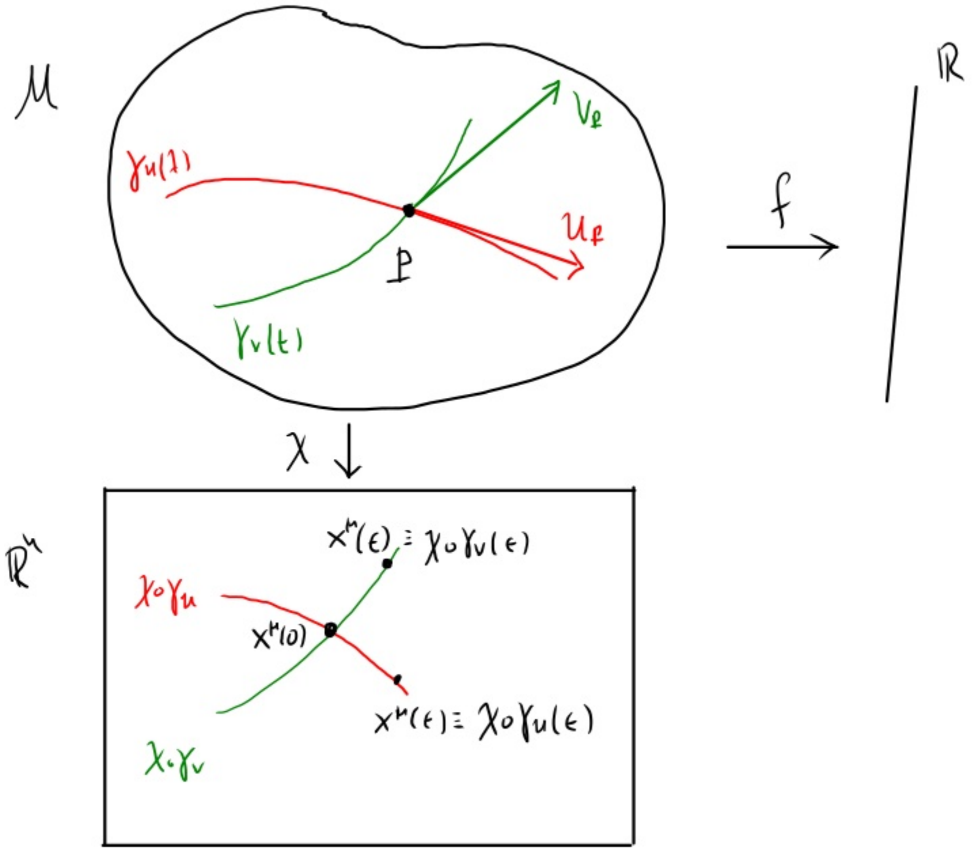
a real function on $\mathcal{M}$

such that:

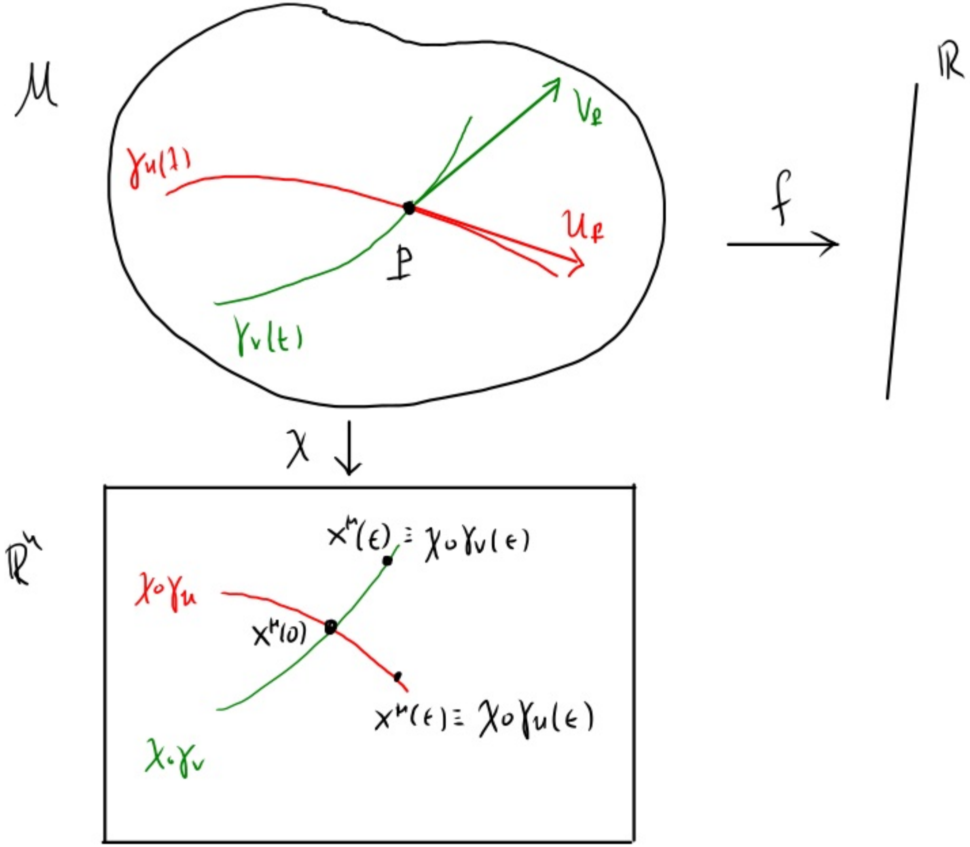
$$V_P(x^\mu) = \left. \frac{dx^\mu}{dt} \right|_P$$



For a coordinate system with

$$\chi : P \mapsto x^{\mu}(P)$$

choose a specific μ



Then

$$x^{\mu} : P \mapsto x^{\mu}(P)$$

a real function on $\mathcal{M}$

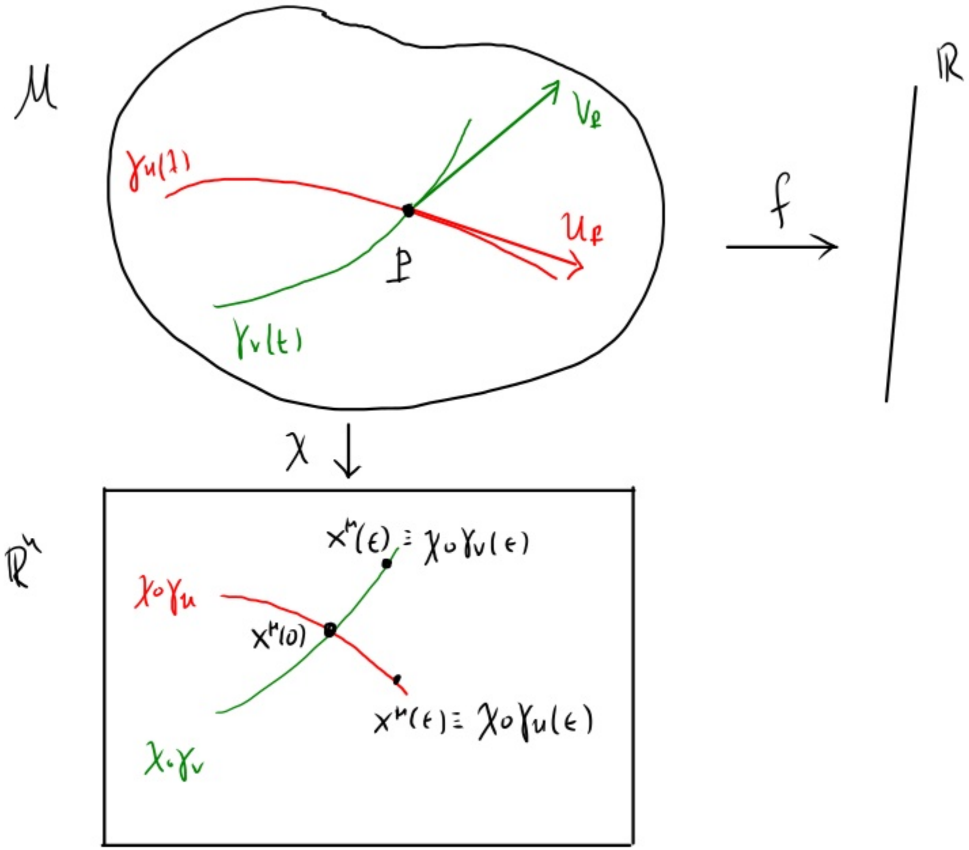
such that:

$$V_P(x^{\mu}) = \frac{dx^{\mu}}{dt}\Big|_P \Rightarrow V_P(f) = \frac{dx^{\mu}}{dt}\frac{\partial f}{\partial x^{\mu}}\Big|_P = V_P(x^{\mu})\,\partial_{\mu} f\,\Big|_P$$

Then

$$U_p(x^\mu) = \frac{dx^\mu}{d\lambda}\bigg|_p \qquad \text{and}$$

$$U_p(f) = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\bigg|_p = U_p(x^\mu)\,\partial_\mu f\,\bigg|_p$$



$$V_p(x^\mu) = \frac{dx^\mu}{dt}\bigg|_p \Rightarrow V_p(f) = \frac{dx^\mu}{dt}\frac{\partial f}{\partial x^\mu}\bigg|_p = V_p(x^\mu)\,\partial_\mu f\,\bigg|_p$$

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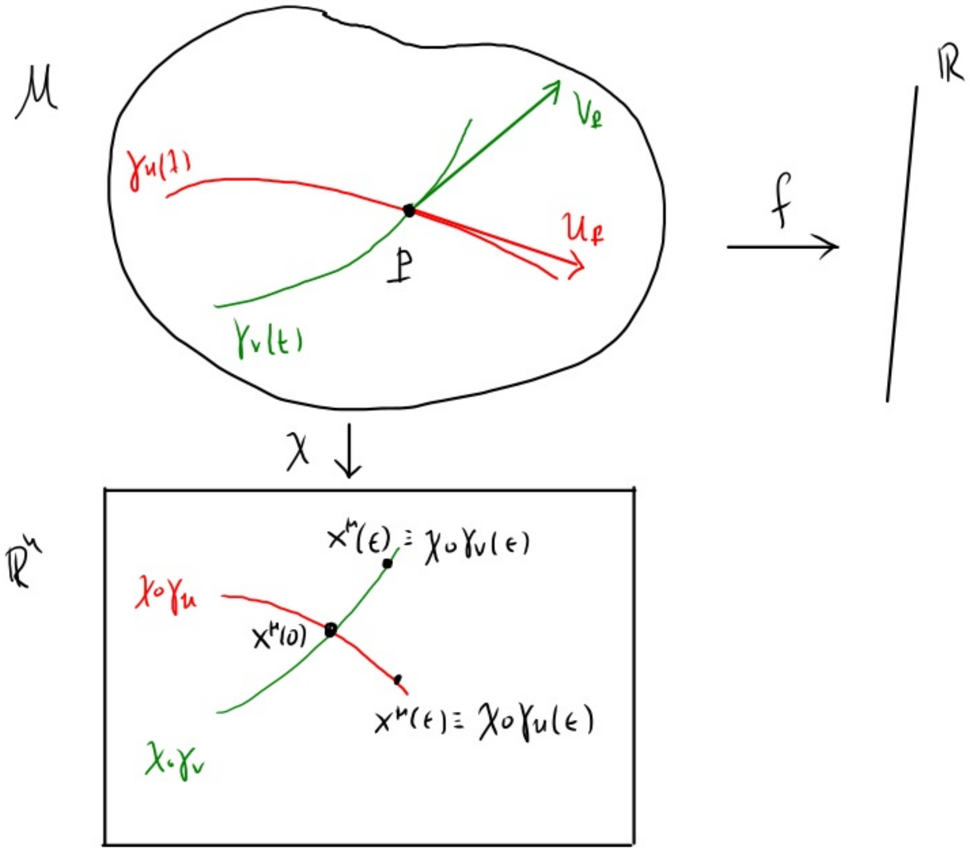
$$U_p(f) = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu}\Big|_p = U_p(x^\mu)\,\partial_\mu f\,\Big|_p$$

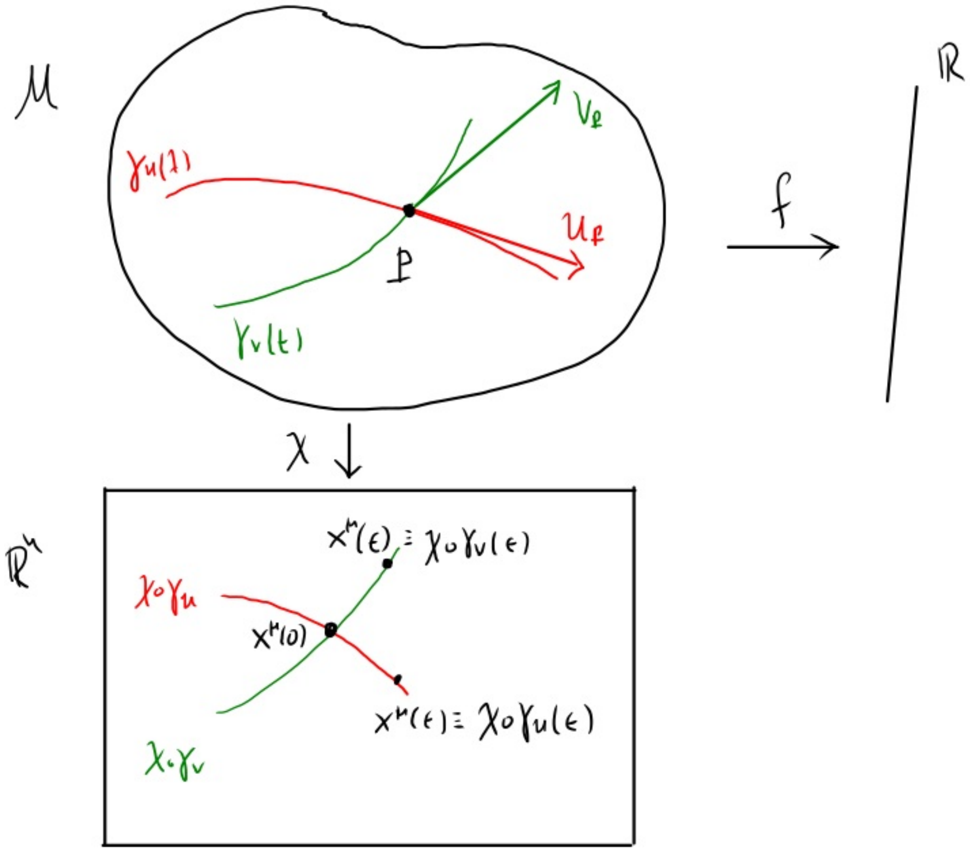
$$V_p = V_p(x^\mu)\,\partial_\mu\,\Big|_p$$

$$U_p = U_p(x^\mu)\,\partial_\mu\,\Big|_p$$



In $\mathbb{R}^4$ we have the curves:

$$x^\mu(t) \equiv \chi \circ \gamma_v(t)$$
$$x^\mu(\lambda) \equiv \chi \circ \gamma_u(\lambda)$$



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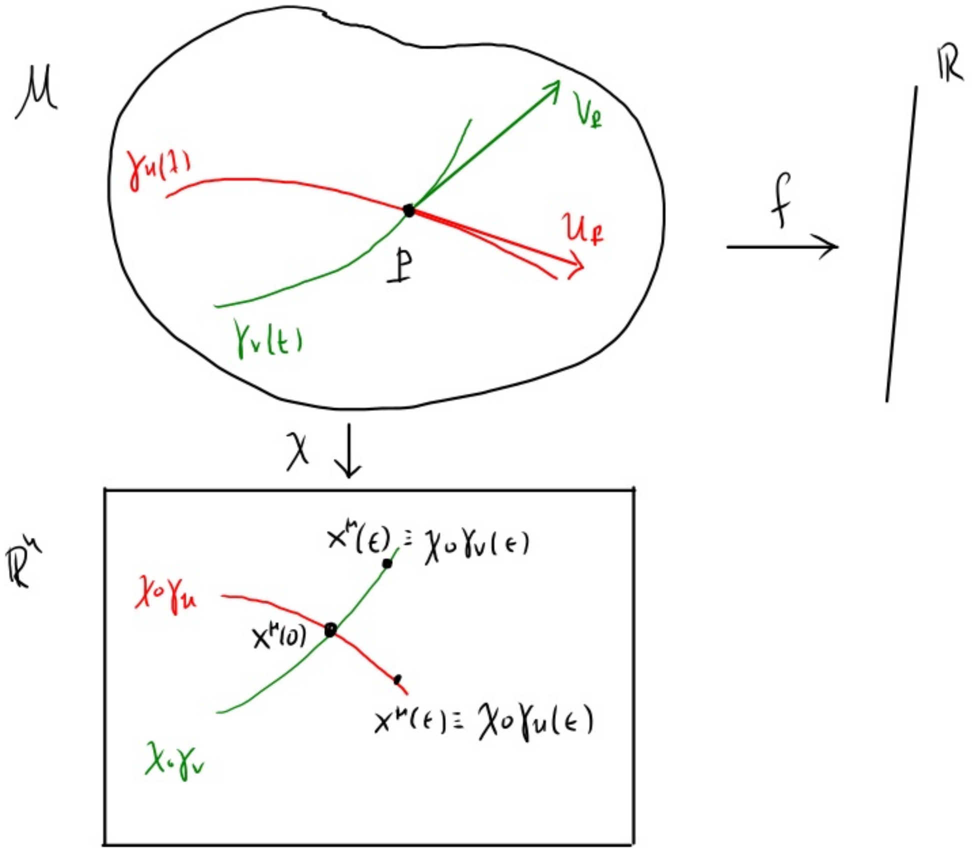

For chosen ε:

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{dt}\right|_0 + \mathcal{O}_v(\epsilon^2)$$

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \mathcal{O}_u(\epsilon^2)$$

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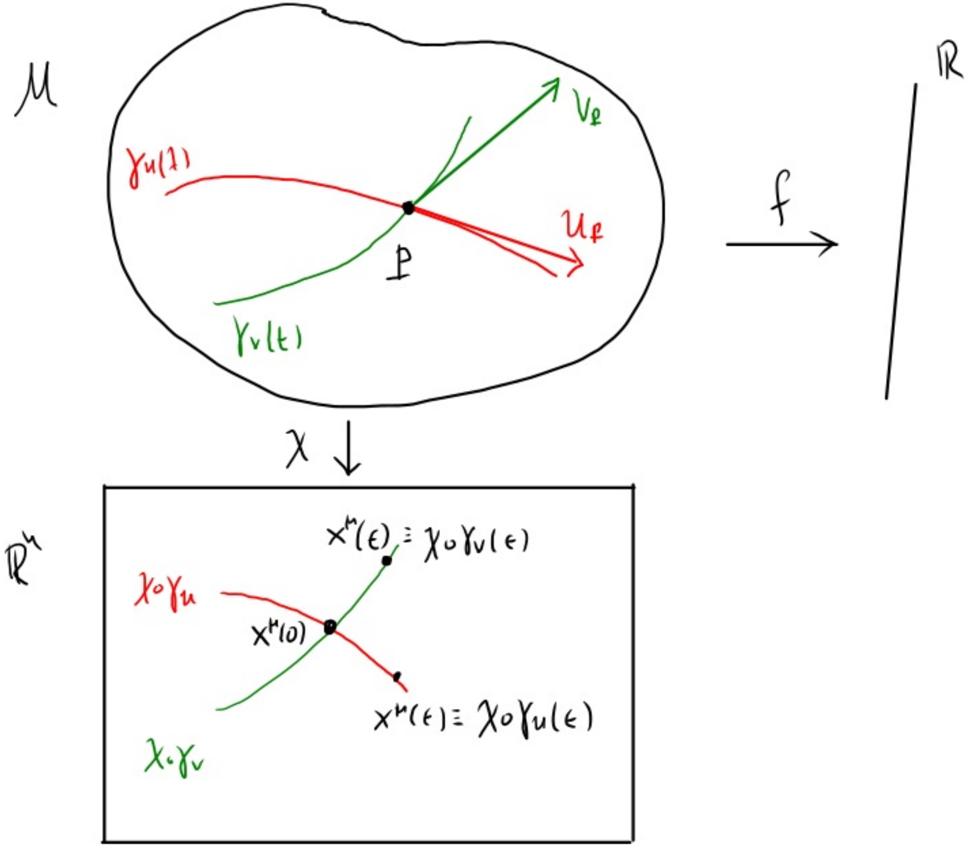

For chosen ε:

move on $\chi \circ \gamma_v$

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{dt}\right|_0 + \mathcal{O}_v(\epsilon^2)$$

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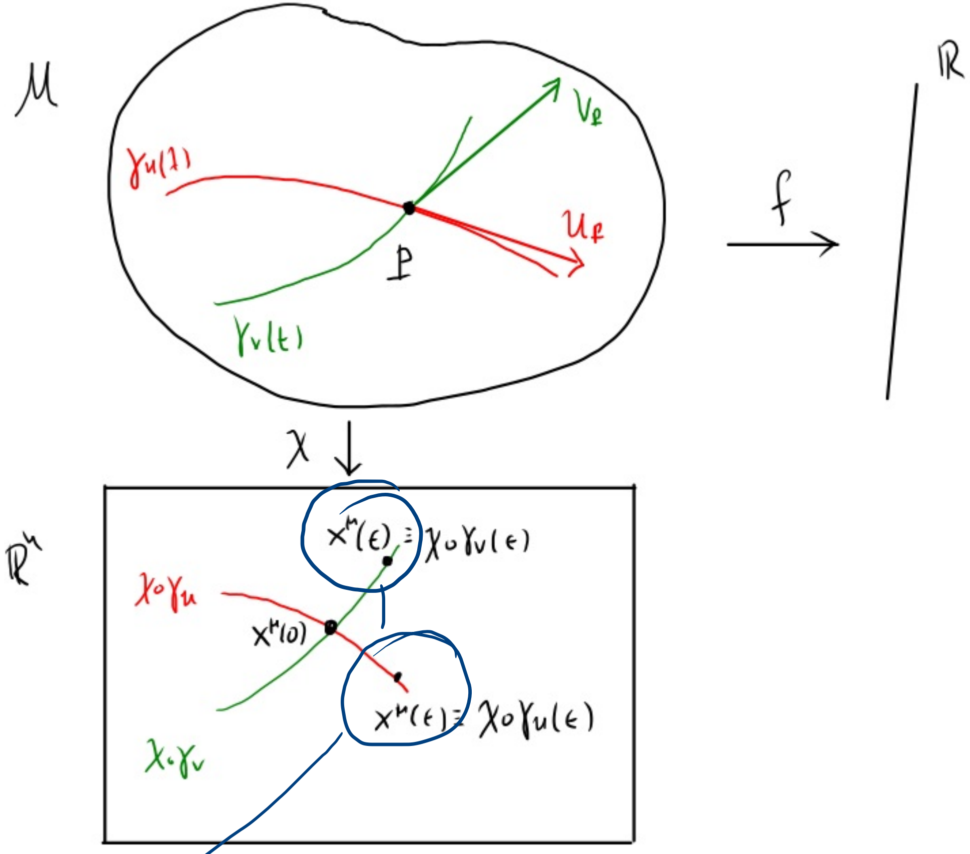
move on $\chi \circ \gamma_u$

In $\mathbb{R}^n$ we have the curves:

$$x^\mu(t) \equiv \chi \circ \gamma_v(t)$$

$$x^\mu(\lambda) \equiv \chi \circ \gamma_u(\lambda)$$



For chosen ε:

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$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \mathcal{O}_u(\epsilon^2)$$

different points of $\mathbb{R}^n$!

In $\mathbb{R}^n$ we have the curves:

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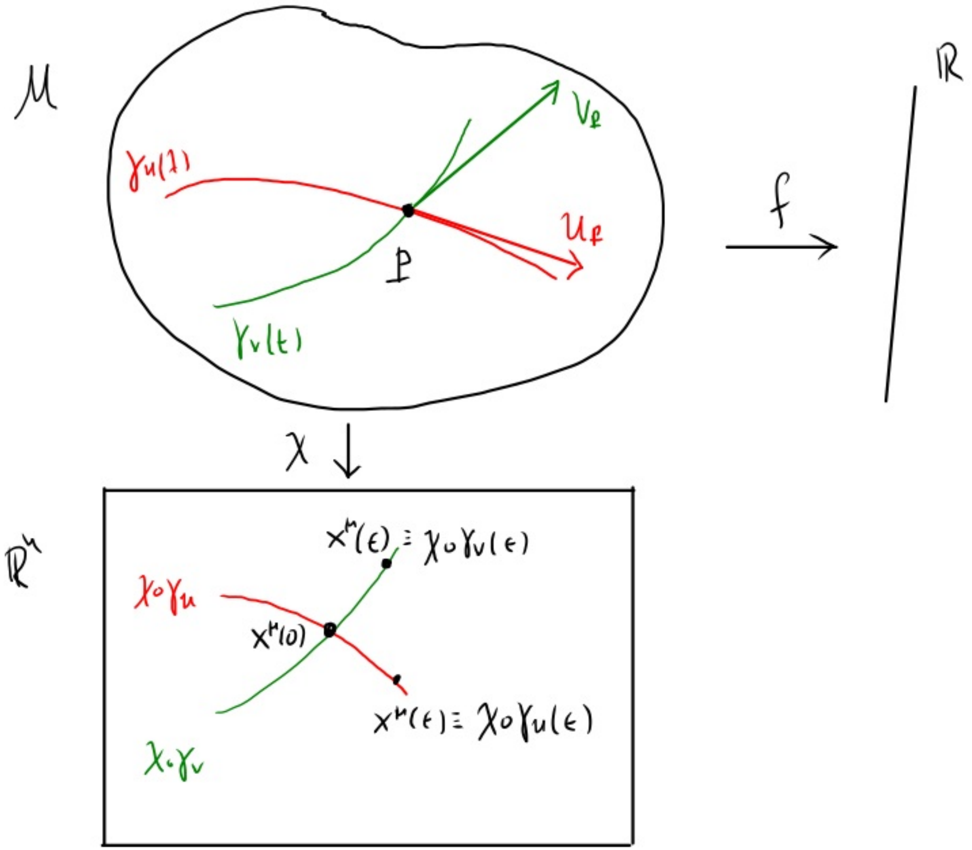

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$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \mathcal{O}_u(\epsilon^2)$$

define any curve $\gamma_w(\epsilon)$ such that $x^\mu(\epsilon) = \chi \circ \gamma_w(\epsilon)$ is

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left[\alpha \left.\frac{dx^\mu}{dt}\right|_0 + \beta \left.\frac{dx^\mu}{d\lambda}\right|_0 \right] + \mathcal{O}_w(\epsilon^2)$$

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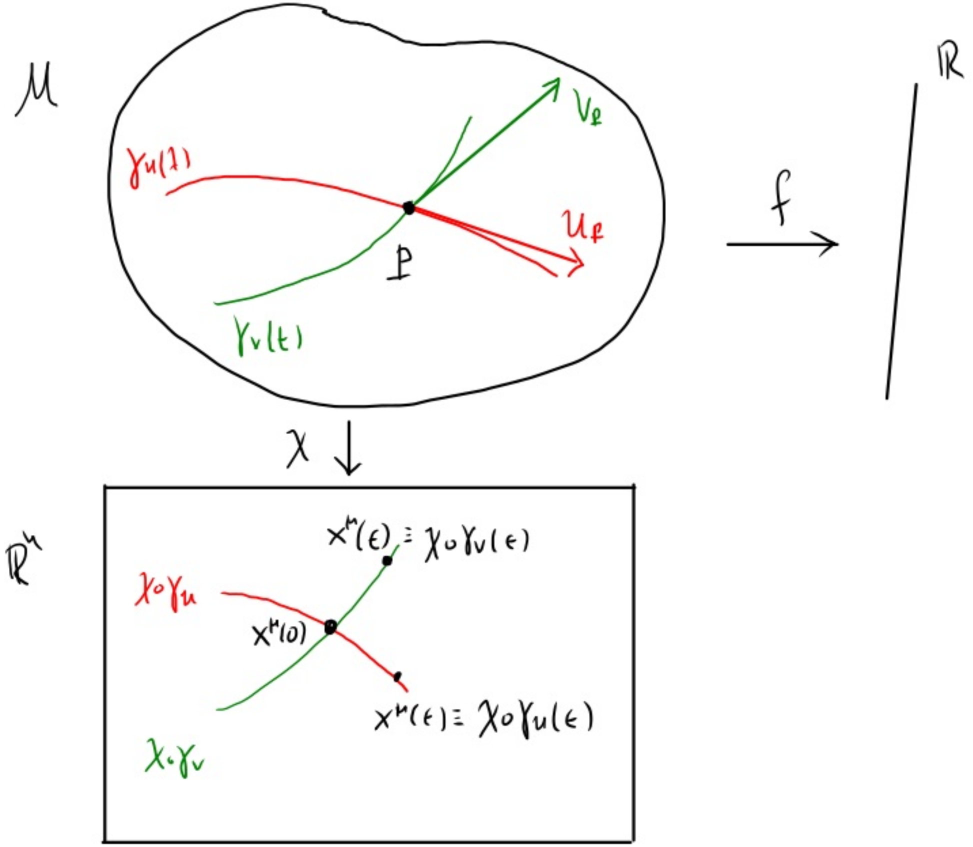

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$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \mathcal{O}_u(\epsilon^2)$$

numbers!

parameter

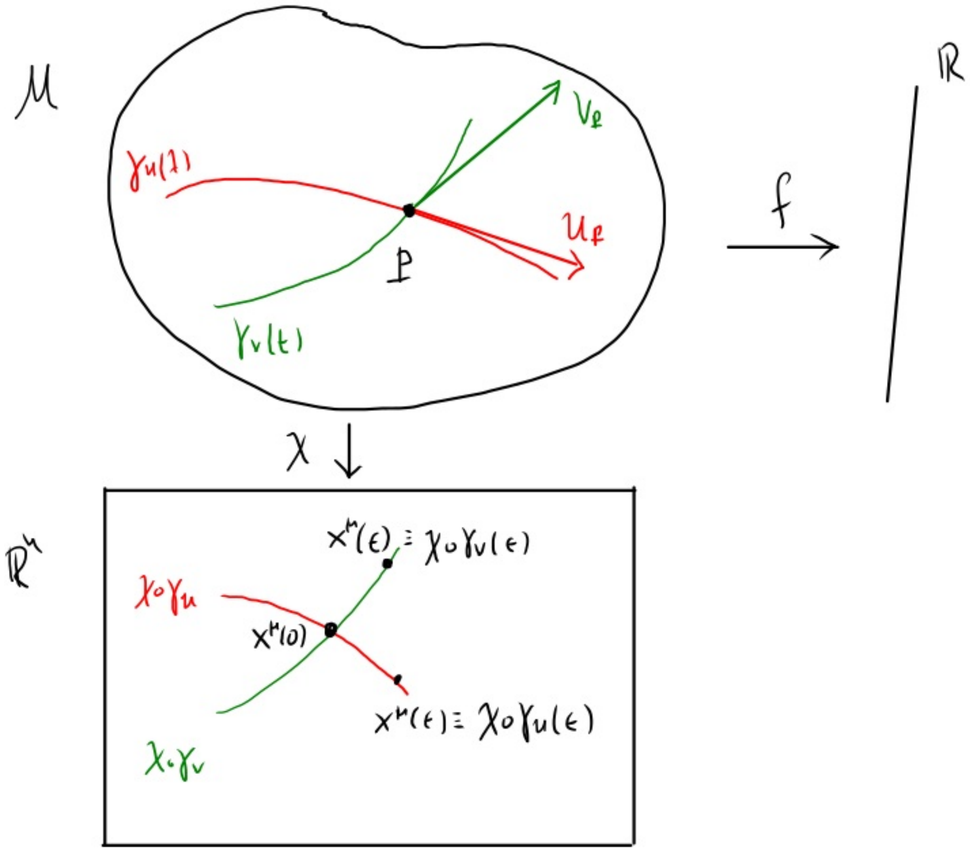
define any curve $\gamma_w(\epsilon)$ such that $x^\mu(\epsilon) = \chi \circ \gamma_w(\epsilon)$ is

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left[\alpha \left.\frac{dx^\mu}{dt}\right|_0 + \beta \left.\frac{dx^\mu}{d\lambda}\right|_0 \right] + \mathcal{O}_w(\epsilon^2)$$

any infinitesimal you like!

The tangent vector of $\gamma_w(\varepsilon)$ is s.t.:

$$W_p(f) = \left.\frac{df}{d\epsilon}\right|_p$$



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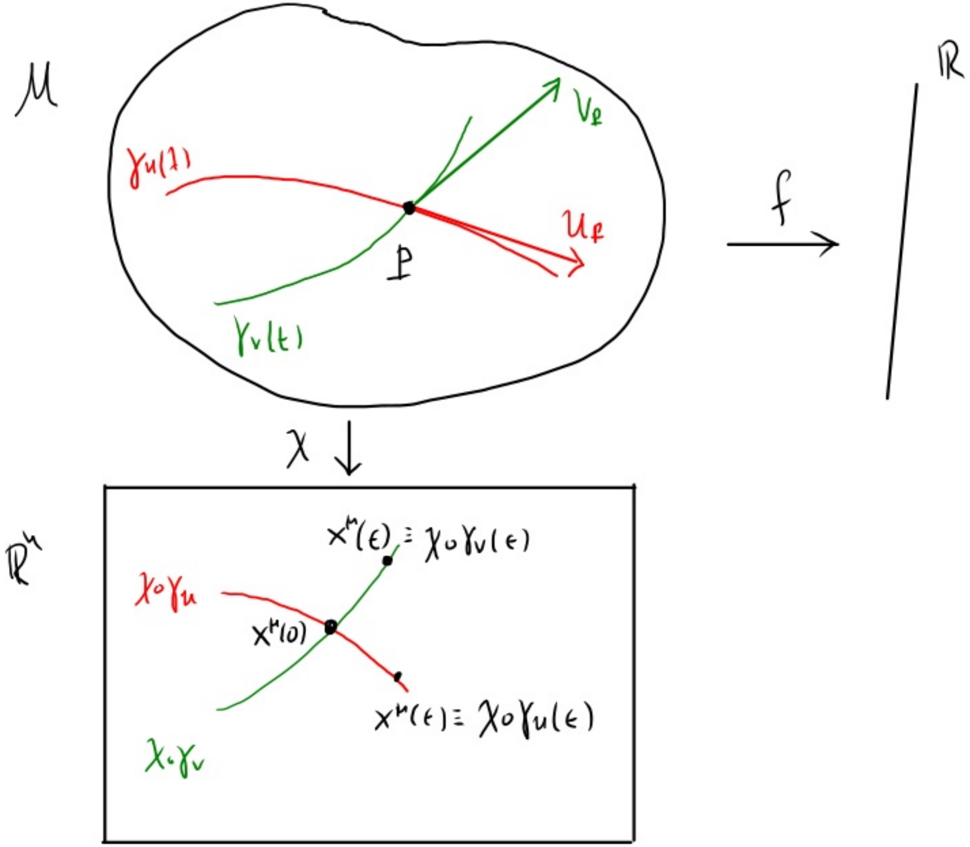

$$W_p(f) = \left.\frac{df}{d\epsilon}\right|_p$$

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---

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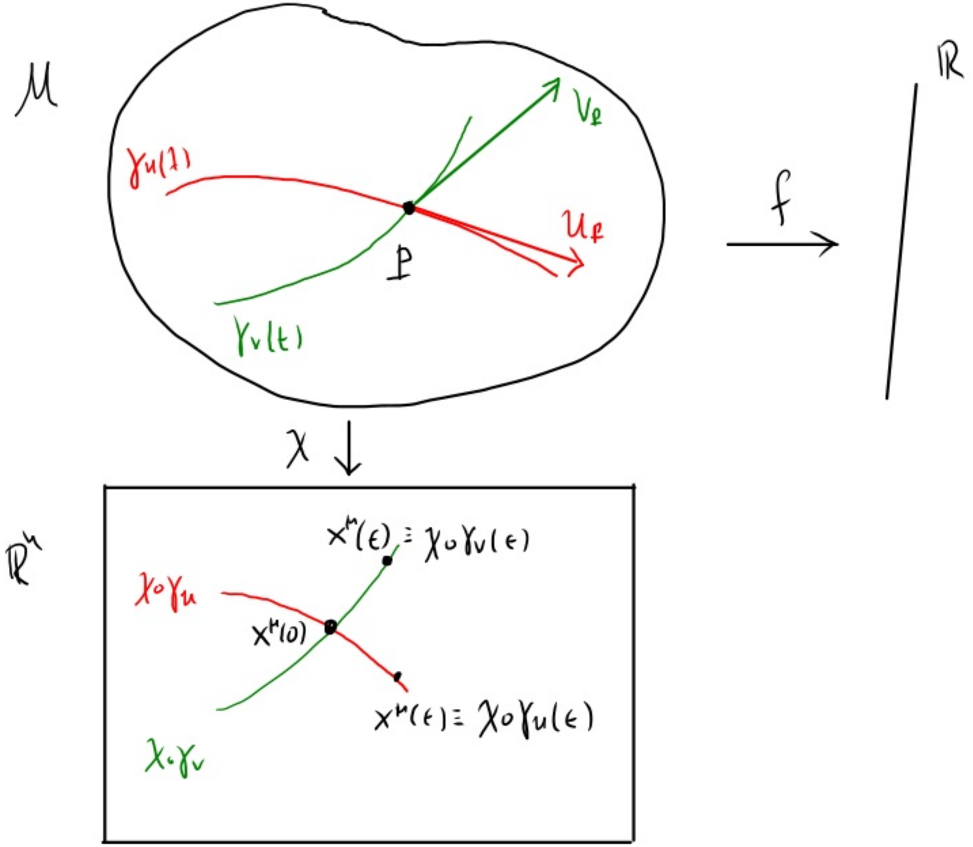

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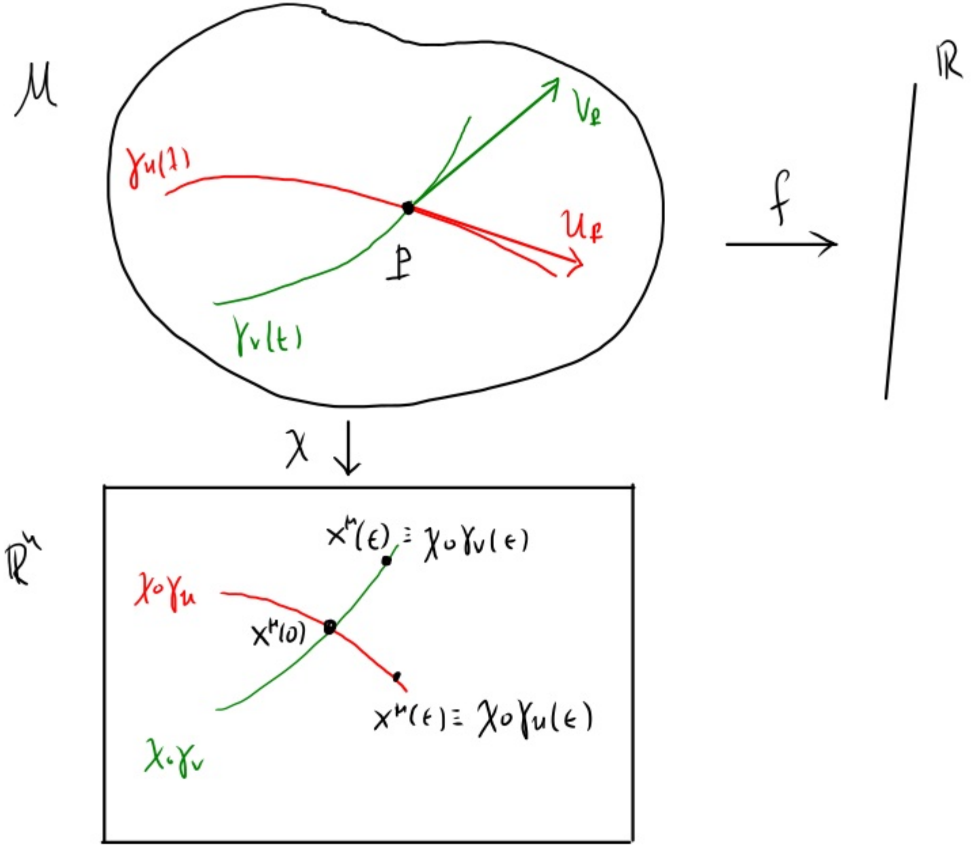

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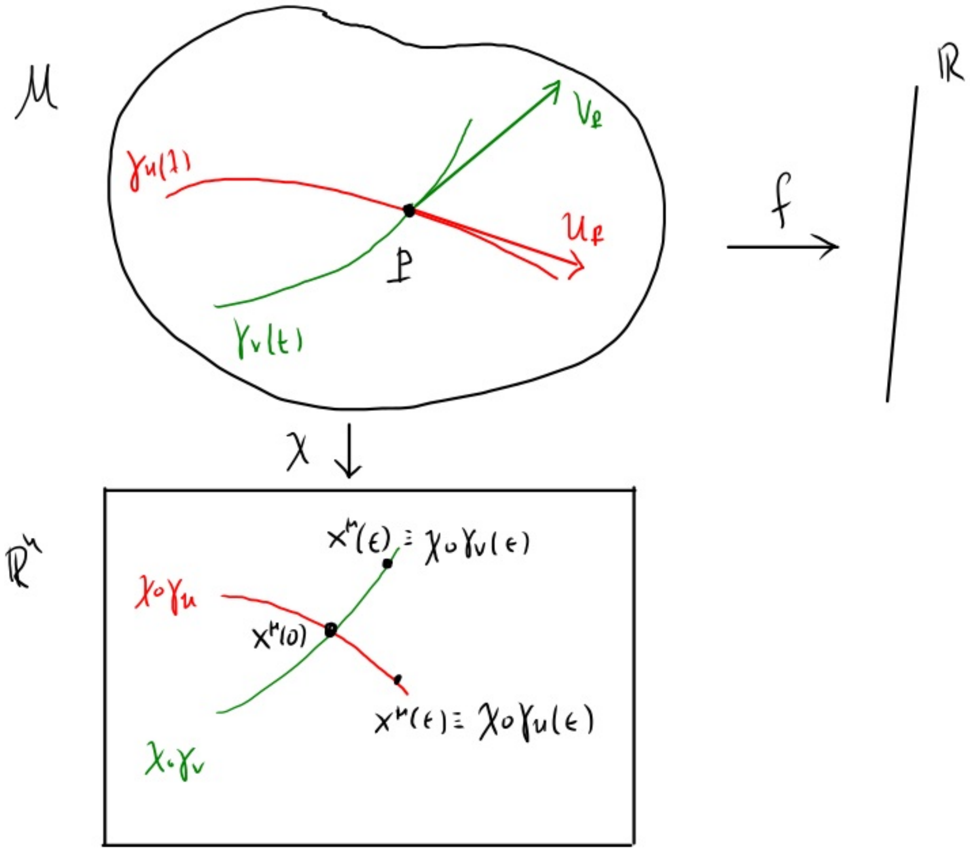
$$= \frac{\partial f}{\partial x^\mu} \frac{dx^\mu(\epsilon)}{d\epsilon}\bigg|_0$$

$$= \frac{\partial f}{\partial x^\mu}\left(\alpha \frac{dx^\mu}{dt}\bigg|_0 + \beta \frac{dx^\mu}{d\lambda}\bigg|_0 \right)$$



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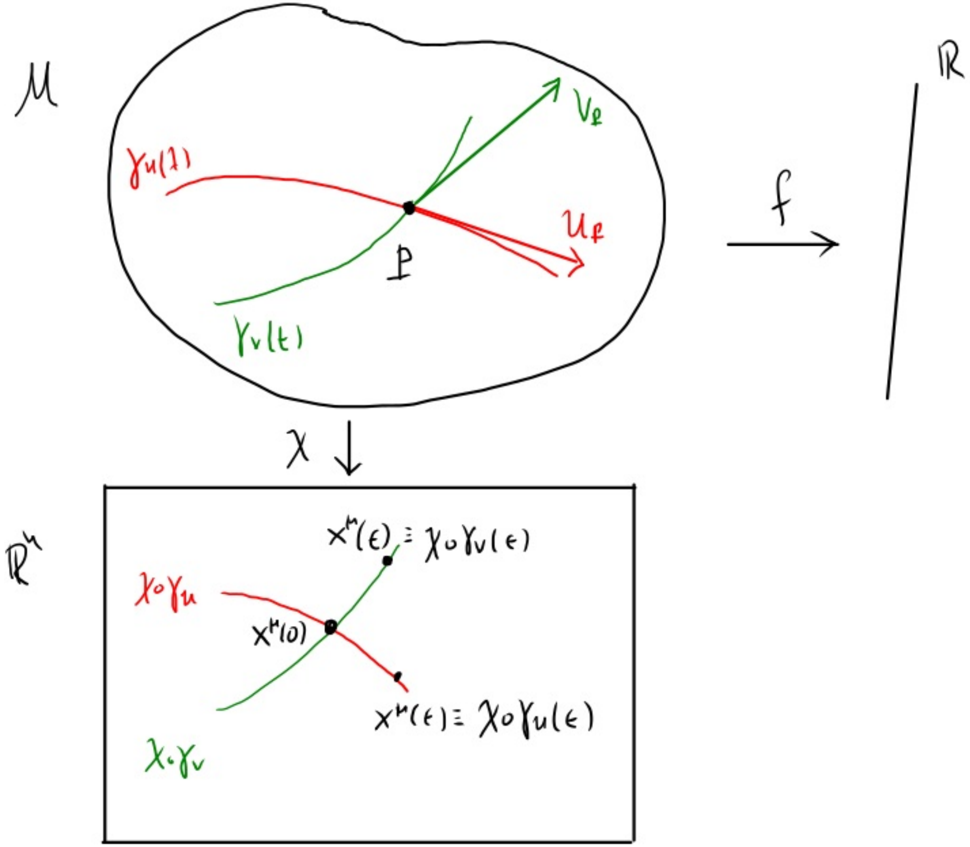

$$W_p(f) = \frac{df}{d\epsilon}\bigg|_p$$

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$$= \frac{\partial f}{\partial x^\mu}\,\frac{dx^\mu(\epsilon)}{d\epsilon}\bigg|_0$$

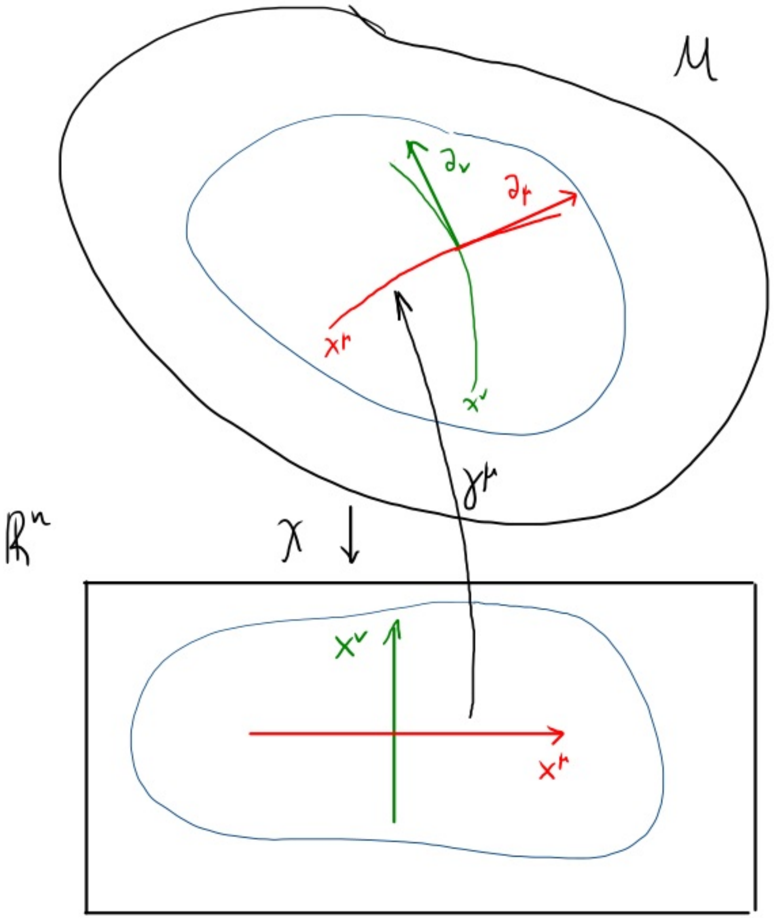
$$= \frac{\partial f}{\partial x^\mu}\left(\alpha\,\frac{dx^\mu}{dt}\bigg|_0 + \beta\,\frac{dx^\mu}{d\lambda}\bigg|_0\right)$$

$$= \alpha\, V_p(f) + \beta\, U_p(f) \quad \overset{\forall f}{\Longrightarrow} \quad W_p = \alpha V_p + \beta U_p$$

Coordinate basis:

Consider the curve:

$$\gamma^\mu : \mathbb{R} \to \mathcal{M}$$
$$x^\mu \to \gamma^\mu(x^\mu)$$



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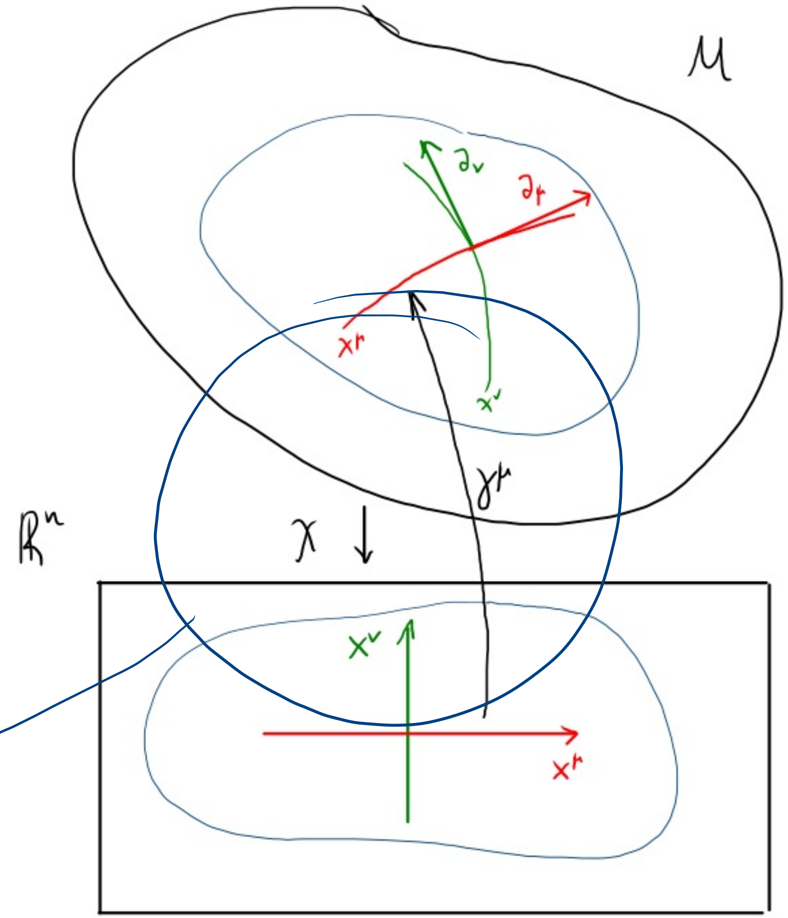


we use χ^{-1} to "ascend"

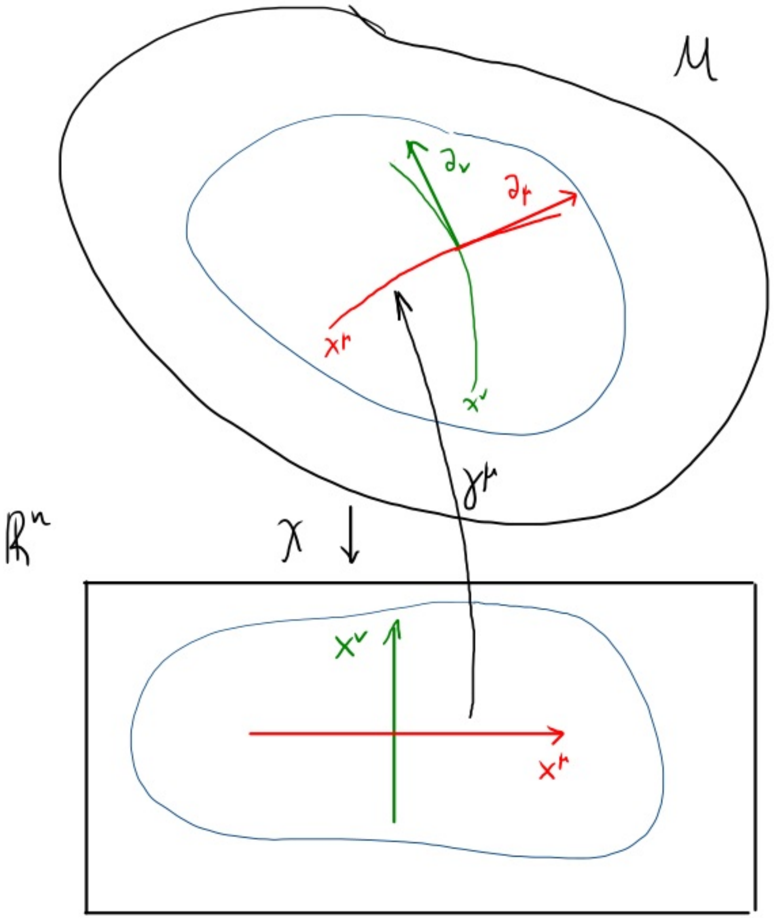
from $\mathbb{R}^n$ to $\mathcal{M}$

Coordinate basis:

Consider the curve:

$$\gamma^{\mu} : \mathbb{R} \to M$$
$$x^{\mu} \to \gamma^{\mu}(x^{\mu})$$

μ is fixed, no summation

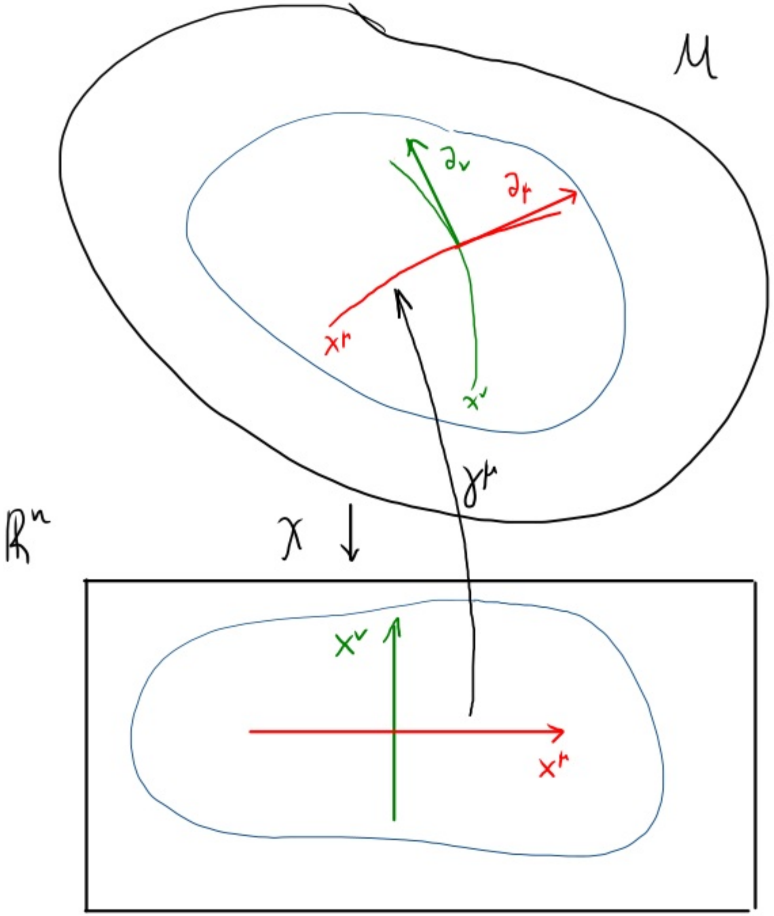


Coordinate basis:

Consider the curve:

$$\gamma^\mu : \mathbb{R} \to M$$
$$x^\mu \to \gamma^\mu(x^\mu)$$

x^μ is the parameter of this curve



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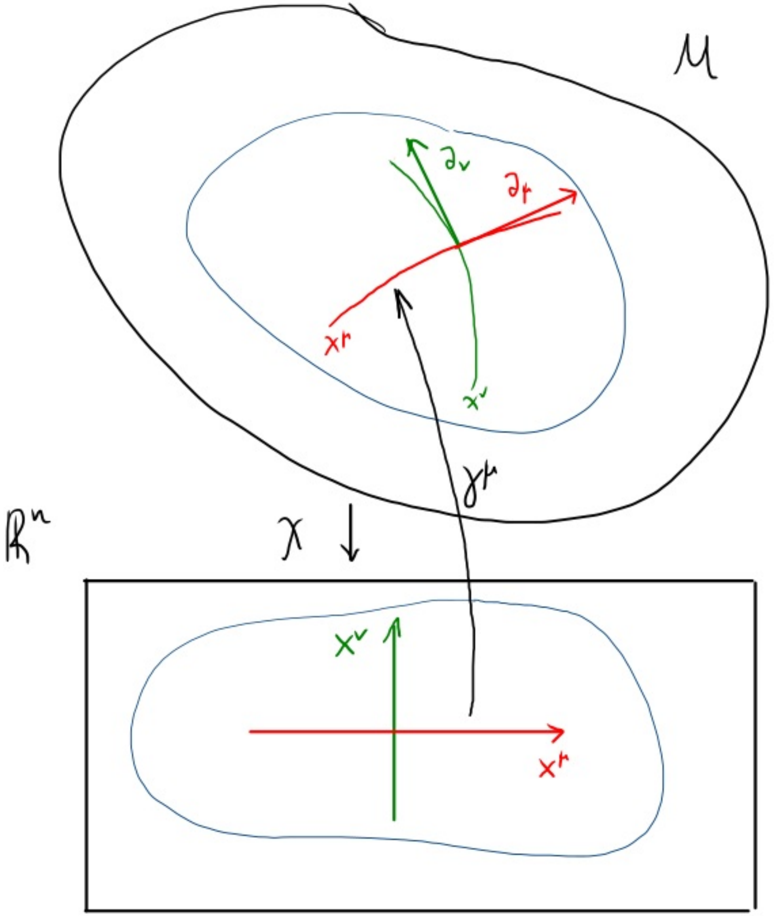

all other x^ν, $\nu \neq \mu$ are held fixed, in $\mathbb{R}^n$ we move parallel to the x^μ-axis

$$\gamma^\mu(x^\mu) = \chi^{-1}(c_0, c_1, \ldots, c_{\mu-1}, x^\mu, c_{\mu+1}, \ldots, c_{n-1})$$

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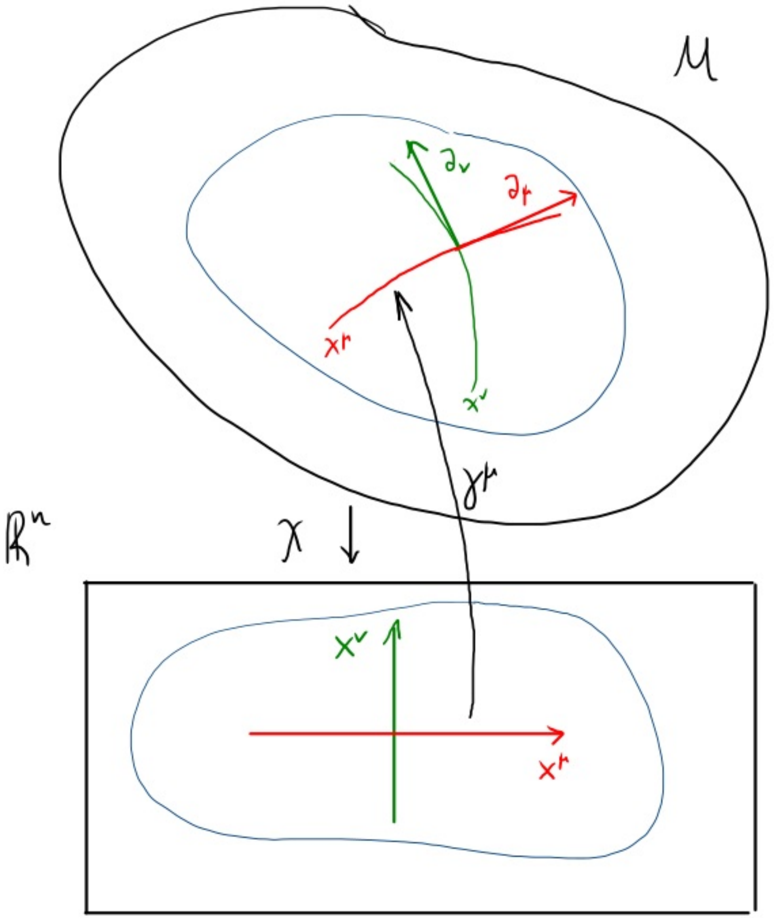

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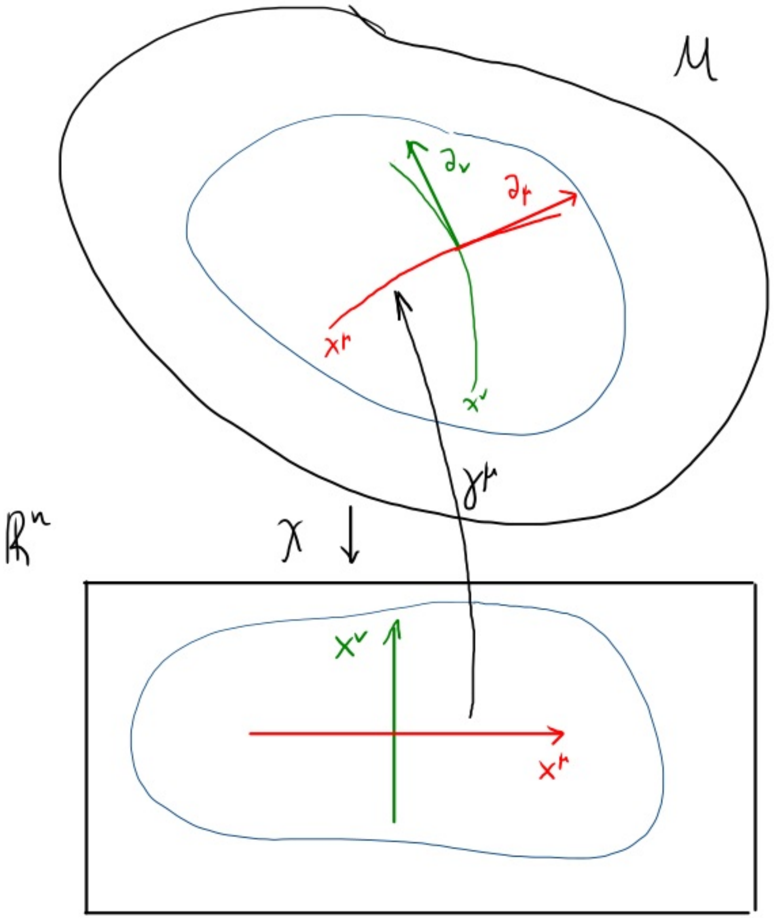
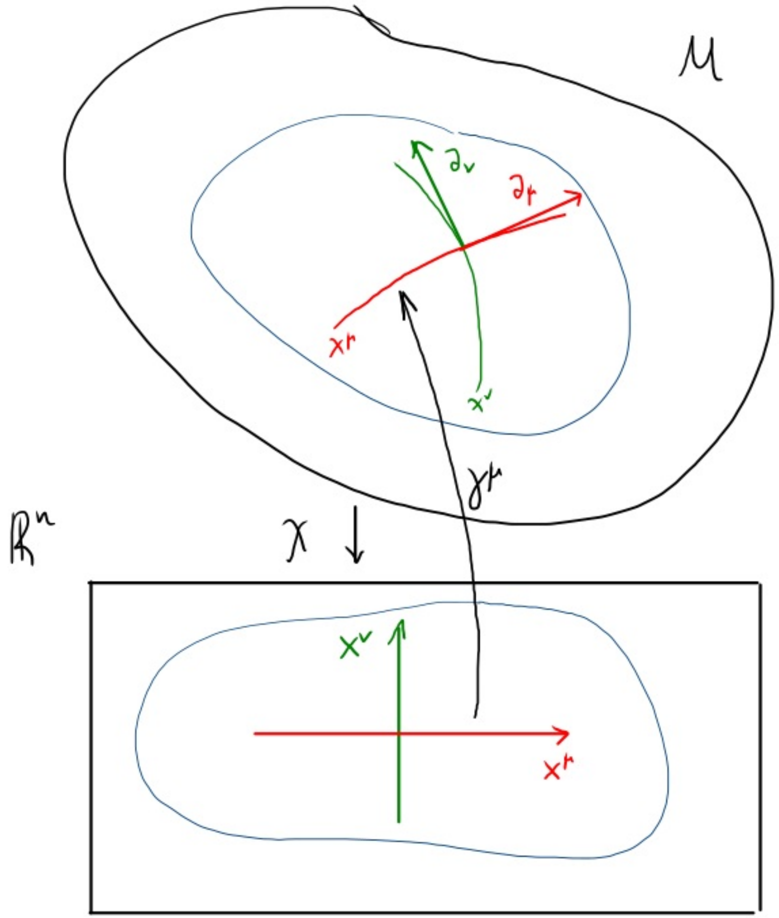


So, taking the derivative "$\frac{df}{dx^\mu}$" corresponds to taking the partial derivative $\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f \implies$ tangent vector is $\partial_\mu \big|_p$

Coordinate basis:

Then:

$$\frac{df}{dx^\mu}\Big|_p = \frac{d}{dx^\mu}\, f \circ \gamma^\mu(x^\mu)\Big|_p$$



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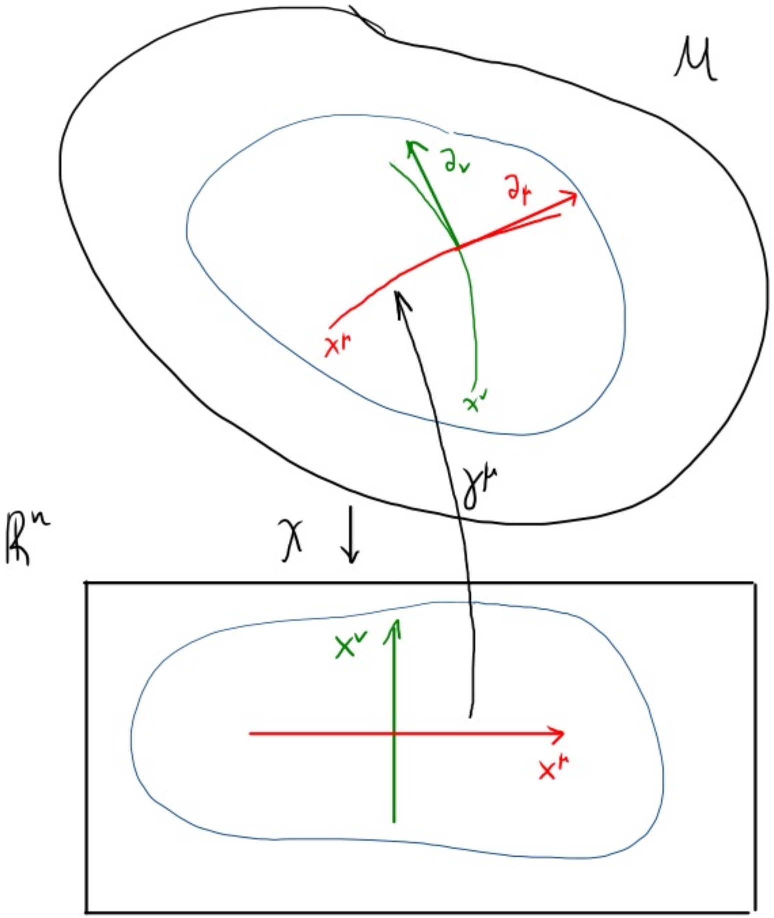
$$\frac{df}{dx^\mu}\bigg|_p = \frac{d}{dx^\mu} f \circ \gamma^\mu (x^\mu) \bigg|_p$$

$$= \frac{\partial}{\partial x^\mu} f \circ \chi^{-1} (x^\nu) \bigg|_p$$



Coordinate basis:

Then:

directional derivative along γ^μ

$$\frac{df}{dx^\mu}\bigg|_p = \frac{d}{dx^\mu} f \circ \gamma^\mu(x^\mu)\bigg|_p$$

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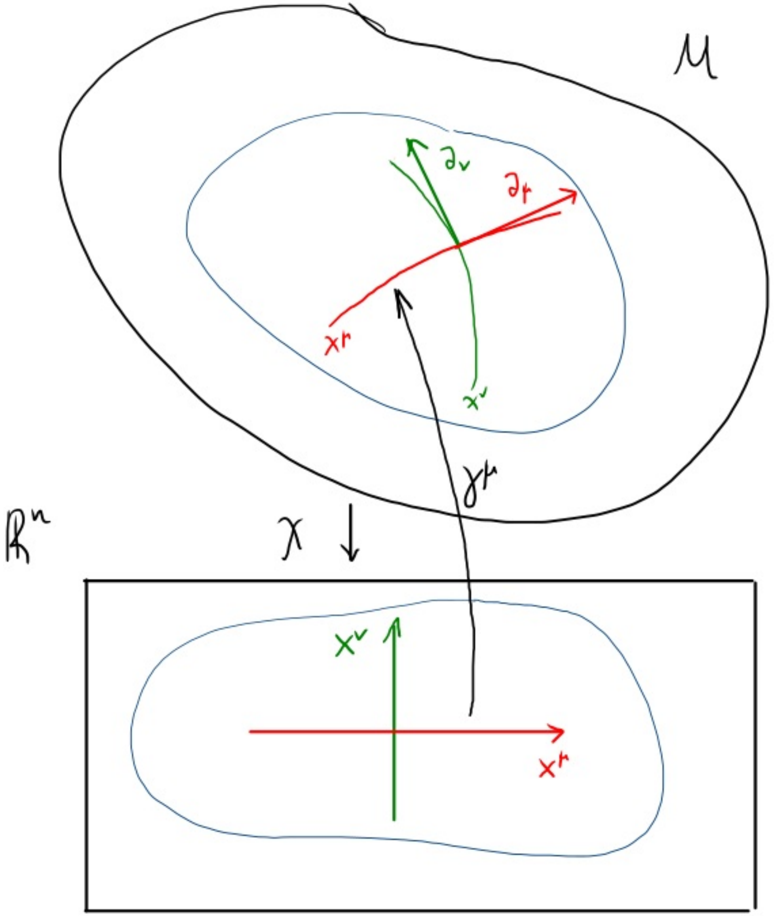
$\hookrightarrow$ definition of partial derivative: we vary x^μ, hold x^ν, $\nu \neq \mu$, fixed



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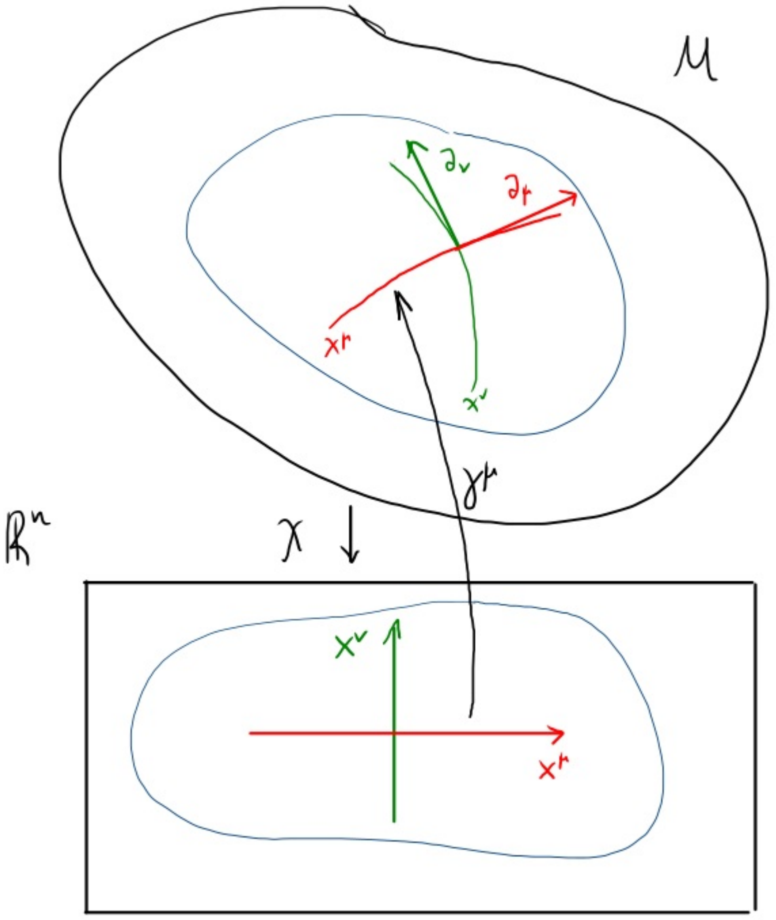
Define $\partial_\mu |_p$ by

$$\partial_\mu f |_p = \frac{\partial f \circ x^{-1}}{\partial x^\mu}\bigg|_p$$



Coordinate basis:

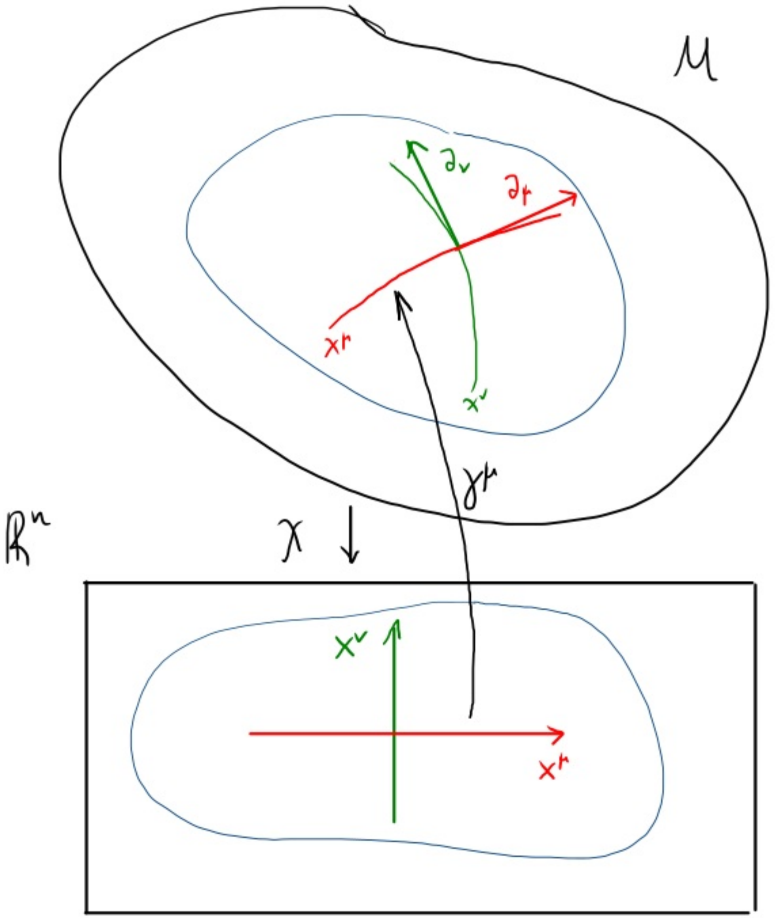
Since

$$V_p(f) = \frac{dx^\mu}{dt} \frac{\partial f}{\partial x^\mu}\Big|_p$$

$$= \frac{dx^\mu}{dt} \partial_\mu f \Big|_p$$

$\Longrightarrow$

$$V_p = \frac{dx^\mu}{dt} \partial_\mu \Big|_p$$



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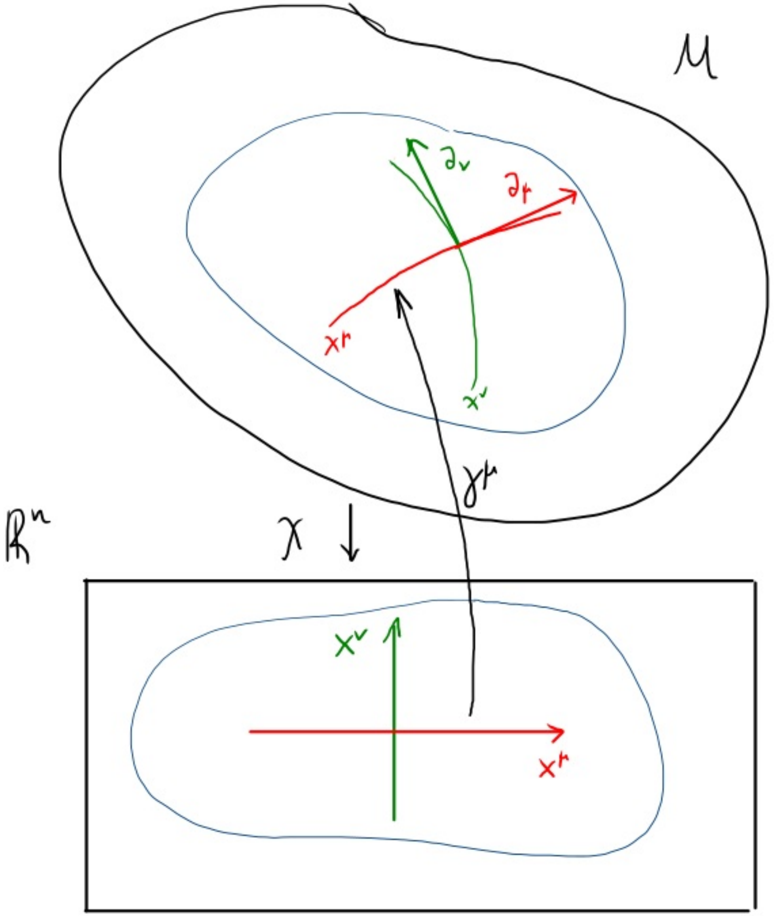
coordinate vectors



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$\Rightarrow$

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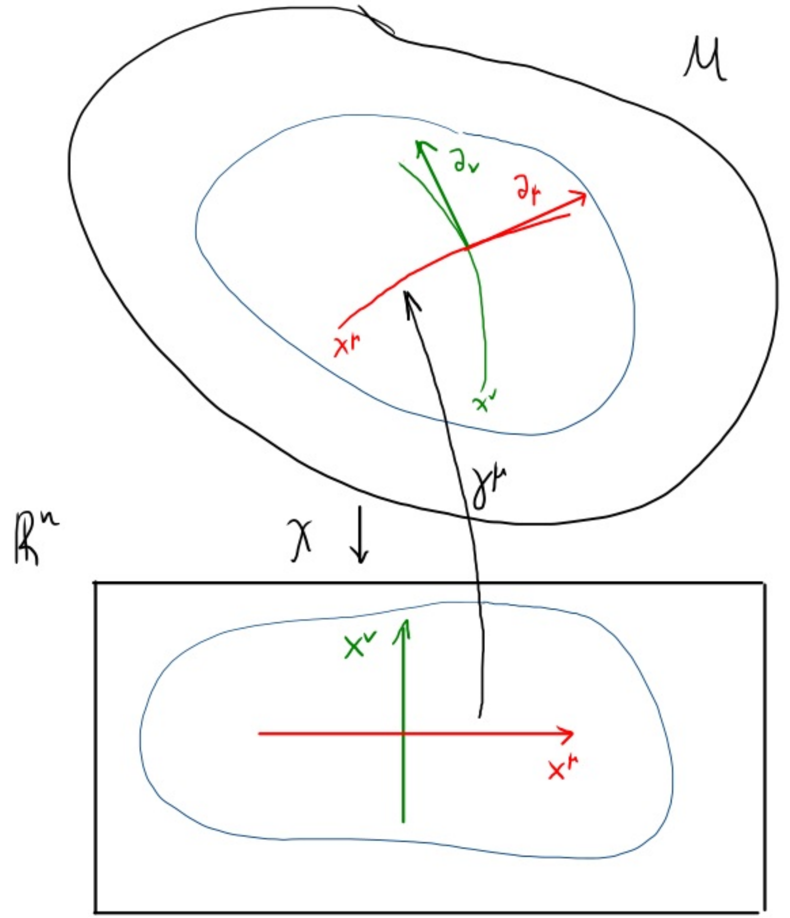
the unique linear combination giving V_p



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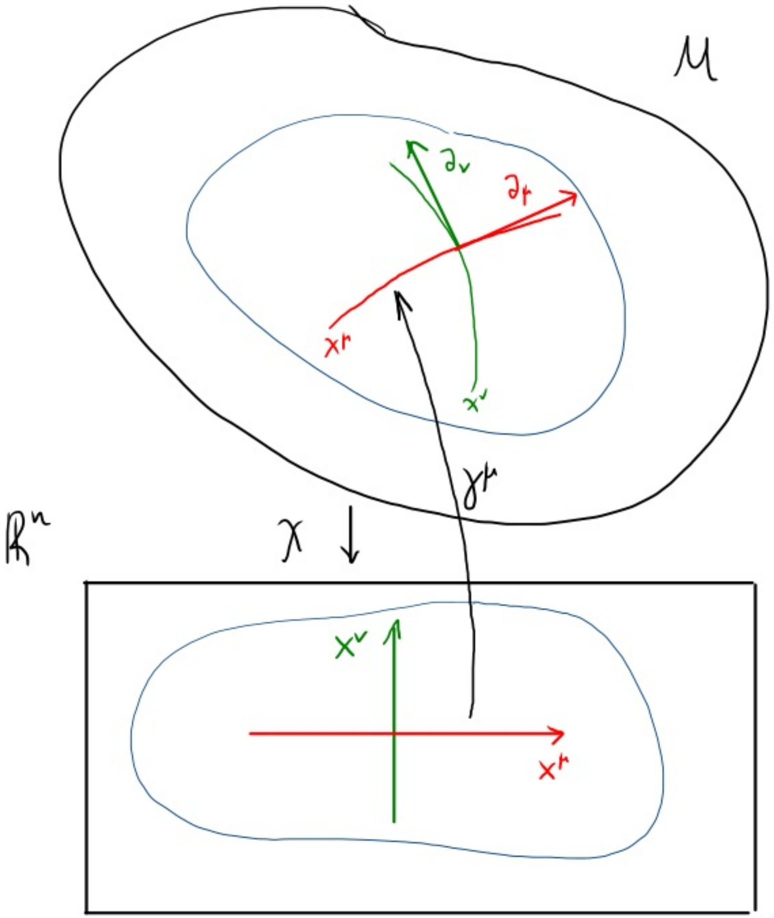
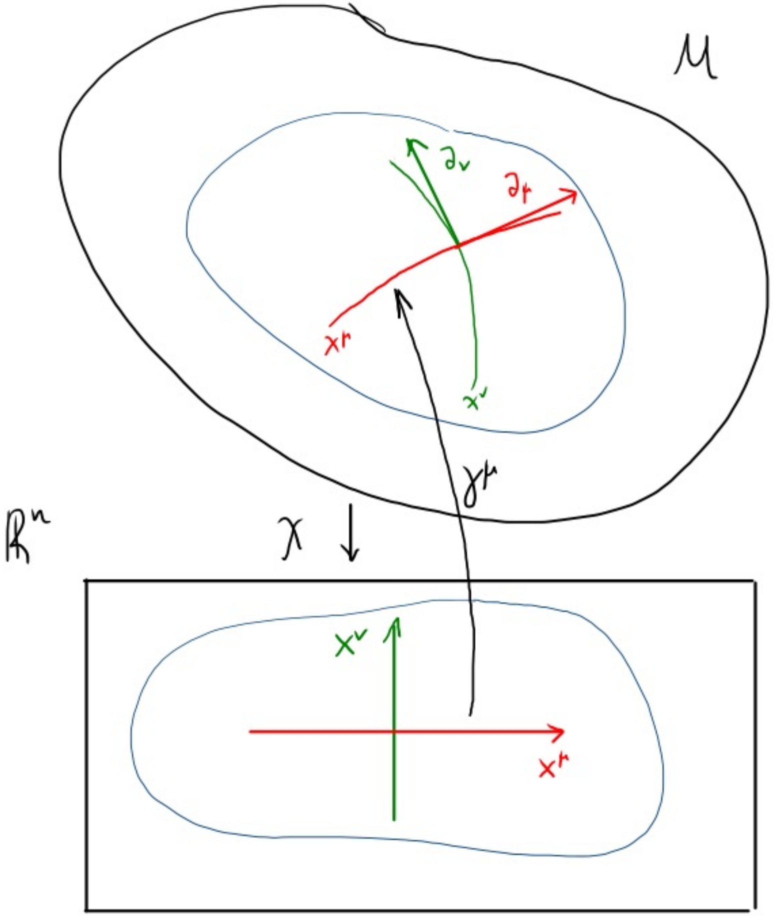


$$\Rightarrow V_p = \frac{dx^\mu}{dt} \partial_\mu \Big|_p$$

an operator acting on any f

Coordinate basis:

$$V = \frac{dx^\mu}{dt} \partial_\mu$$

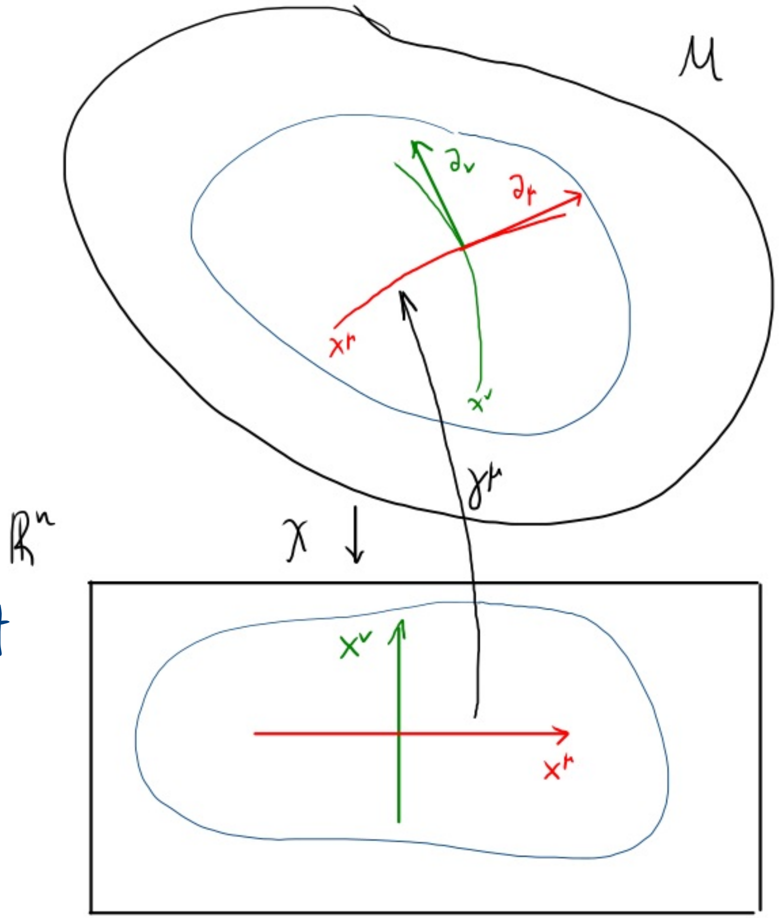


Coordinate basis:

$$V = \frac{dx^\mu}{dt}\,\partial_\mu = V^\mu \partial_\mu$$

components of V in the coordinate basis $\{\partial_\mu\}$

true $\forall\, P$ in chart
remove P from notation for simplicity



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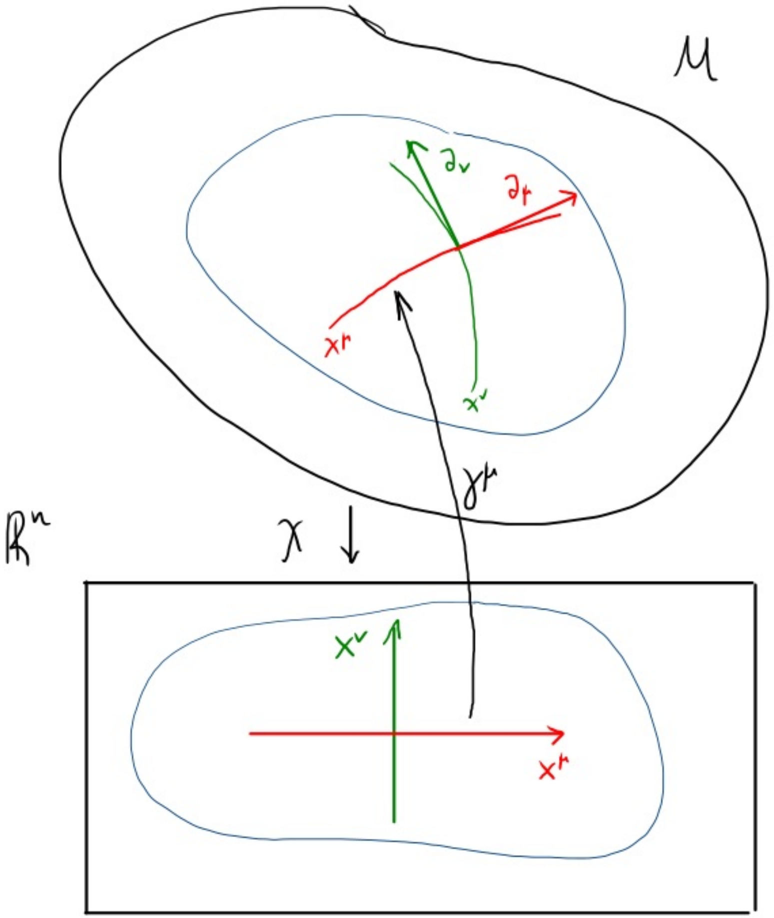


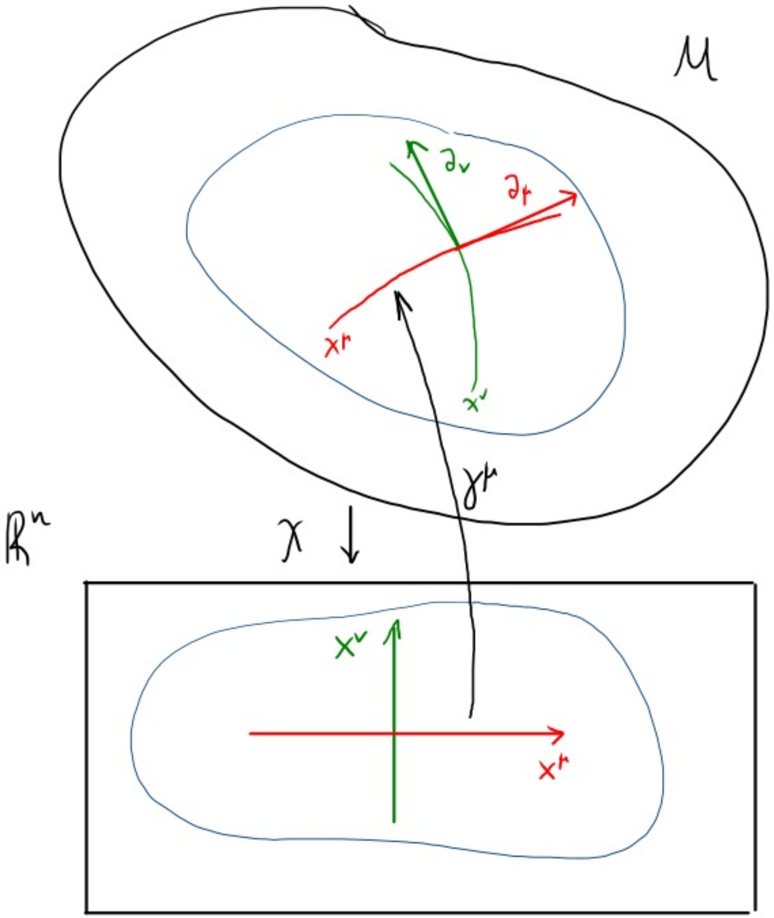
$V^\mu = \dfrac{dx^\mu}{dt}$ components of V in $\{\partial_\mu\}$

$\{\partial_\mu\}_{\mu=0,\dots,n-1}$ a basis $\Rightarrow$ T_pM is n-dimensional vector space

Coordinate basis:

- $\{\partial_\mu\}$ is derived from chosen coordinates

  $\rightarrow$ coordinate basis



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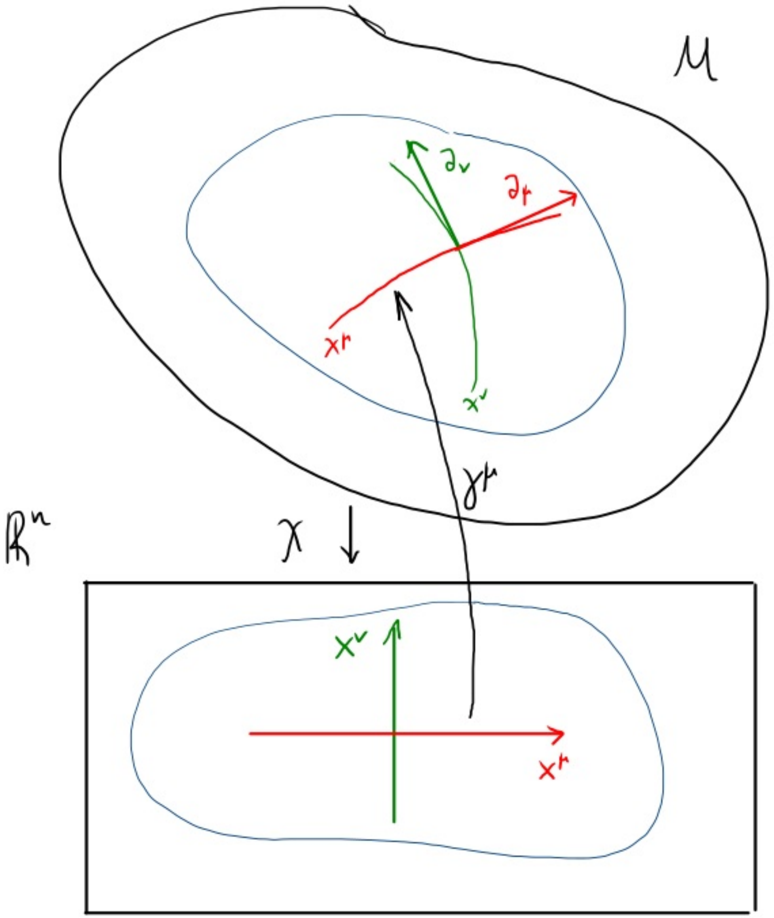
- Not all bases in $T_p M$ are coordinate bases.

Any $e_\alpha = \Lambda_\alpha{}^\mu \partial_\mu$ is a basis if $\operatorname{rank} \Lambda_\alpha{}^\mu = n$



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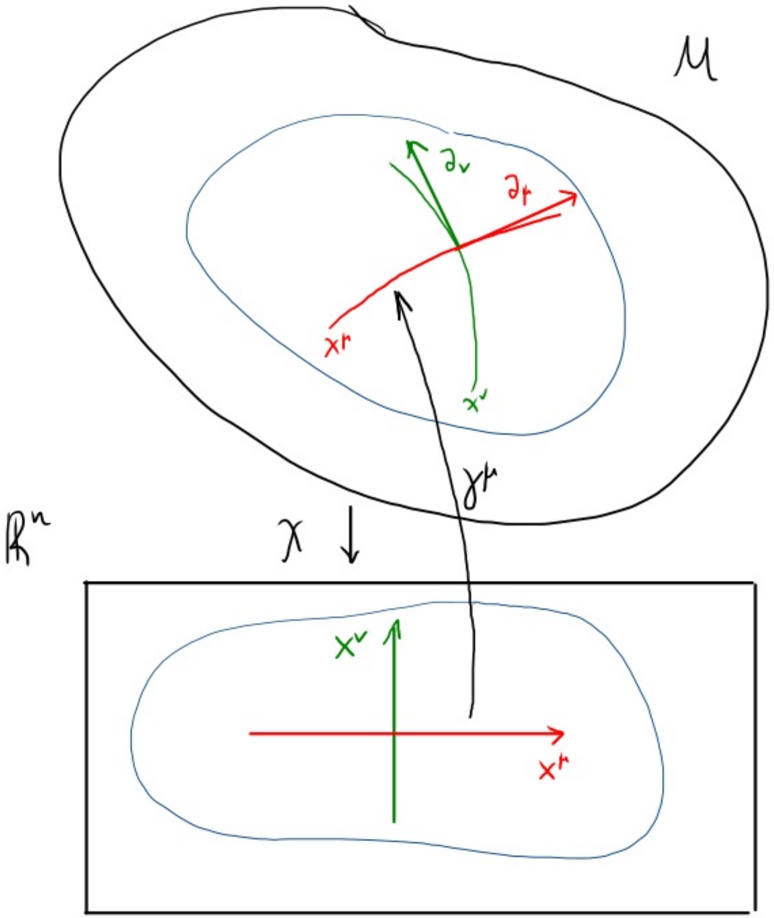

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  Any $e_\alpha = \Lambda_\alpha{}^\mu \partial_\mu$ is a basis if $\operatorname{rank} \Lambda_\alpha{}^\mu = n$

- If there is a metric + inner product, $\{\partial_\mu\}$ may not be orthonormal

- Change of basis $\Rightarrow$ change of components

(the vector stays the same, the components change)

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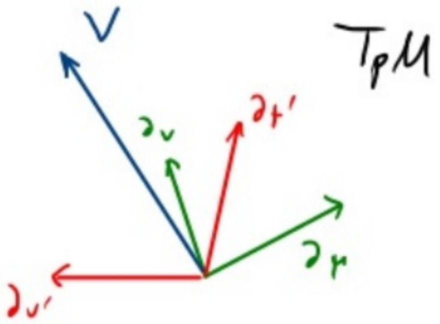
notice placement of indices!

$V = V^{\mu} \partial_{\mu}$

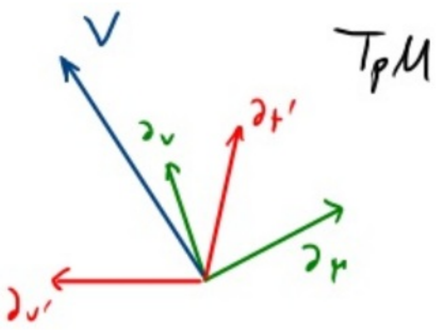
$V = V^{\mu'} \partial_{\mu'}$

same vector
different bases



$$V = V^{\mu} \partial_{\mu}$$

$$\left.\begin{aligned} V &= V^{\mu'} \partial_{\mu'} \\ \partial_{\mu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu} \end{aligned}\right\} \Rightarrow V = V^{\mu'} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$



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$$\left.\begin{array}{l} V = V^{\mu'}\partial_{\mu'} \\ \\ \partial_{\mu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}}\,\partial_{\mu} \end{array}\right\} \Rightarrow V = \underbrace{\left(V^{\mu'}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right)}_{V^{\mu}}\partial_{\mu} \quad \Rightarrow \quad V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\,V^{\mu'}$$

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$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}$$

Inverting the above system of linear equations, and using

$$\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} \right) = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \right)^{-1}$$

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$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}$$

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$ renamed dummy index $\nu' \to \mu'$

$$\frac{\partial x^{\nu'}}{\partial x^{\mu}} V^{\mu} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} = \delta^{\nu'}{}_{\mu'} V^{\mu'} = V^{\nu'}$$

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index matching

For any other basis $\{e_\alpha\}$ we have

$$e_\alpha = \Lambda_\alpha{}^\mu \partial_\mu \quad , \quad \Lambda_\alpha{}^\mu \text{ invertible}$$

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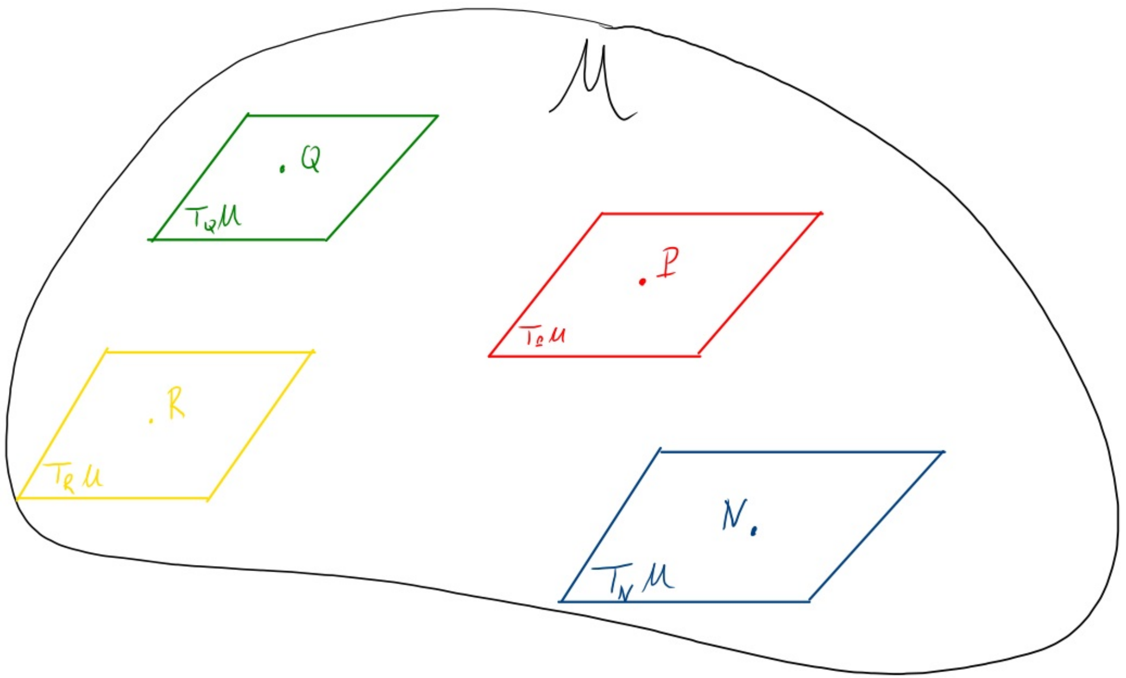
$$V = V^\mu \partial_\mu$$

$$\Rightarrow \quad V^\mu = V^\alpha \Lambda_\alpha{}^\mu$$

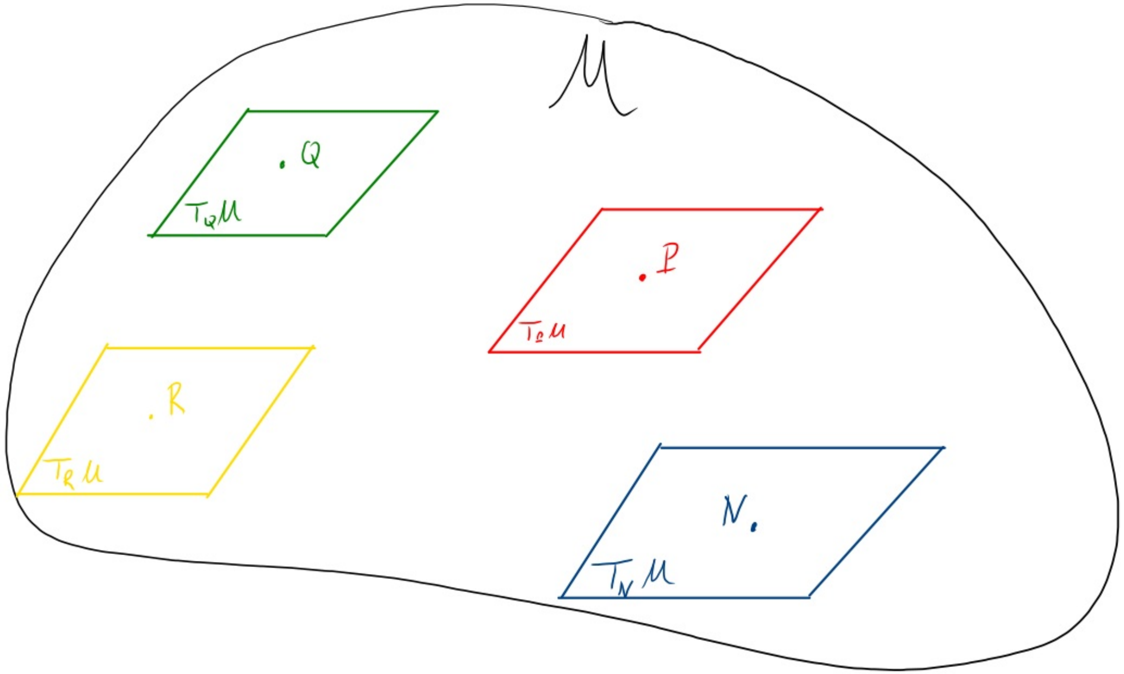
Vector Fields

- Consider vectors for all $P \in M$



Vector Fields

- Consider vectors for all $P \in M$
- Each point P has its $T_P M$ "hanging" above it (tangent bundle)



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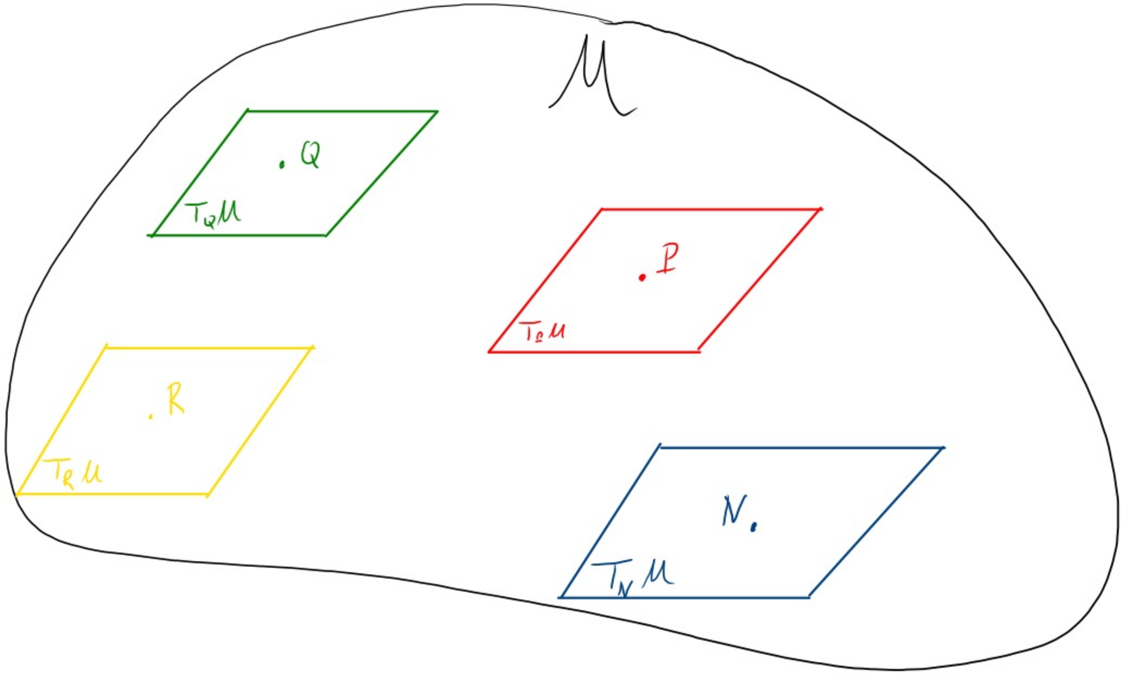

- Consider vectors for all $P \in \mathcal{M}$
- Each point P has its $T_P\mathcal{M}$ "hanging" above it (tangent bundle)
- If we choose a vector from each $T_P\mathcal{M}$ in a smooth way, we obtain a (smooth) vector field

$\forall f \in \mathcal{F}(\mathcal{M})$, $V(f) = \frac{df}{dt}$ is a smooth function on $\mathcal{M}$

Vector Fields

- $\{\partial_\mu\}$ are smooth vector fields

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  a coordinate basis of $T_P M$ $\forall$ P in the chart

$$V = V^\mu \partial_\mu$$

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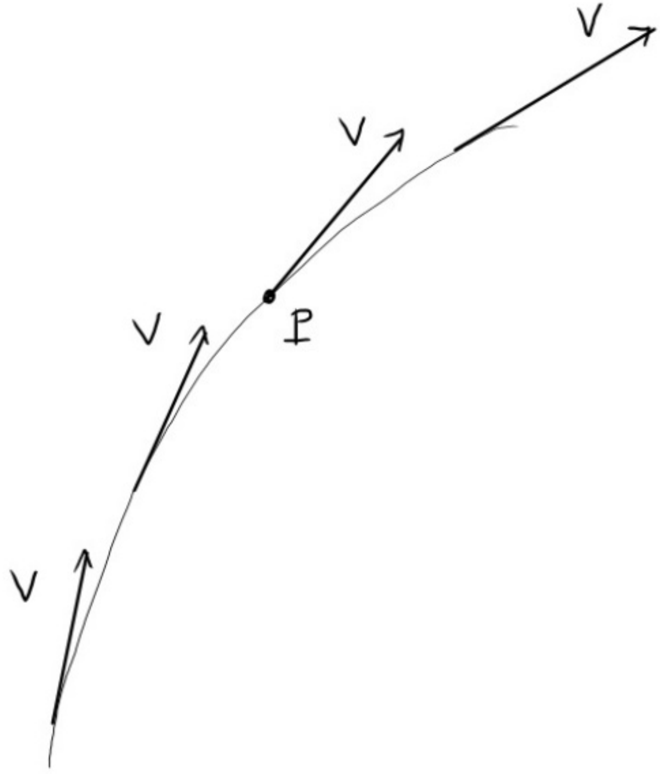
$\Rightarrow$ a smooth V has smooth components in a coordinate basis

Integral Curves:

For each point P in a chart

$$\frac{dx^\mu}{dt} = \underbrace{V^\mu(x^\nu)}_{\text{value of component } V^\mu \text{ at point w/coordinates } \{x^\nu\}}$$



t : parameter of curve to which V is tangent at its points

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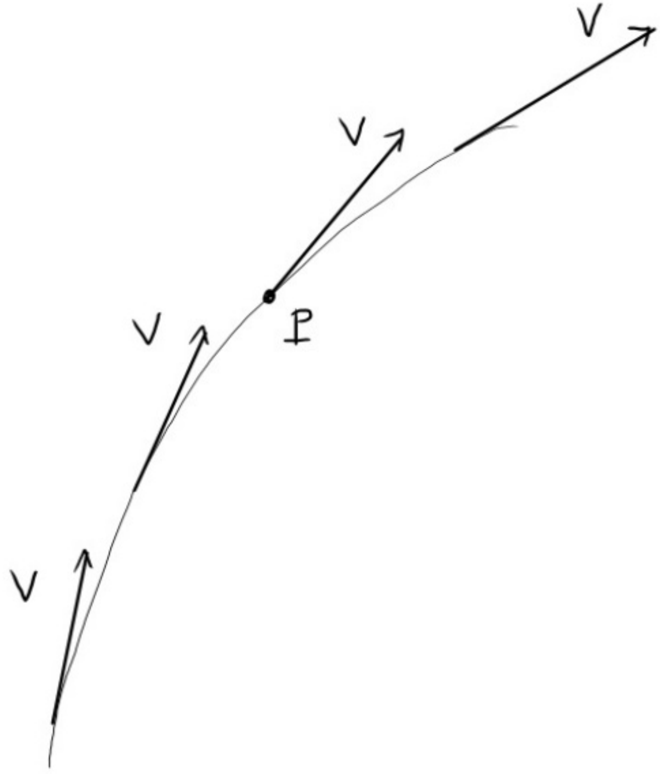


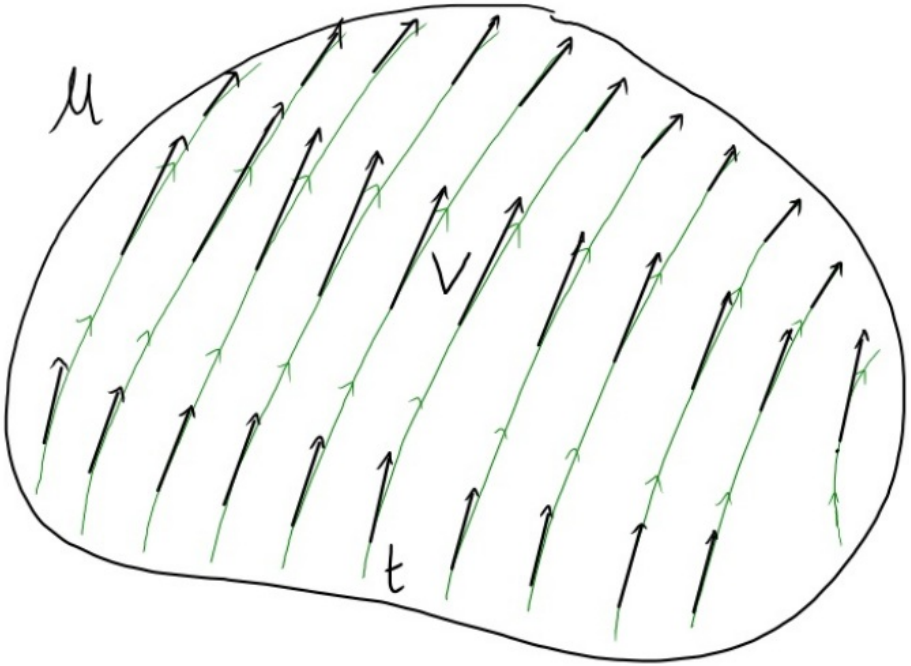
$$\frac{dx^\mu}{dt} = \underbrace{V^\mu}_{\text{value of component } V^\mu \text{ at point w/coordinates } \{x^\nu\}}(x^\nu) \qquad (1)$$

t : parameter of curve to which V is tangent at its points

If $x^\mu(0)$ are the coordinates of P, then (1) has a unique solution

$\Rightarrow \exists$ unique integral curve of v. field V going through P

The integral curves of a nonvanishing vector field on $U \subseteq M$, "fill" U



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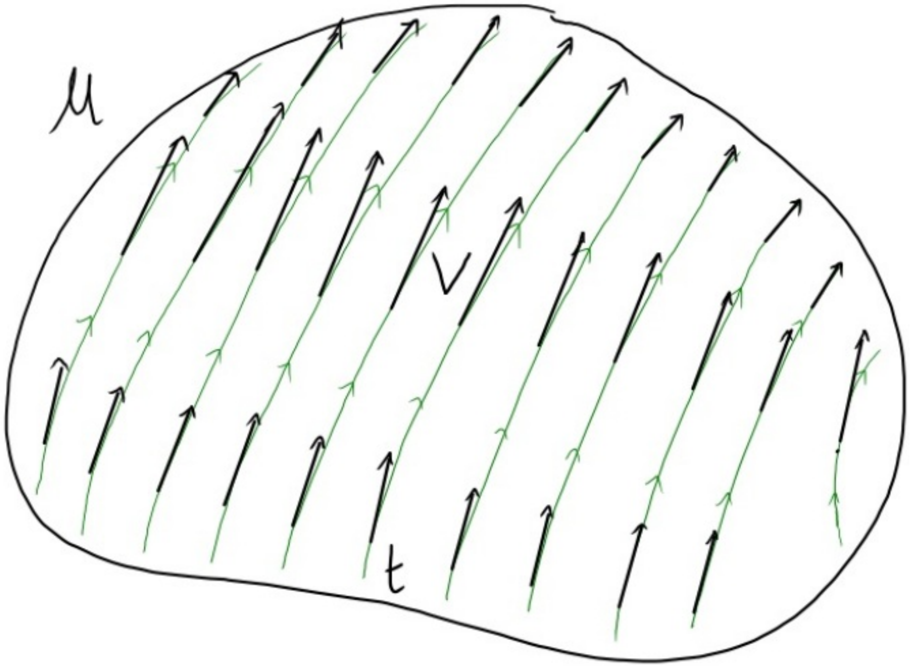

- They pass through each $P \in U$

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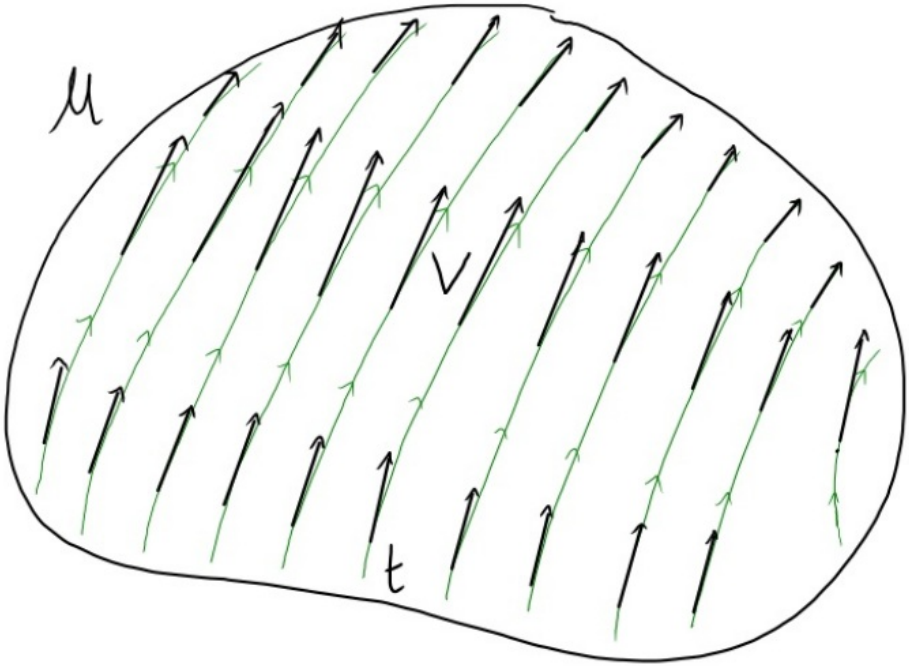

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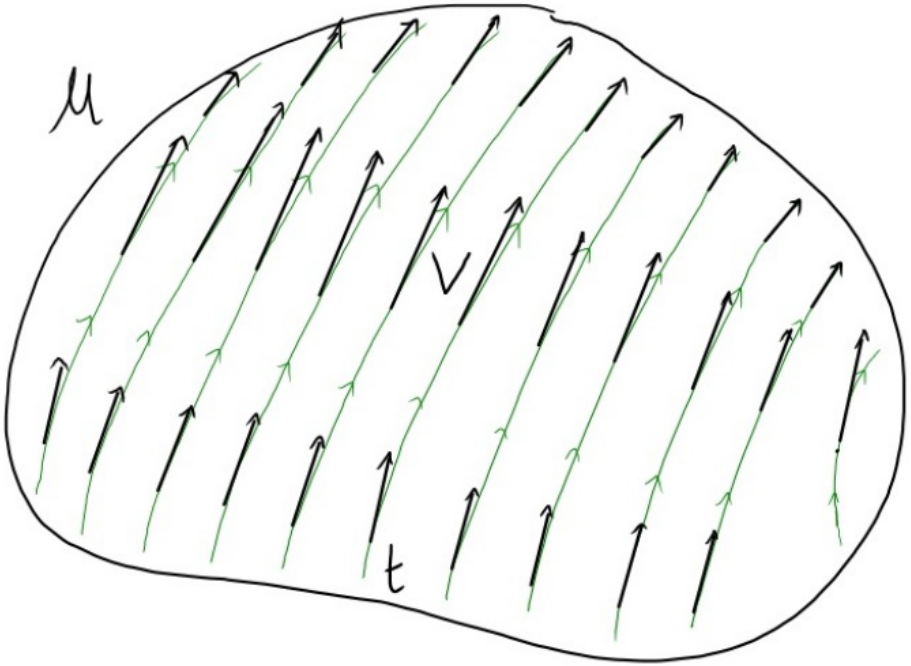

- They pass through each $P \in U$
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$\Rightarrow$ they form a "congruence"