- Diffeomorphisms
  - the symmetry of GR

- Lie Derivatives
  - 1-parameter family of diffeos generated by a vector field

We have seen that

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \qquad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \omega_{\mu'}$$

xfm of components of vectors under change of coordinate basis

same, for 1-forms

We have seen that

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$\downarrow$ xfm of components of vectors under change of coordinate basis

$\downarrow$ same, for 1-forms

Can be used as a definition of vectors/1-forms

We have seen that

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \qquad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \omega_{\mu'}$$

Define the matrix:

$$\left((\phi^*)^{\mu}{}_{\mu'}\right) = \begin{pmatrix} \frac{\partial x^1}{\partial x^{1'}} & \frac{\partial x^1}{\partial x^{2'}} & \cdots & \frac{\partial x^1}{\partial x^{n'}} \\ \frac{\partial x^2}{\partial x^{1'}} & \frac{\partial x^2}{\partial x^{2'}} & \cdots & \frac{\partial x^2}{\partial x^{n'}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x^{1'}} & \frac{\partial x^n}{\partial x^{2'}} & \cdots & \frac{\partial x^n}{\partial x^{n'}} \end{pmatrix}$$

row column

We have seen that

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$$V^{\mu} = (\phi^*)^{\mu}{}_{\mu'} V^{\mu'} \Rightarrow V = \phi^* \cdot V'$$

$$V = \begin{pmatrix} V^1 \\ \vdots \\ V^n \end{pmatrix} \qquad V' = \begin{pmatrix} V^{1'} \\ \vdots \\ V^{n'} \end{pmatrix}$$

We have seen that

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \qquad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \omega_{\mu'}$$

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$$\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\nu'}} = \delta^{\mu'}{}_{\nu'} \Rightarrow$$

$$\frac{\partial x^{\mu'}}{\partial x^{\mu}} \cdot (\phi^*)^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}$$

We have seen that

$$V^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'} \qquad \omega_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \omega_{\mu'}$$

$$\left((\phi^{*-1})^{\mu'}{}_{\mu}\right) = \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^{1}} & \frac{\partial x^{1'}}{\partial x^{2}} & \cdots & \frac{\partial x^{1'}}{\partial x^{n}} \\ \frac{\partial x^{2'}}{\partial x^{1}} & \frac{\partial x^{2'}}{\partial x^{2}} & \cdots & \frac{\partial x^{2'}}{\partial x^{n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^{n'}}{\partial x^{1}} & \frac{\partial x^{n'}}{\partial x^{2}} & \cdots & \frac{\partial x^{n'}}{\partial x^{n}} \end{pmatrix}$$

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$$\omega_{\mu} = \omega_{\mu'} (\phi^{*-1})^{\mu'}{}_{\mu} \Rightarrow \omega = \omega' (\phi^{*})^{-1}$$

$$\omega = (\omega_1 \cdots \omega_n)$$

$$\omega' = (\omega_1' \cdots \omega_n')$$

$$\frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\nu'}} = \delta^{\mu'}{}_{\nu'} \Rightarrow$$

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$$\omega = (\omega_1 \cdots \omega_n)$$

$$\omega' = (\omega_1' \cdots \omega_n')$$

$$T^{\mu\nu}{}_{\rho} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\rho'}}{\partial x^{\rho}} T^{\mu'\nu'}{}_{\rho'}$$

$$= (\phi^{*})^{\mu}{}_{\mu'} (\phi^{*})^{\nu}{}_{\nu'} T^{\mu'\nu'}{}_{\rho'} (\phi^{*-1})^{\rho'}{}_{\rho}$$

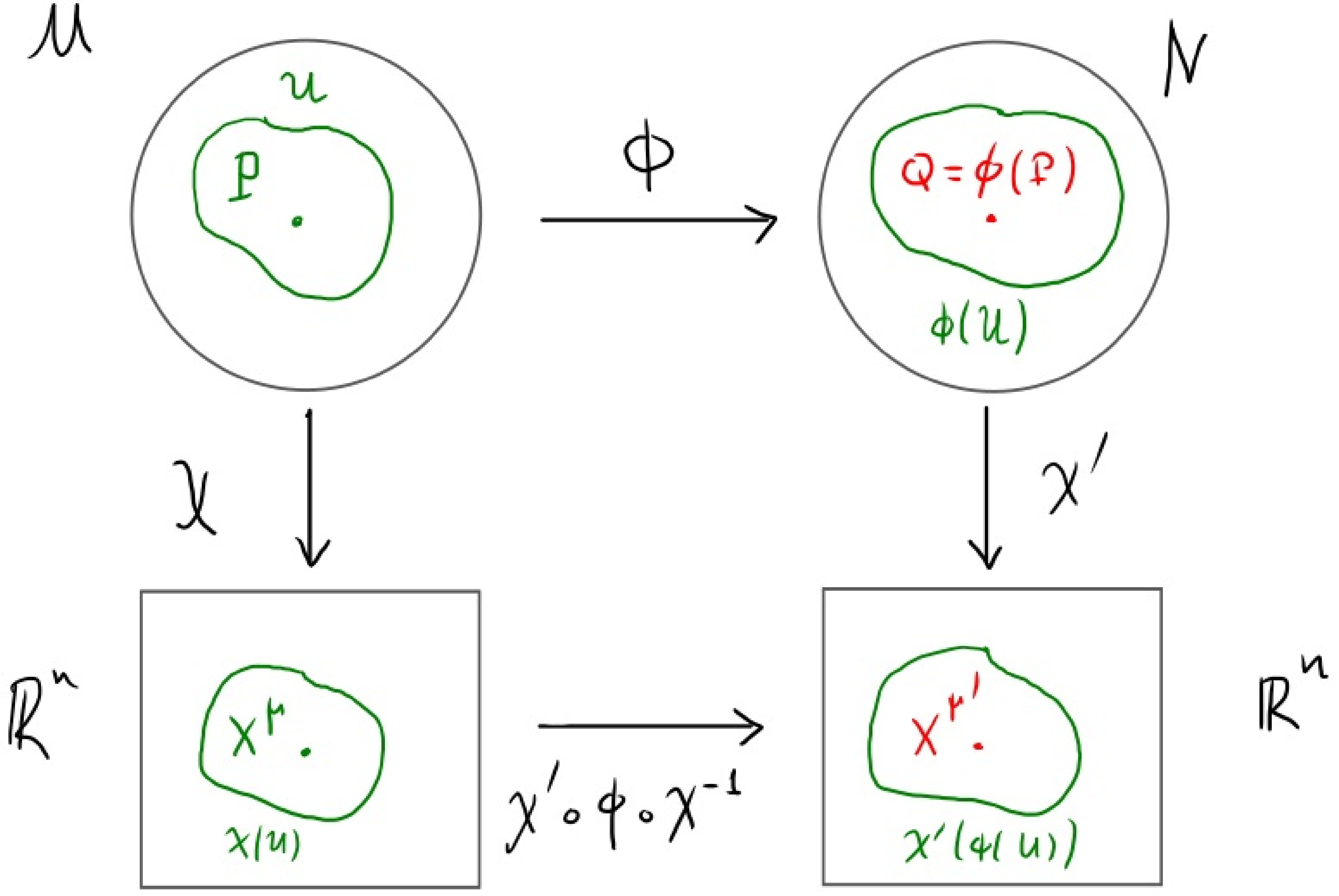
Smooth maps

A map $\phi : \mathcal{M} \to \mathcal{N}$ is smooth, if for every pair of charts, the function

$$\chi' \circ \phi \circ \chi^{-1} : \chi(\mathcal{U}) \to \chi'(\phi(\mathcal{U}))$$

$$x^{\mu} \mapsto x^{\mu'}$$

is smooth.



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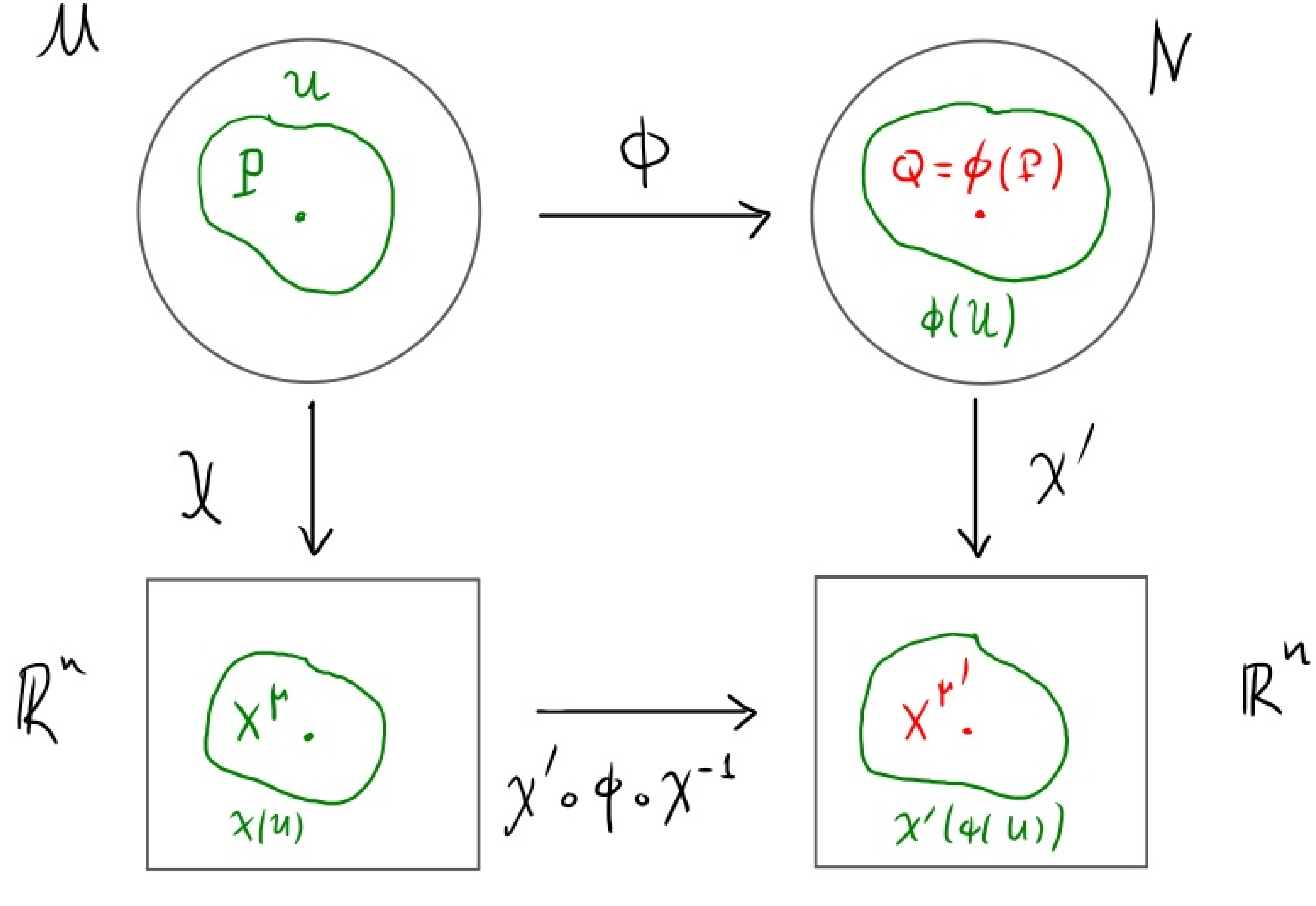


It simply means that $x^{\mu'} = x^{\mu'}(x^{\nu})$,

$\dfrac{\partial x^{\mu'}}{\partial x^{\mu}}$, $\dfrac{\partial^2 x^{\mu'}}{\partial x^{\mu} \partial x^{\nu}}$, ... are smooth

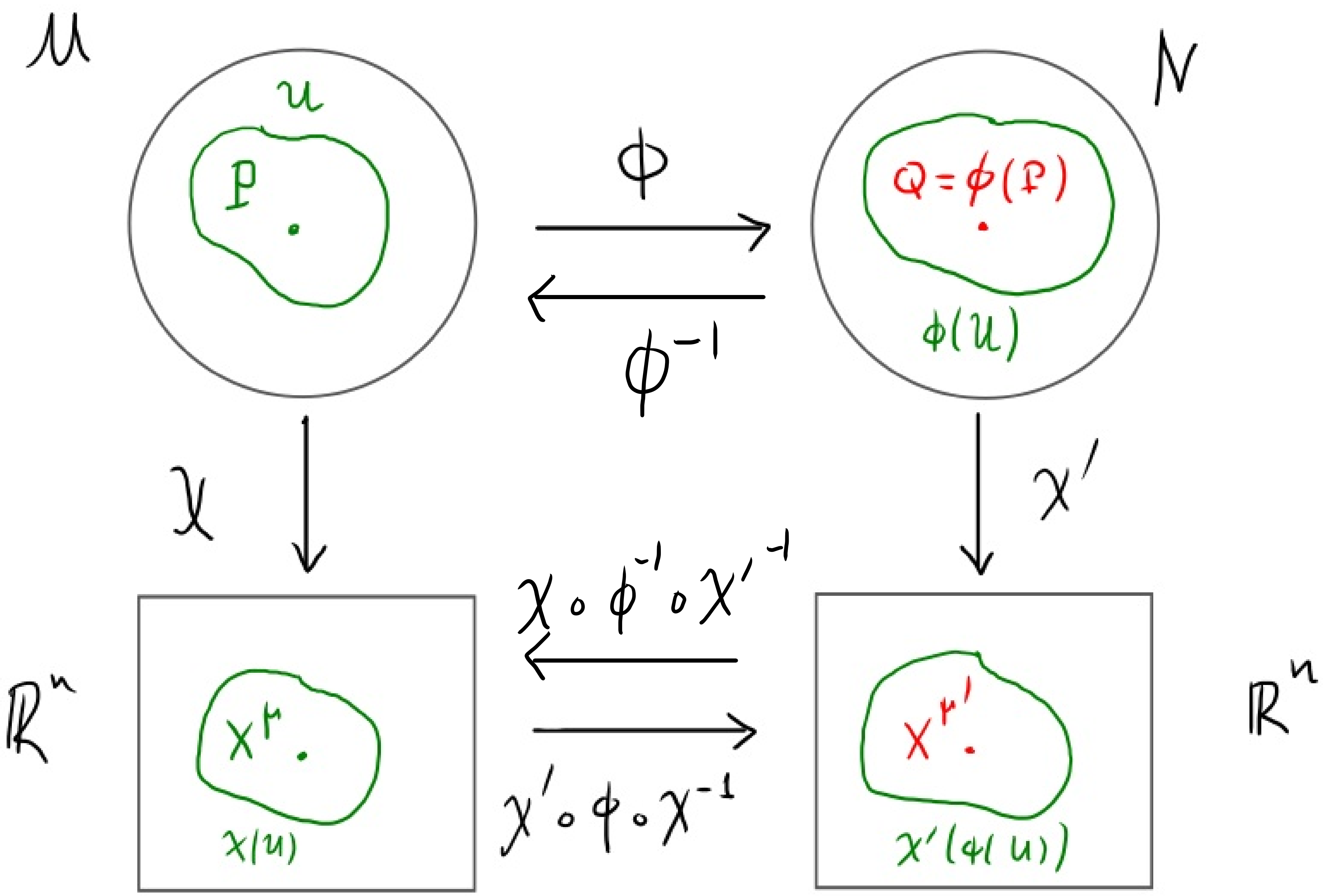
Diffeomorphisms

A map $\phi : M \to N$ is a diffeomorphism, if for every pair of charts, the function

$$x' \circ \phi \circ x^{-1} : x(U) \to x'(\phi(U))$$

$$x^{\mu} \mapsto x^{\mu'}$$

is smooth, and ϕ is a bijection (1-1, onto, invertible)

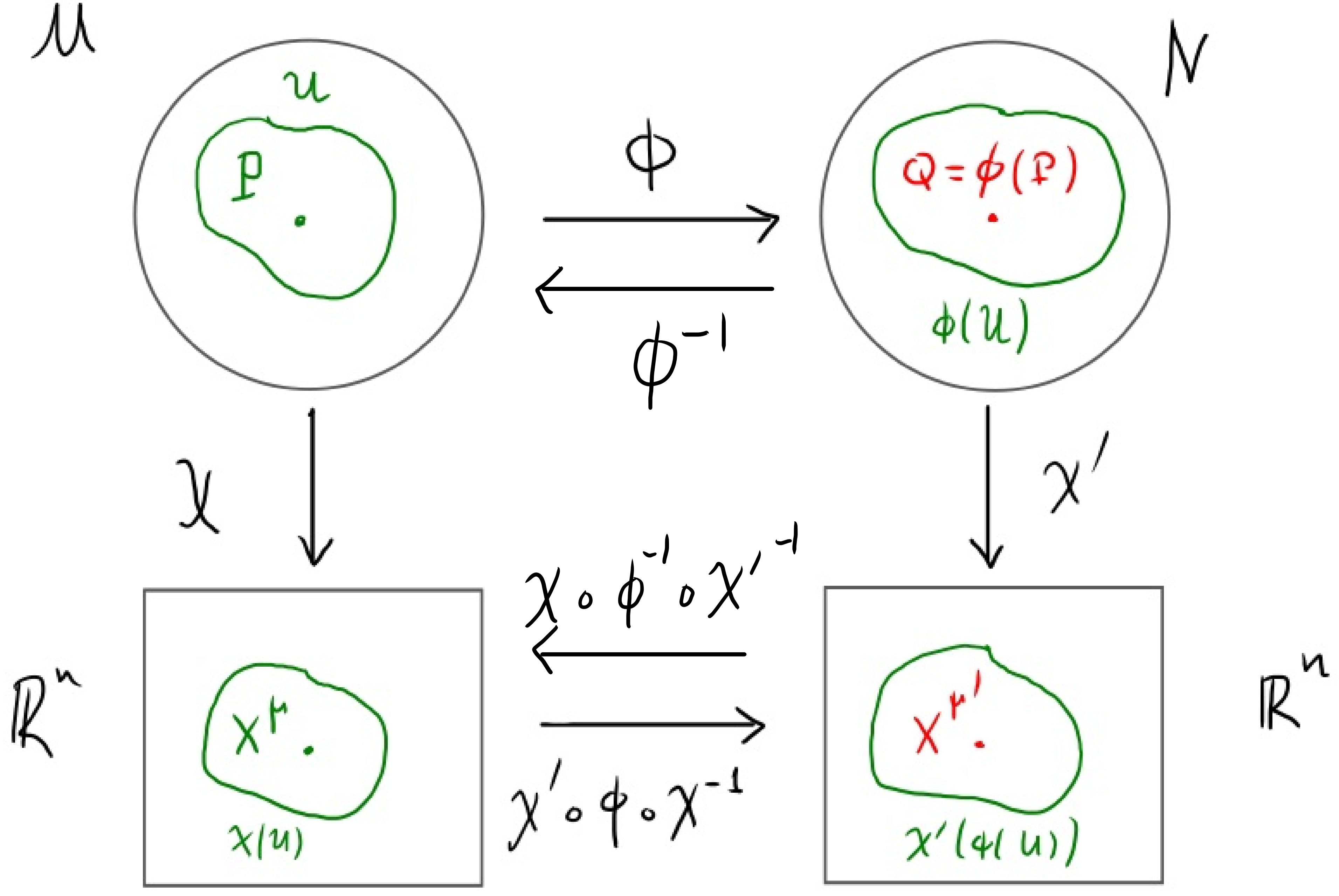


Diffeomorphisms

A map $\phi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism if for every pair of charts, the function

$$\chi' \circ \phi \circ \chi^{-1} : \chi(\mathcal{U}) \to \chi'(\phi(\mathcal{U}))$$

$$x^\mu \mapsto x^{\mu'}$$



is smooth, and ϕ is a bijection (1-1, onto, invertible)

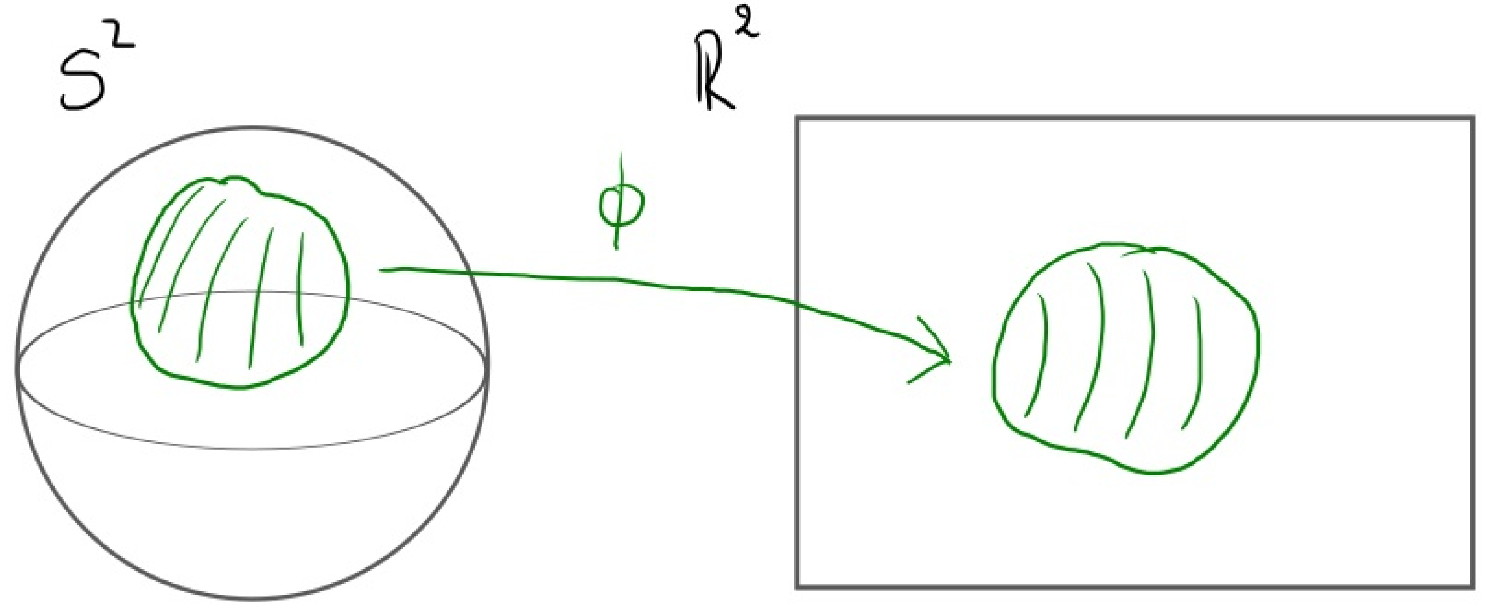
* ϕ also a homeomorhism
* $\mathcal{M}$ and $\mathcal{N}$ topologically and differentiably indistinguishable

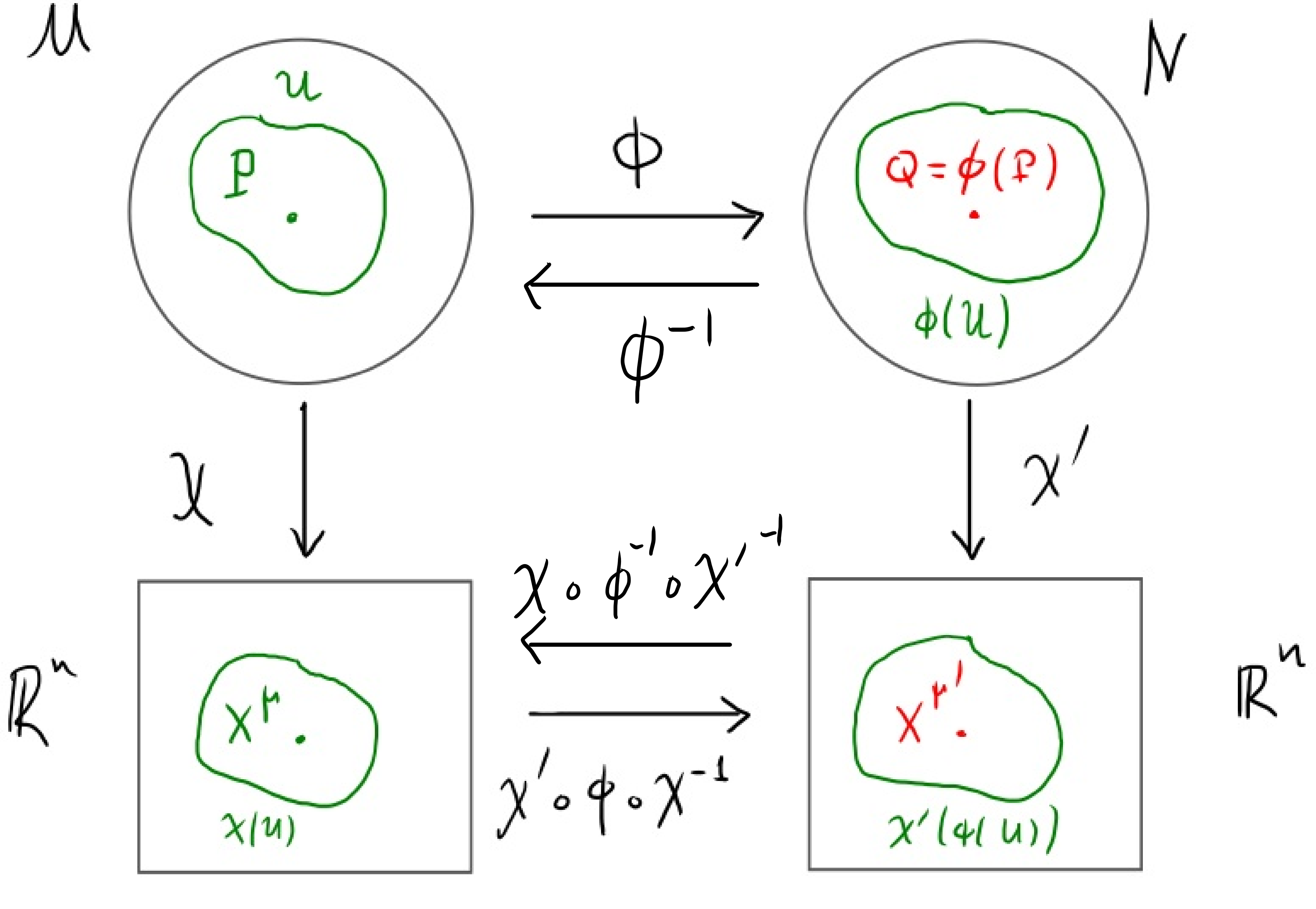
Diffeomorphisms

* $M \cong N$ are diffeomorphic

* Manifolds of the same dimension can be locally diffeomorphic, but not globally

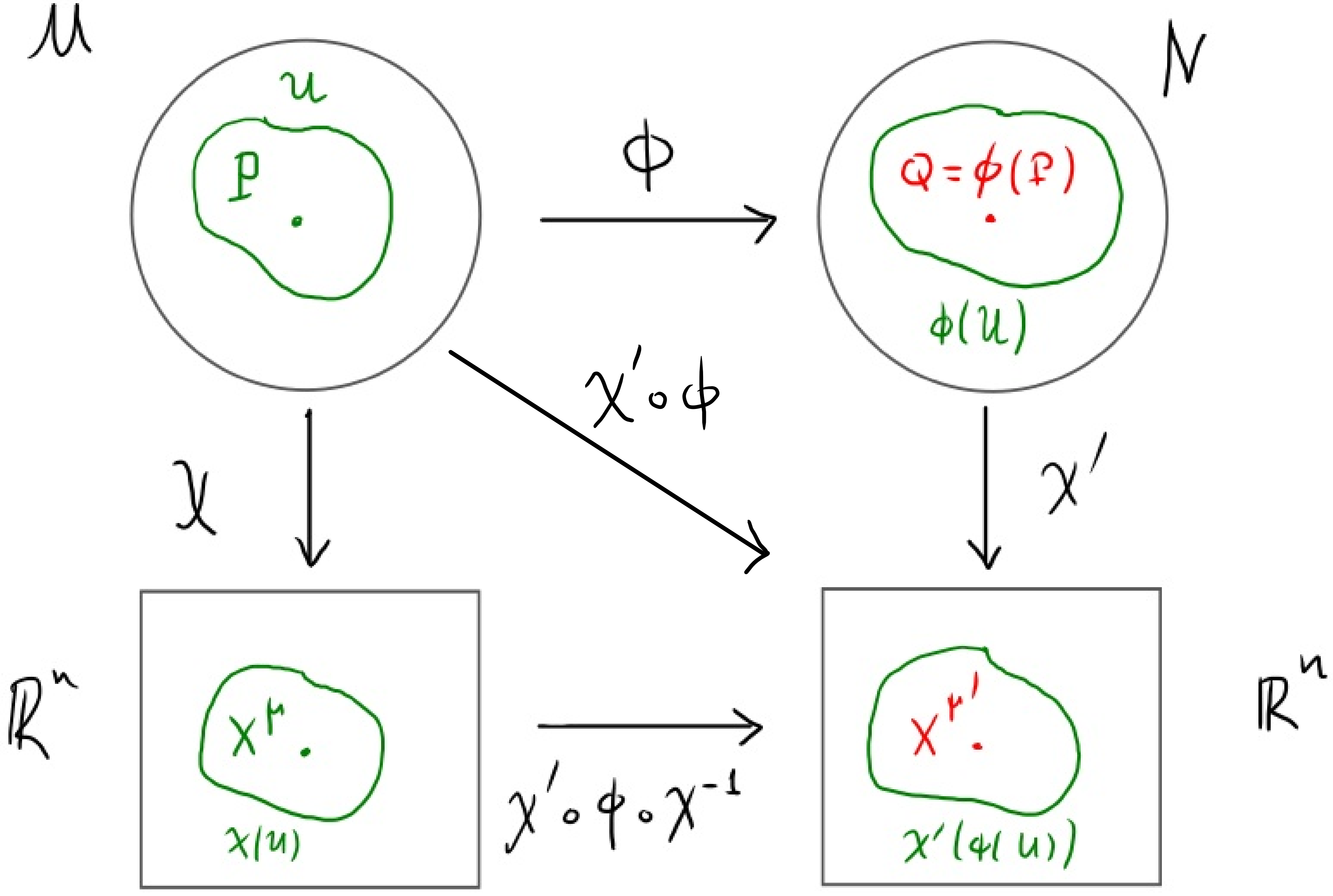
e.g. S^2 and $\mathbb{R}^2$ (*)





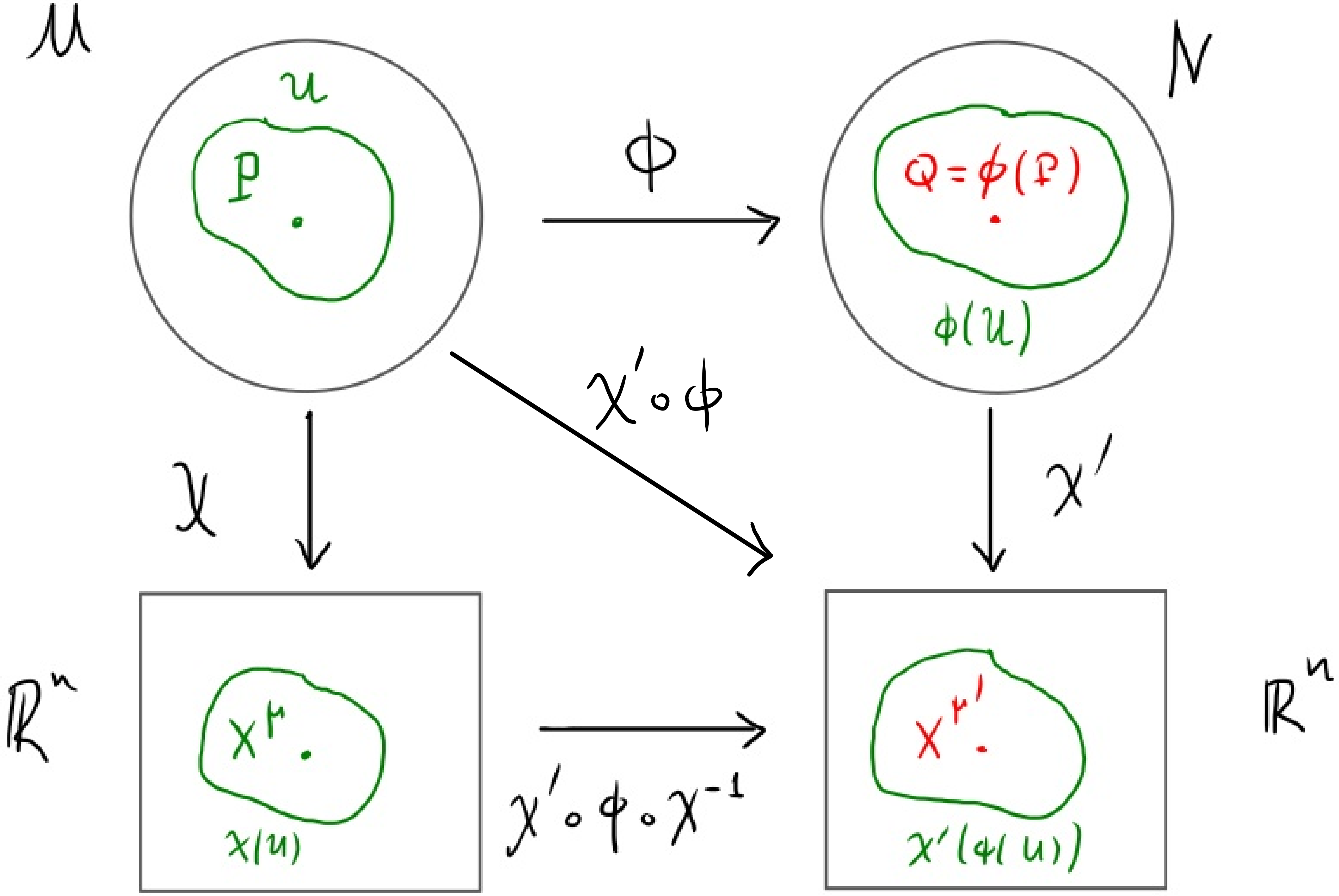
(*) Not all diff manifolds of same dim are locally diffeomorphic. E.g. there are locally inequivalent differential structures for $\mathbb{R}^4$

Diffeomorphisms



* $M \cong N$ are diffeomorphic
* Manifolds of the same dimension can be locally diffeomorphic, but not globally
* A diffeomorphism defines a coordinate system $(U, \chi' \circ \phi)$ with coordinates $x^{\mu'}$ and transition function $\chi' \circ \phi \circ \chi^{-1}$

Diffeomorphisms



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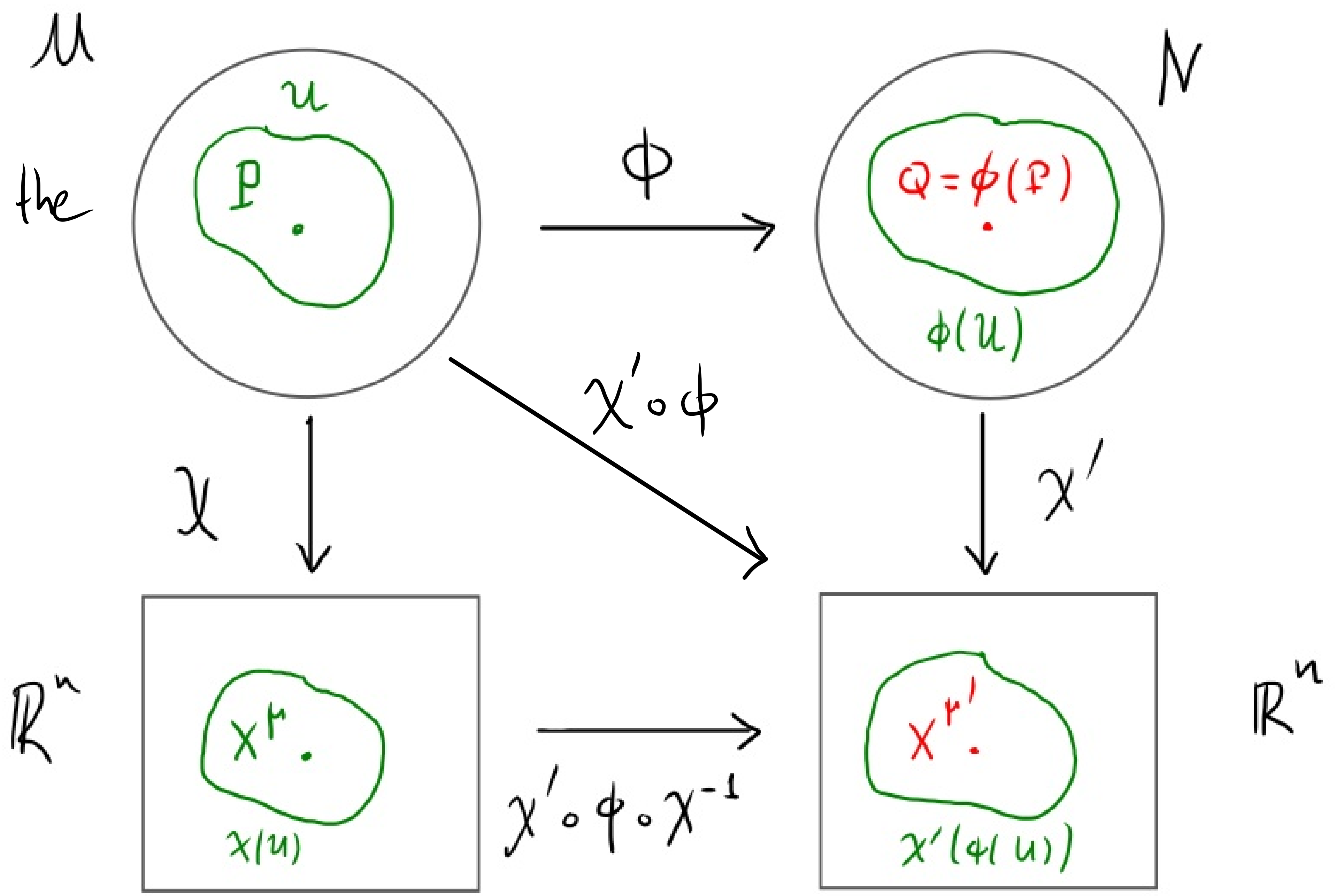
* (Diffeomorphism invariant theory) $\Leftrightarrow$ (Coordinate xfm invariant theory)

Diffeomorphisms

* Diffeomorphisms $\phi: M \to M$ form the diffeomorphism group of M



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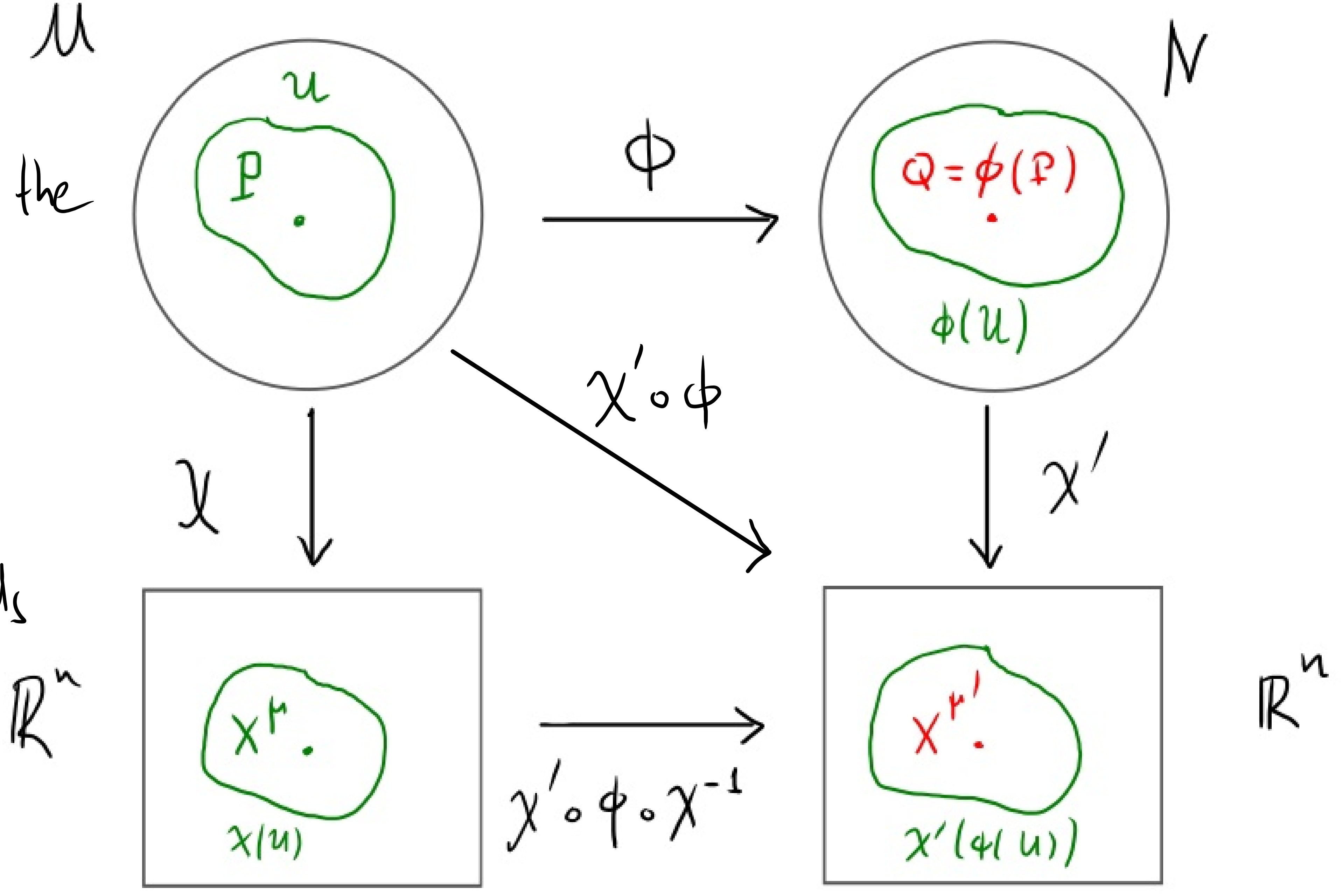
Diffeomorphisms

* Diffeomorphisms $\phi: M \to M$ form the diffeomorphism group of M
  - infinitesimal diffeos
    $$x^\mu \to x^\mu + \epsilon V^\mu(x)$$
    generated by Lie algebra of v.fields
    $$V = V^\mu(x)\,\partial_\mu$$



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Diffeomorphisms

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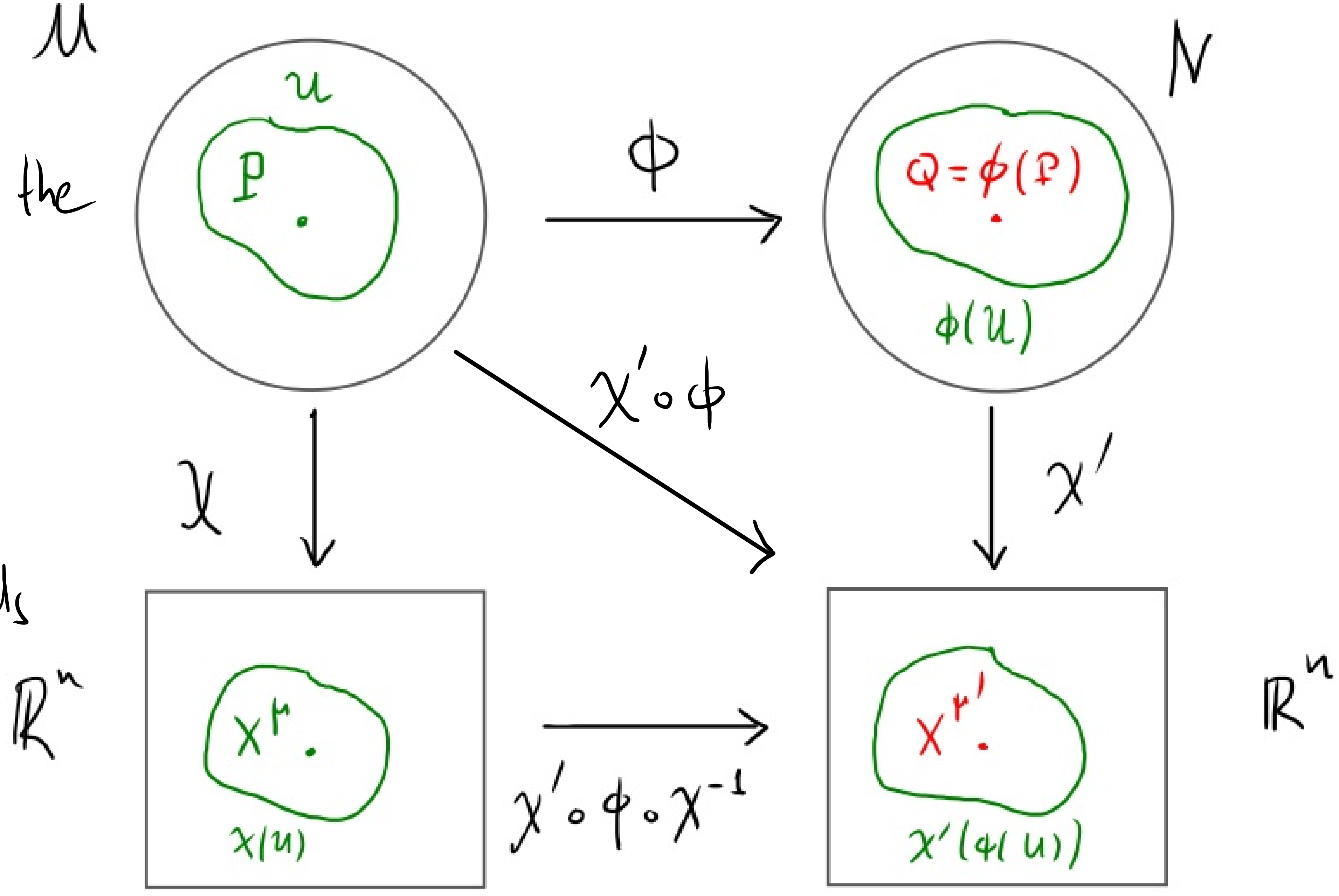


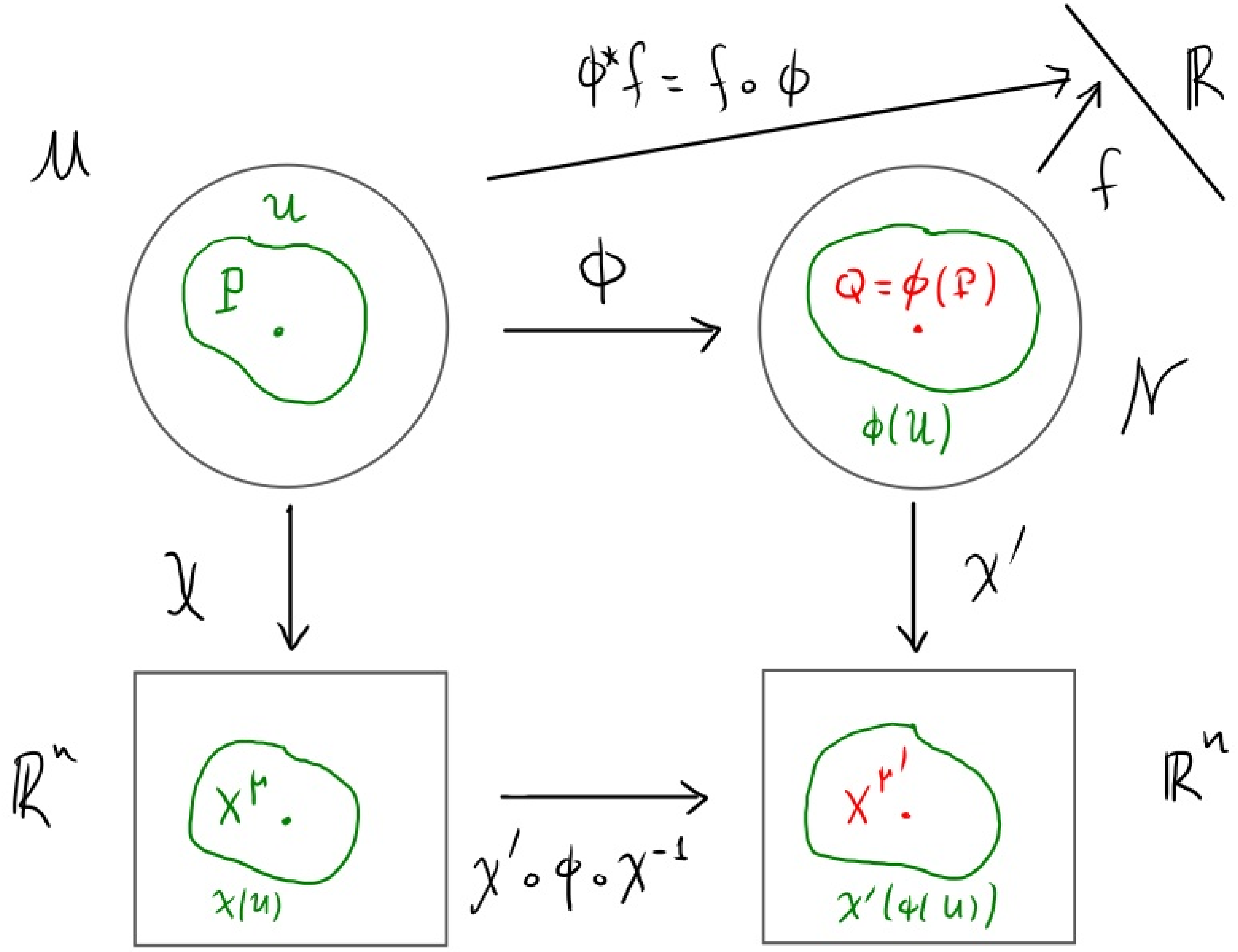
* $x^\mu \to x^{\mu'}$ can be regarded as

(α) a diffeomorhism (active transformation)

(β) a coordinate xfm (passive 〃)

Pullback of a function

Let $f : N \to \mathbb{R}$ a smooth fu



Pullback of a function

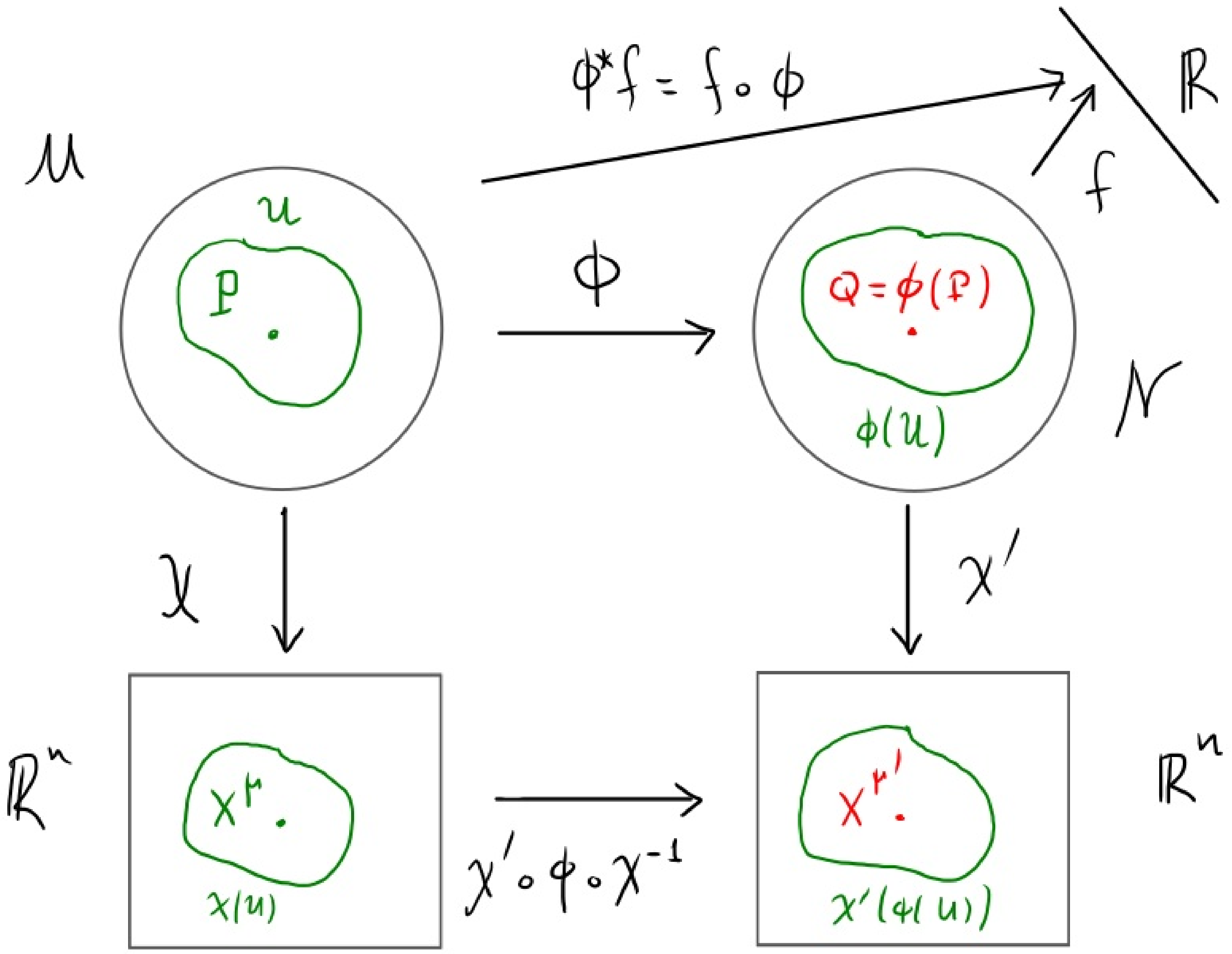
Let $f : N \to \mathbb{R}$ a smooth fn,

then the function

$$\phi^* f = f \circ \phi : M \to \mathbb{R}$$

is also a smooth function on M

$\phi^* f$: the pullback of f on M



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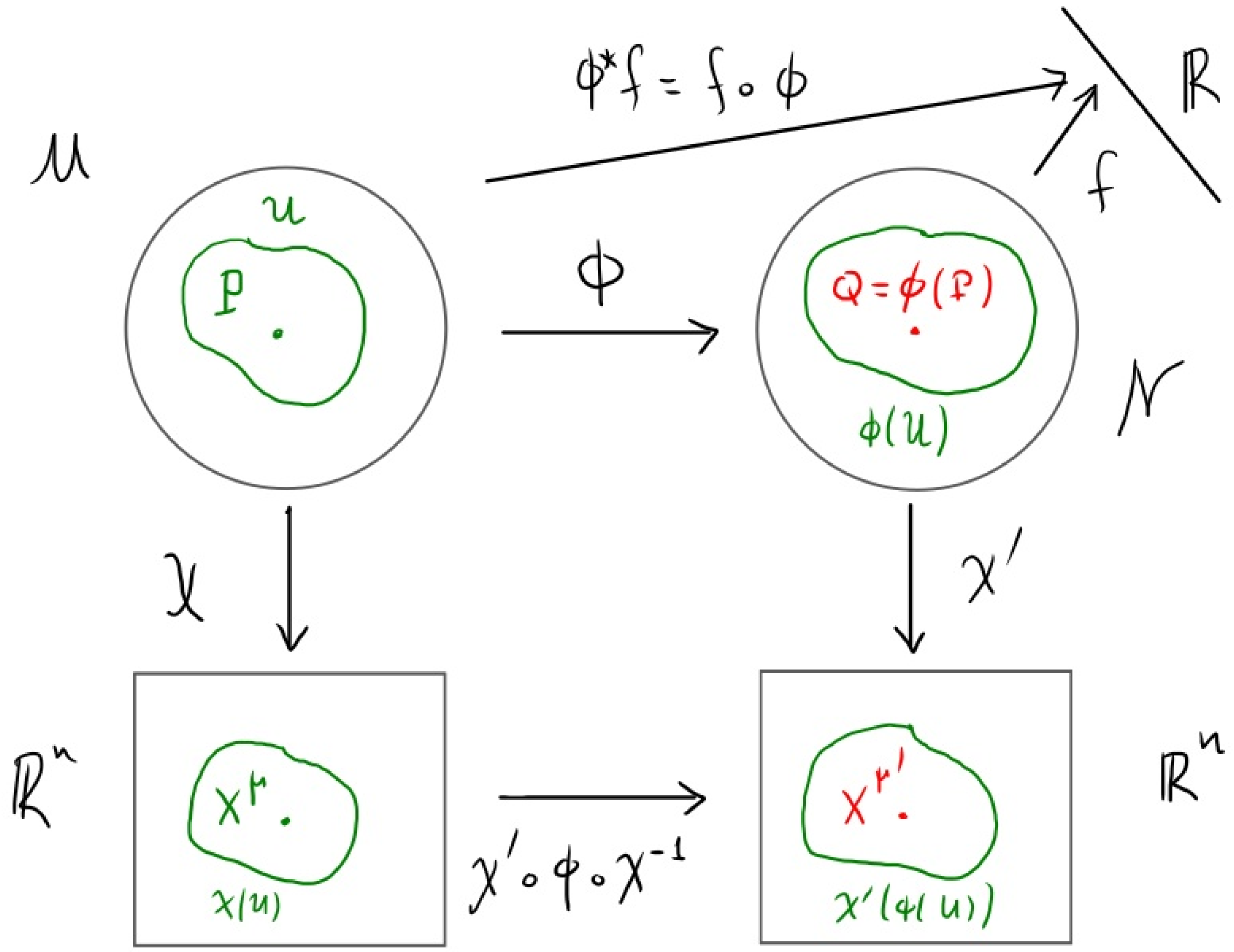

$$f \text{ smooth on } N \Leftrightarrow f \circ \chi'^{-1} \text{ smooth} \Leftrightarrow \underbrace{\phi^* f \circ \chi^{-1}}_{\text{smooth}} = f \circ \phi \circ \chi^{-1} = \underbrace{\overbrace{f \circ \chi'^{-1}}^{\text{smooth}} \circ \overbrace{\chi' \circ \phi \circ \chi^{-1}}^{\text{smooth}}}_{\text{composition of smooth is smooth}}$$

Pullback of a function

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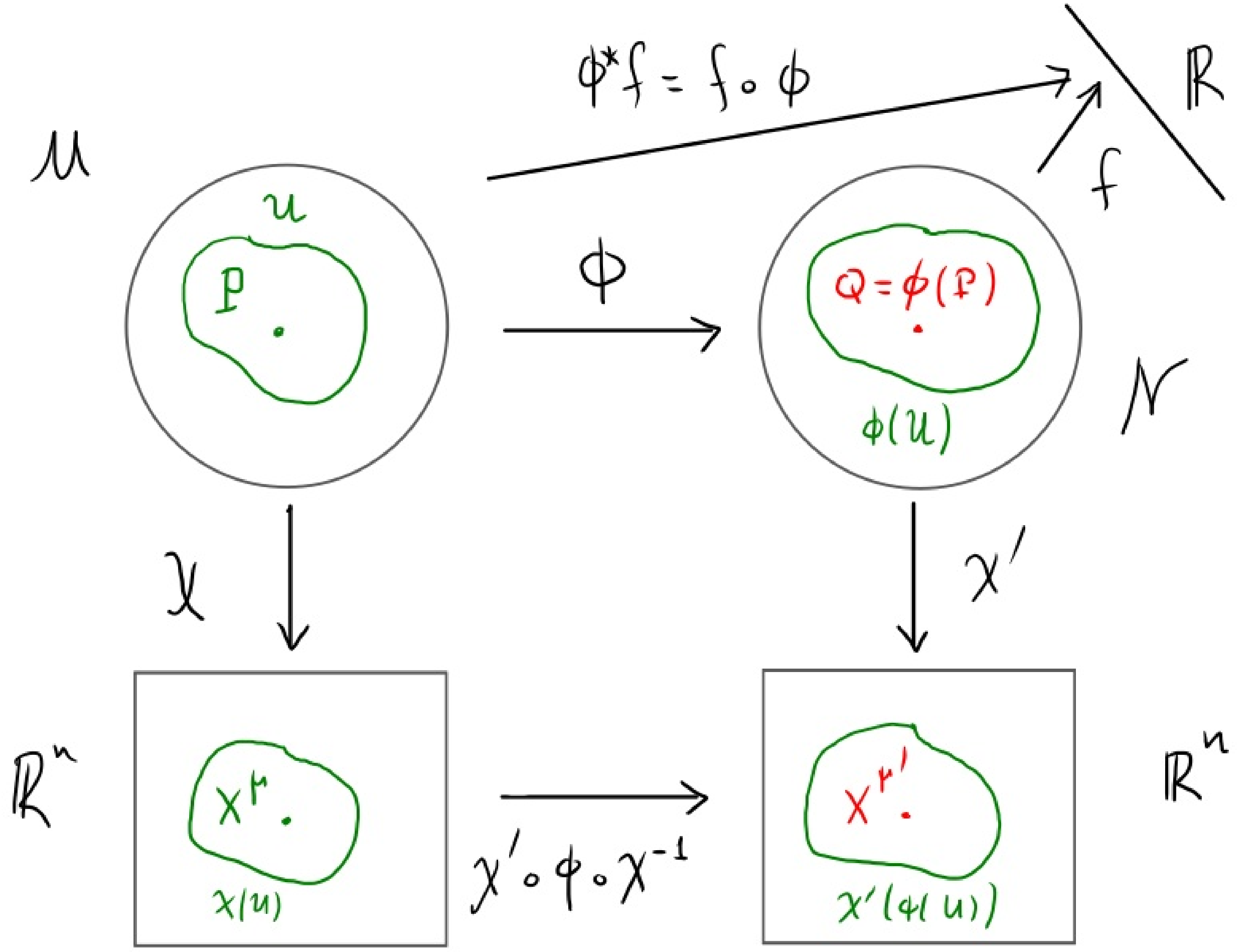


$$f \text{ smooth on } N \Leftrightarrow f \circ \chi'^{-1} \text{ smooth} \Leftrightarrow \underbrace{\phi^* f \circ \chi^{-1}}_{\text{smooth}} = f \circ \phi \circ \chi^{-1} = f \circ \chi'^{-1} \circ \chi' \circ \phi \circ \chi^{-1}$$

$$\Leftrightarrow \phi^* f \text{ smooth}$$

Pullback of a function

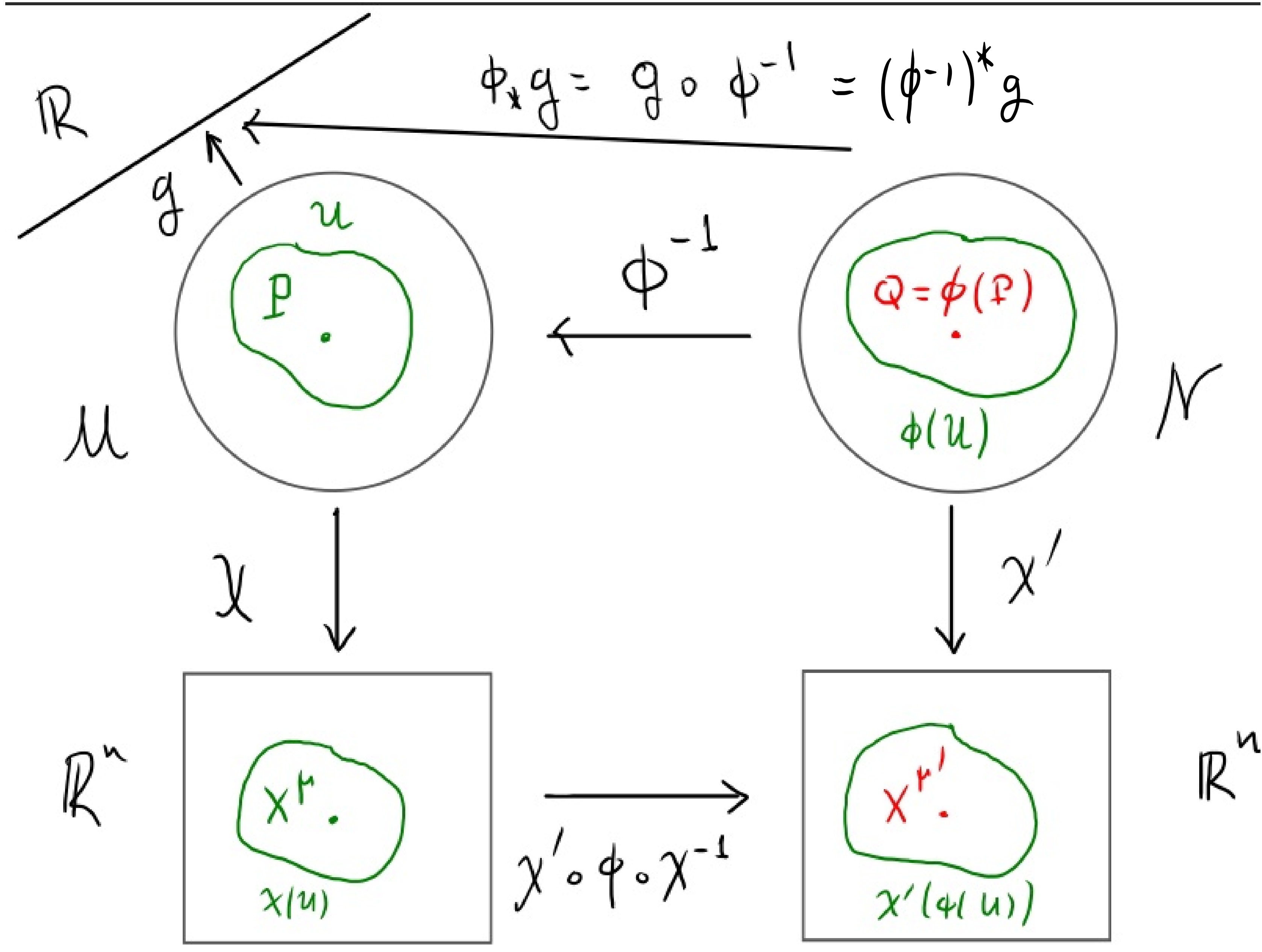
Let $f : N \to \mathbb{R}$ a smooth fn,

then the function

$$\phi^* f = f \circ \phi : M \to \mathbb{R}$$

Similarly if $g : M \to \mathbb{R}$

$$(\phi^{-1})^* g = g \circ \phi^{-1} : N \to \mathbb{R}$$



We also define $\phi_* g = (\phi^{-1})^* g$, the pushforward of g to N

Pullback of a vector V

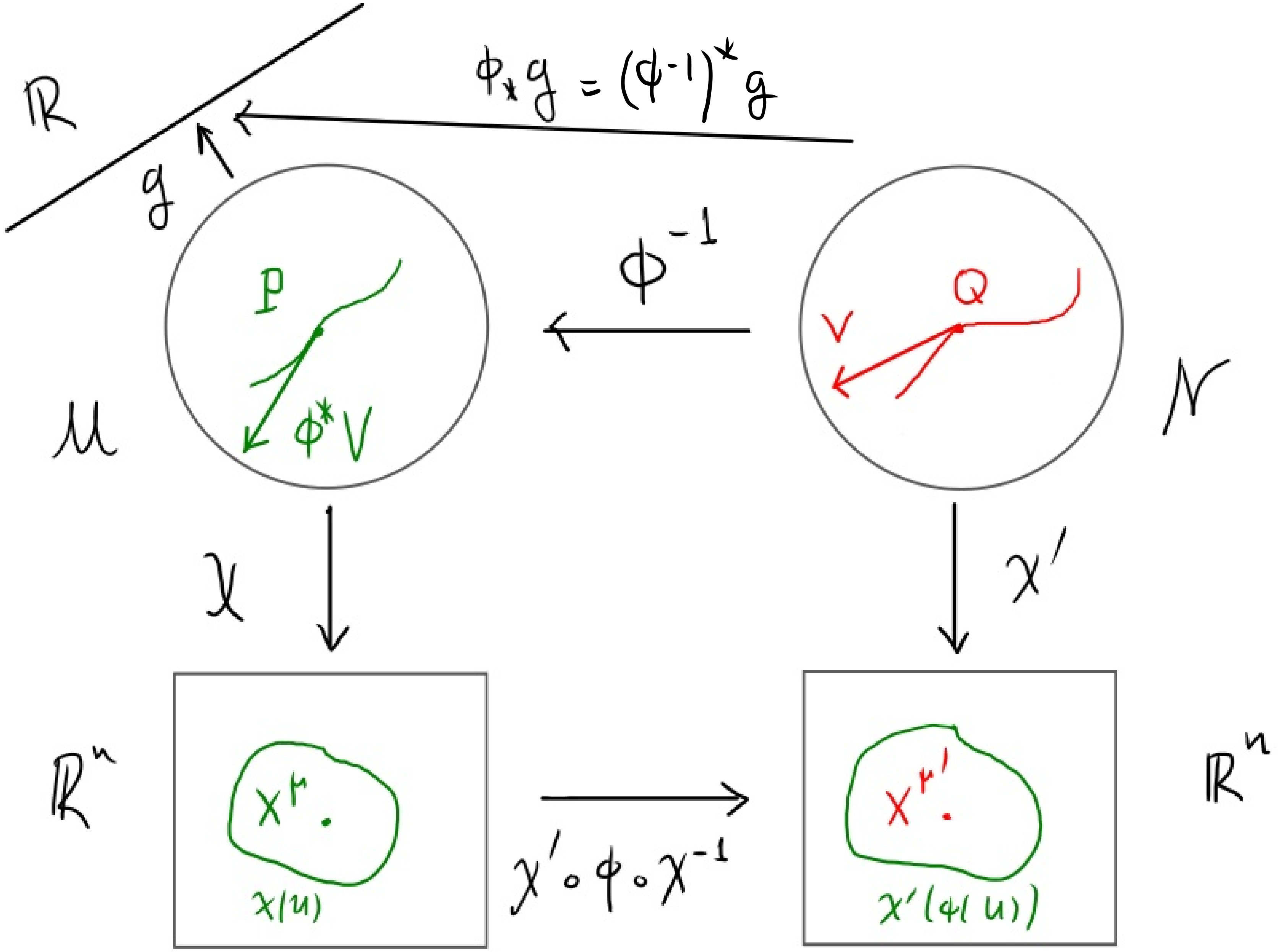
Let $V \in T_Q N$, then define

$\phi^* V \in T_P M$ by

$$\phi^* V(g) = V\left((\phi^{-1})^* g\right)$$

any smooth function $g \in \mathcal{F}(M)$, defines $(\phi^{-1})^* g \in \mathcal{F}(N)$

V is known, so its action on $(\phi^{-1})^* g$ is also known



Pullback of a vector V

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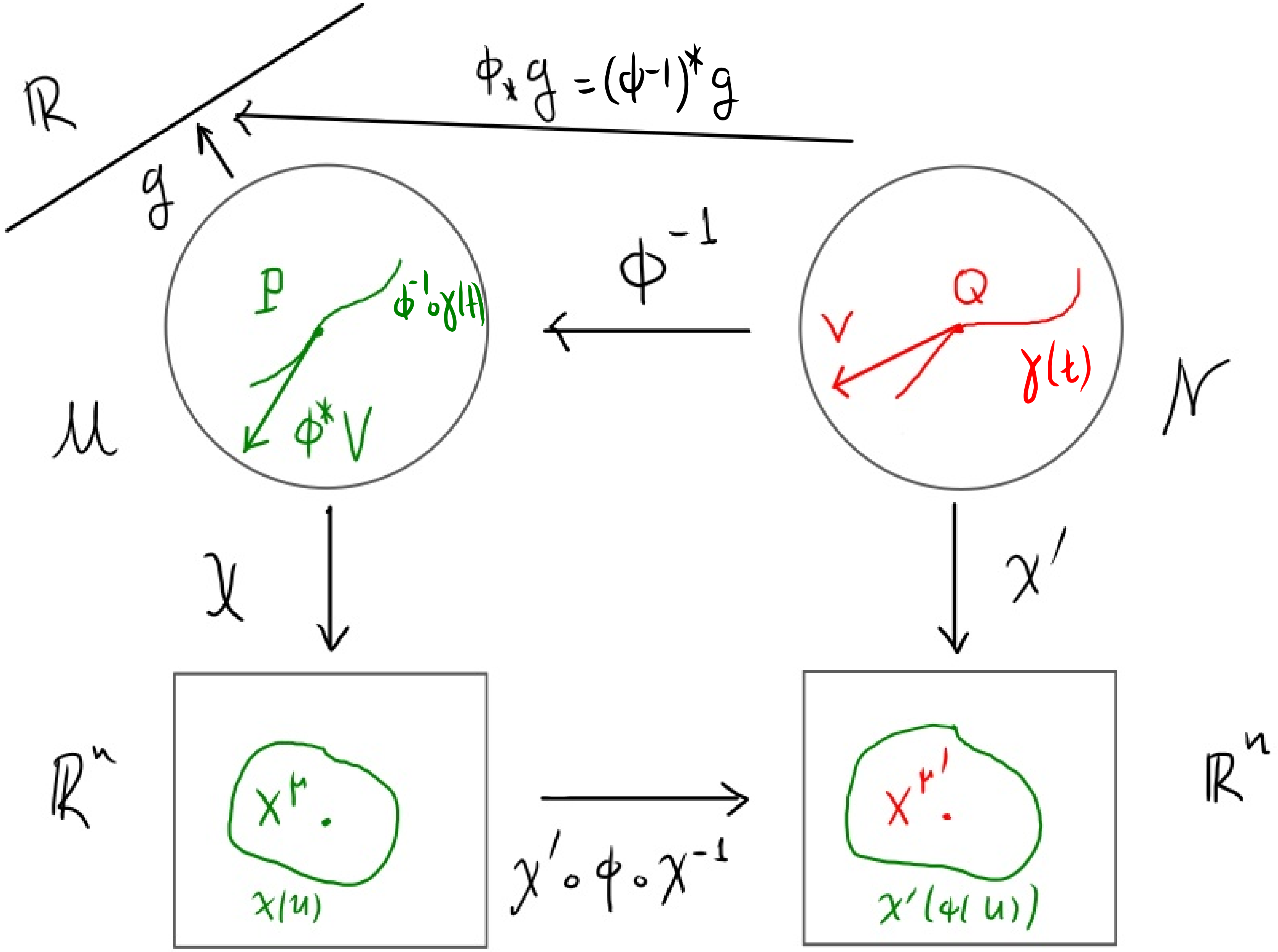
$$= V(g \circ \phi^{-1})$$



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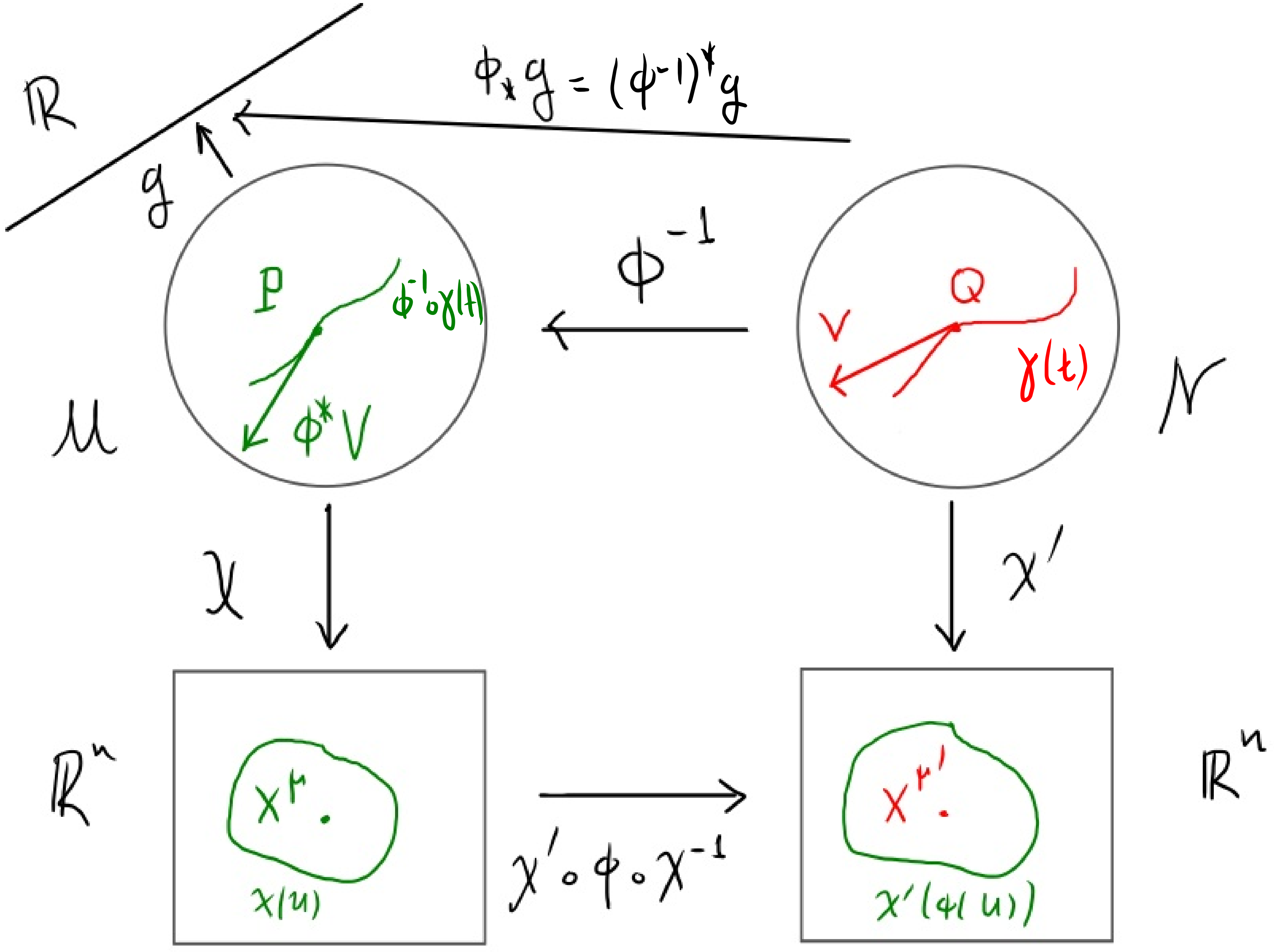
$$\phi^* V(g) = V\left((\phi^{-1})^* g\right)$$

$$= V\left(g \circ \phi^{-1}\right)$$

$$= \frac{d}{dt}\, g \circ \phi^{-1} \circ \gamma(t)$$



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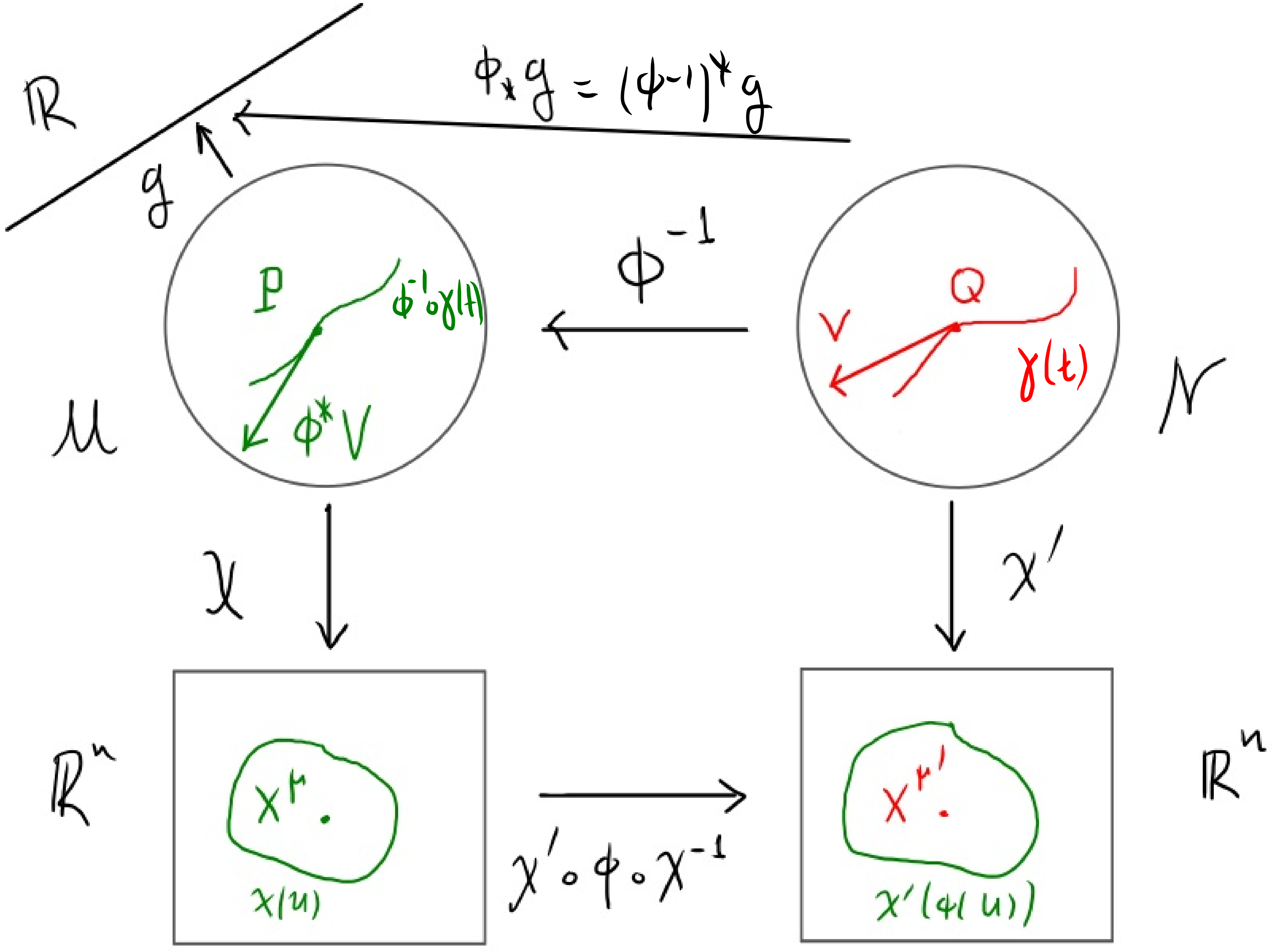
$$\phi^* V(g) = V\left((\phi^{-1})^* g\right)$$

$$= V\left(g \circ \phi^{-1}\right)$$

$$= \frac{d}{dt}\, g \circ \phi^{-1} \circ \gamma(t)$$

$$= \frac{d}{dt}\, g \circ x^{-1} \circ x \circ \phi^{-1} \circ x'^{-1} \circ x' \circ \gamma(t)$$



Pullback of a vector V

Let $V \in T_Q N$, then define

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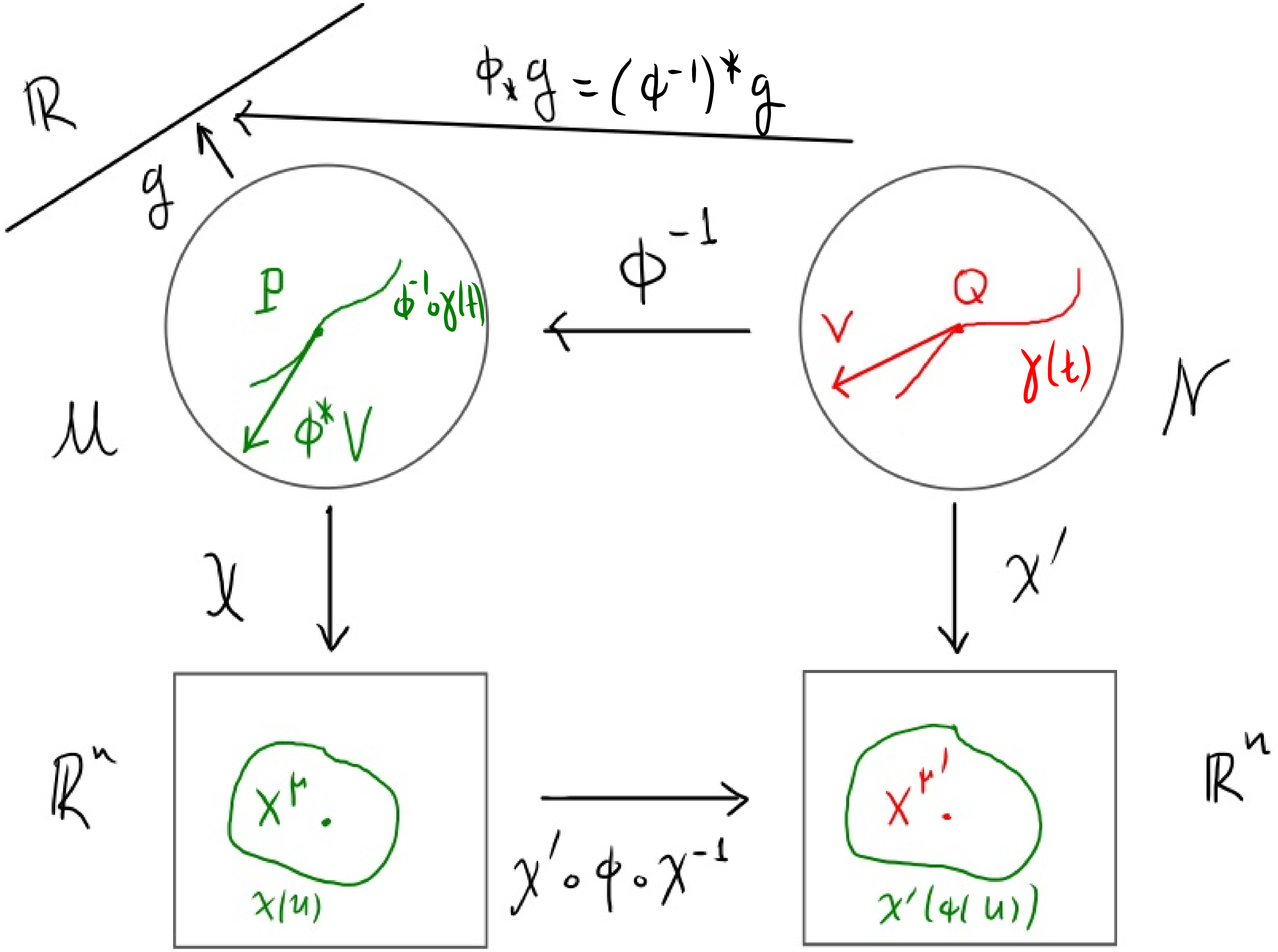
$$\phi^* V(g) = V\left((\phi^{-1})^* g\right)$$

$$= V\left(g \circ \phi^{-1}\right)$$

$$= \frac{d}{dt}\, g \circ \phi^{-1} \circ \gamma(t)$$

$$= \frac{d}{dt}\, \underbrace{g \circ \chi^{-1}}_{g(x^{\mu})} \circ \underbrace{\chi \circ \phi^{-1} \circ \chi'^{-1}}_{x^{\mu}(x^{\mu'})} \circ \underbrace{\chi' \circ \gamma}_{x^{\mu'}}(t)$$



Pullback of a vector V

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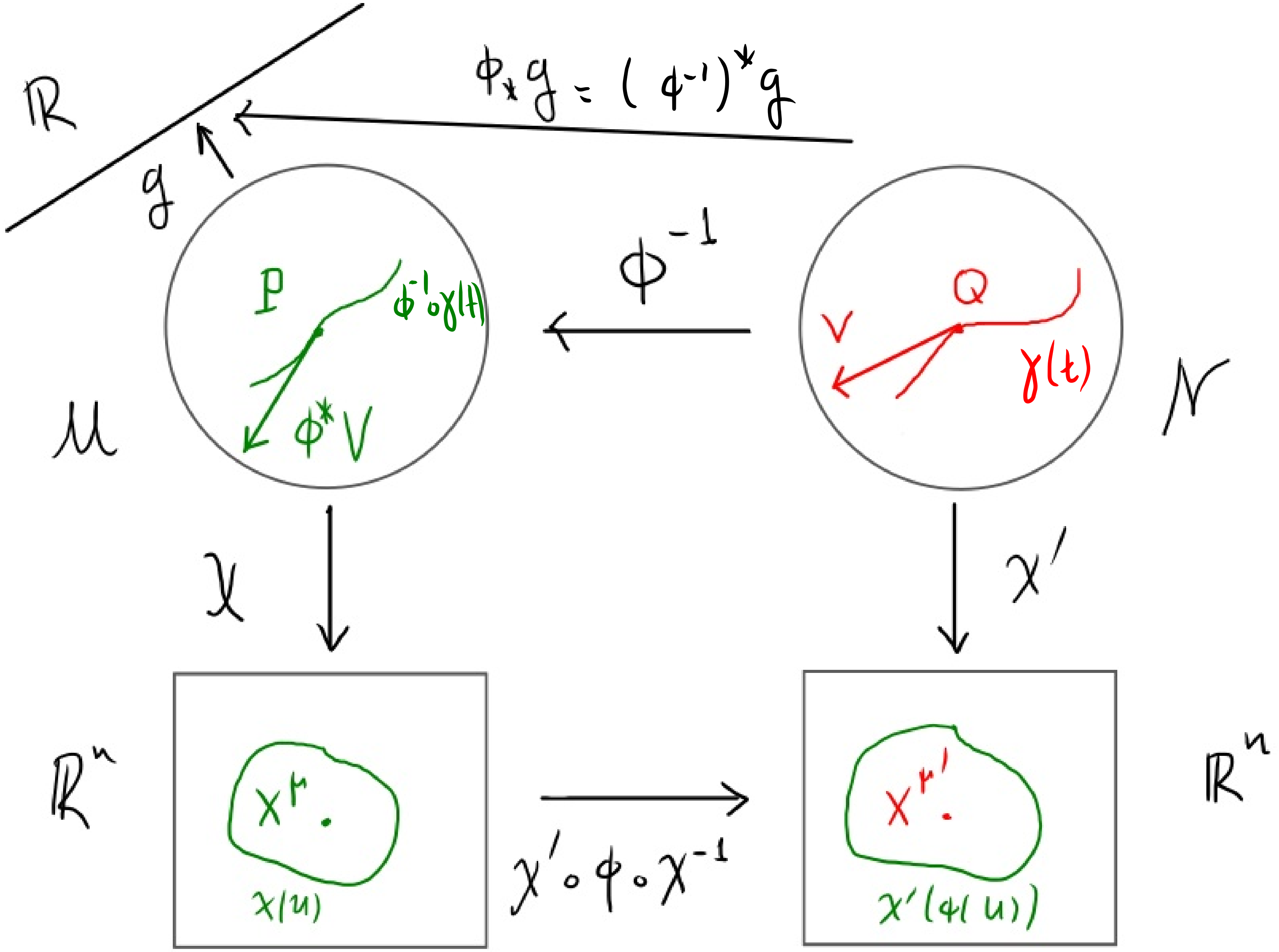
$$\phi^* V(g) = V\left((\phi^{-1})^* g\right)$$

$$= V(g \circ \phi^{-1})$$

$$= \frac{d}{dt} \, g \circ \phi^{-1} \circ \gamma(t)$$

$$= \frac{d}{dt} \, \underbrace{g \circ \chi^{-1}}_{g(x^\mu)} \circ \underbrace{\chi \circ \phi^{-1} \circ \chi'^{-1}}_{x^\mu(x^{\mu'})} \circ \underbrace{\chi' \circ \gamma}_{x^{\mu'}(t)} (t)$$

$$= \frac{\partial g(x^\mu)}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{d x^{\mu'}}{dt}$$

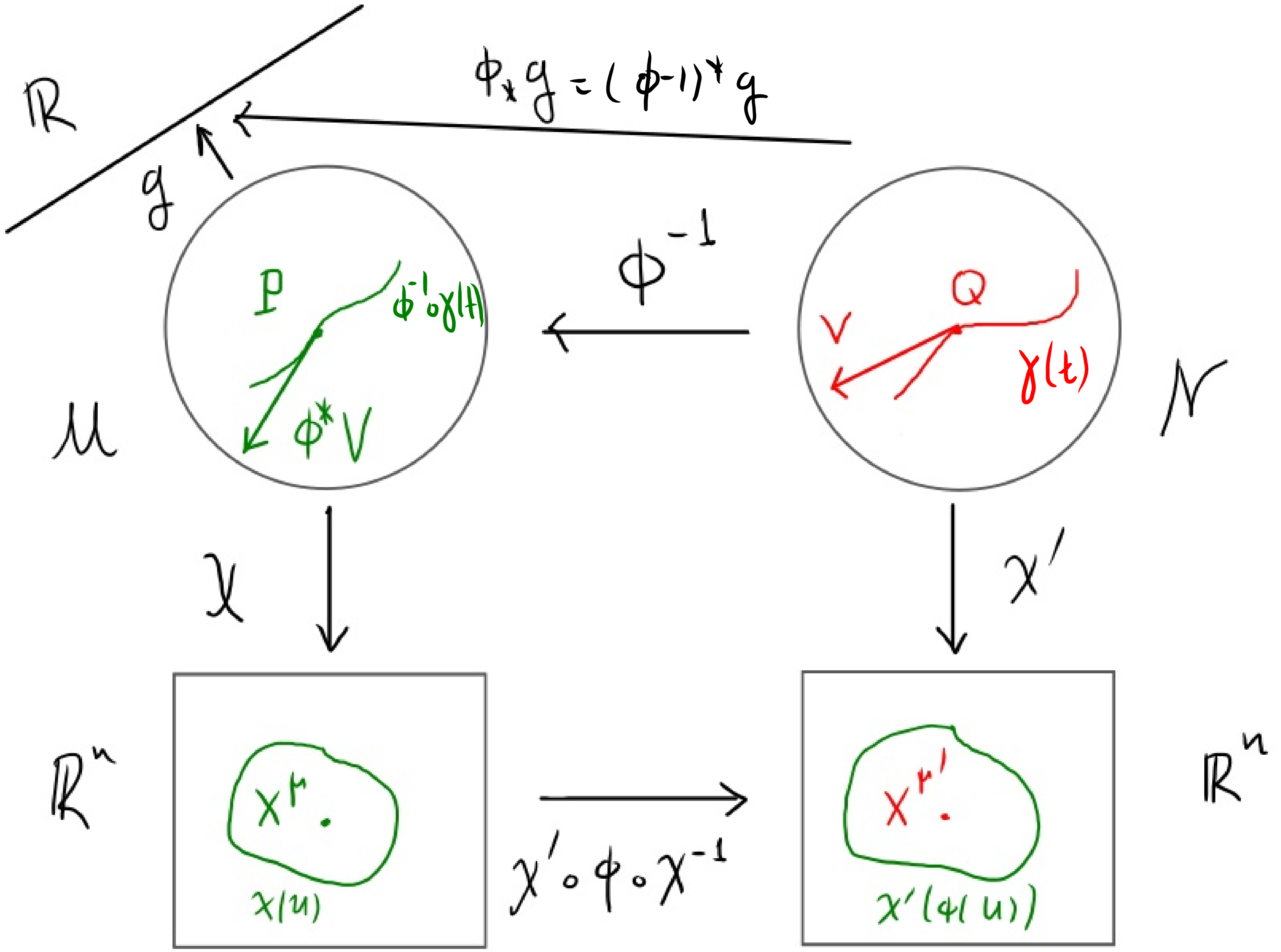


Pullback of a vector V

Let $V \in T_Q N$, then define

$\phi^* V \in T_P M$ by

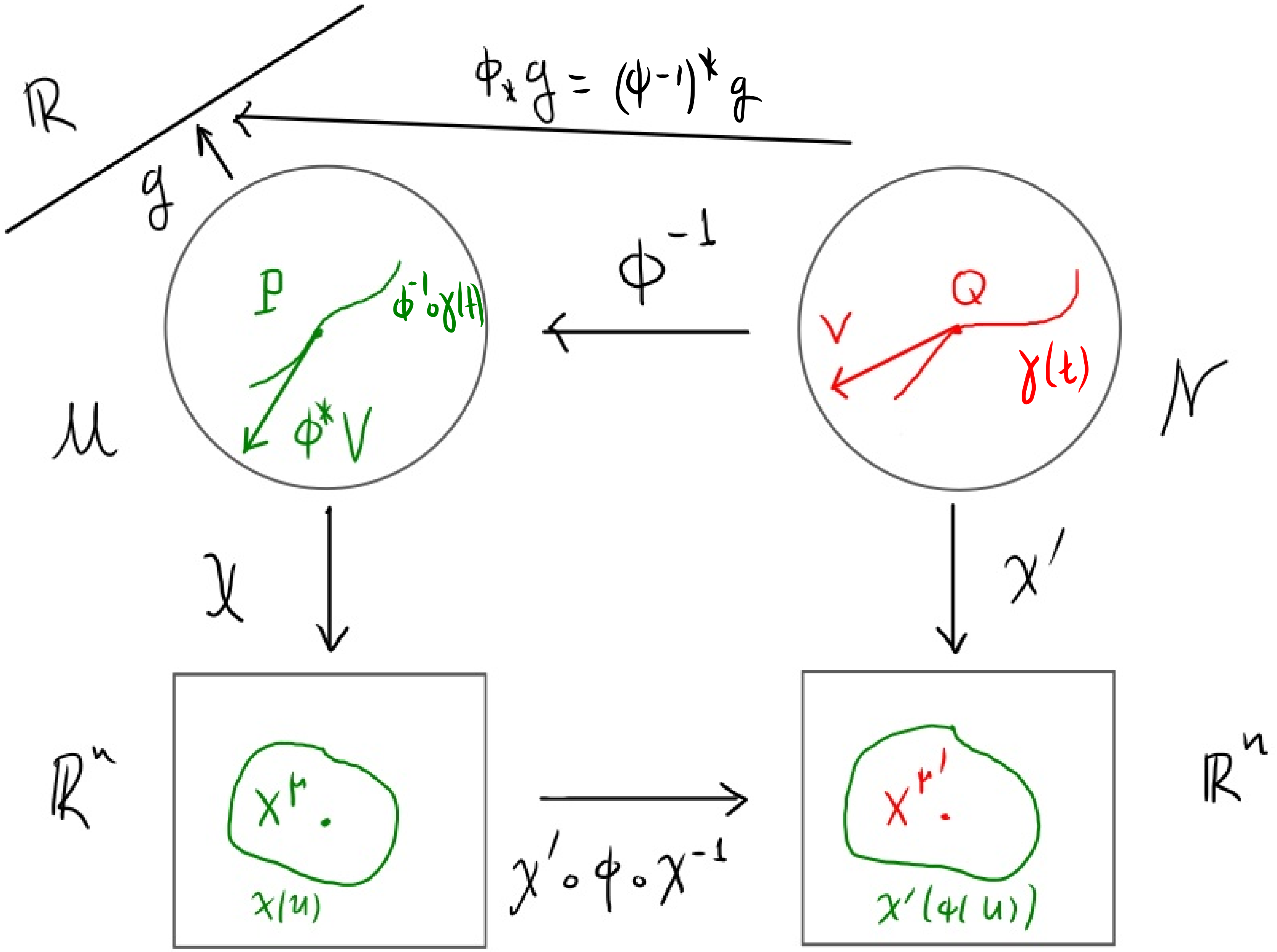
$$
\begin{aligned}
\phi^* V(g) &= V\left((\phi^{-1})^* g\right) \\
&= V\left(g \circ \phi^{-1}\right) \\
&= \frac{d}{dt}\, g \circ \phi^{-1} \circ \gamma(t) \\
&= \frac{d}{dt}\, \underbrace{g \circ \chi^{-1}}_{g(x^\mu)} \circ \underbrace{\chi \circ \phi^{-1} \circ \chi'^{-1}}_{x^\mu(x^{\mu'})} \circ \underbrace{\chi' \circ \gamma}_{x^{\mu'}(t)}(t) \\
&= \frac{\partial g(x^\mu)}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{d x^{\mu'}}{dt} = \partial_\mu g \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot V^{\mu'}
\end{aligned}
$$



Pullback of a vector V

Let $V \in T_Q N$, then define

$\phi^* V \in T_P M$



$$\Rightarrow \phi^* V(g) = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \partial_\mu g \quad \forall g \in \mathcal{F}(M)$$

$$\Rightarrow \phi^* V = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}}_{V^\mu} \partial_\mu$$

$$\phi^* V(g) = \frac{\partial g(x^\mu)}{\partial x^\mu} \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{d x^{\mu'}}{dt} = \partial_\mu g \cdot \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot V^{\mu'}$$

Pullback of a vector V

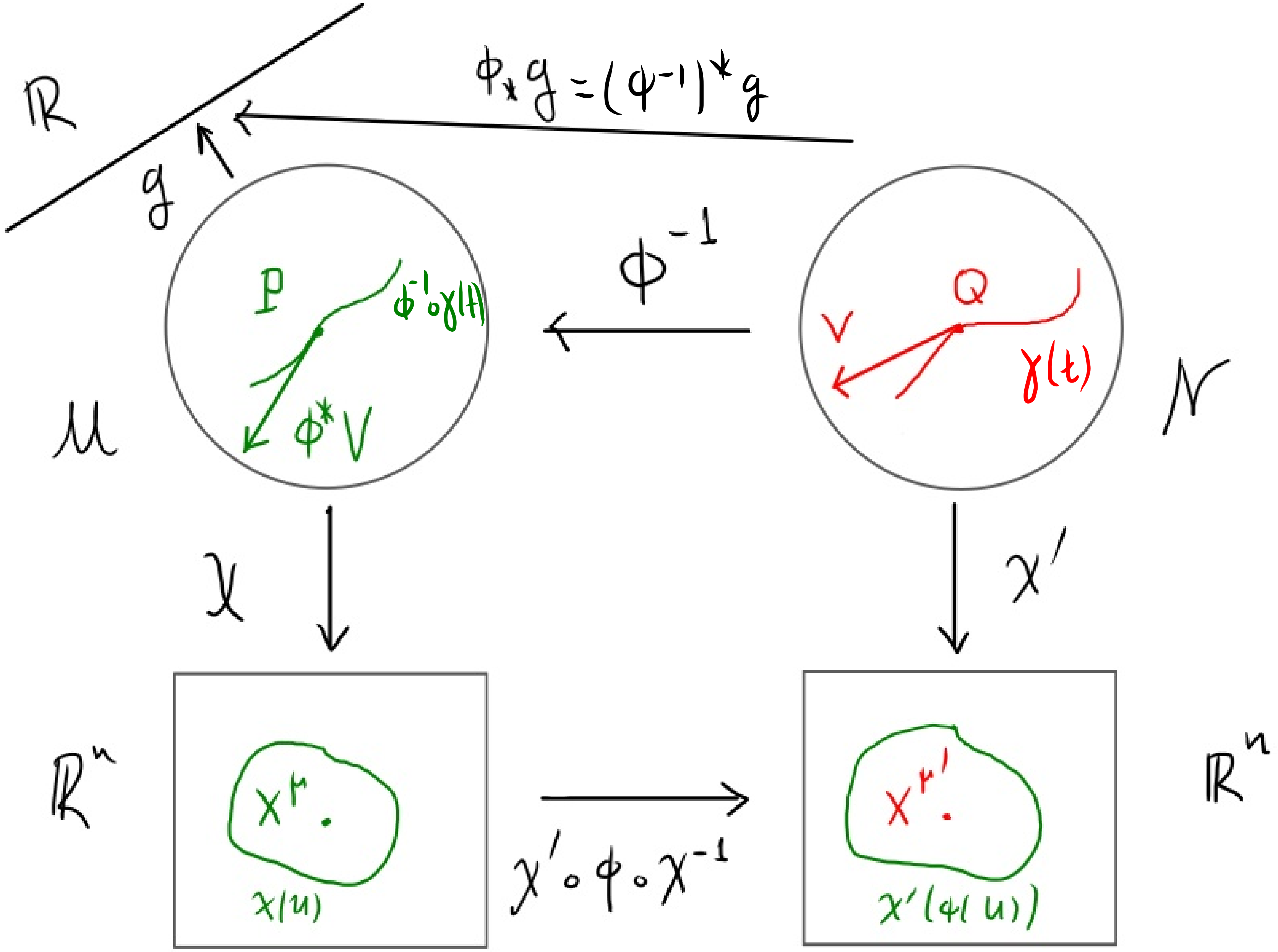
Let $V \in T_Q N$, then define

$\phi^* V \in T_P M$

$\Rightarrow \phi^* V(g) = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'} \partial_\mu g \quad \forall g \in \mathcal{F}(M)$

$\Rightarrow \phi^* V = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}}_{V^\mu} \partial_\mu$

$\Rightarrow (\phi^* V)^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}$



- like a coordinate xfm!
- active xfm vs pasive xfm

Pullback of 1-form

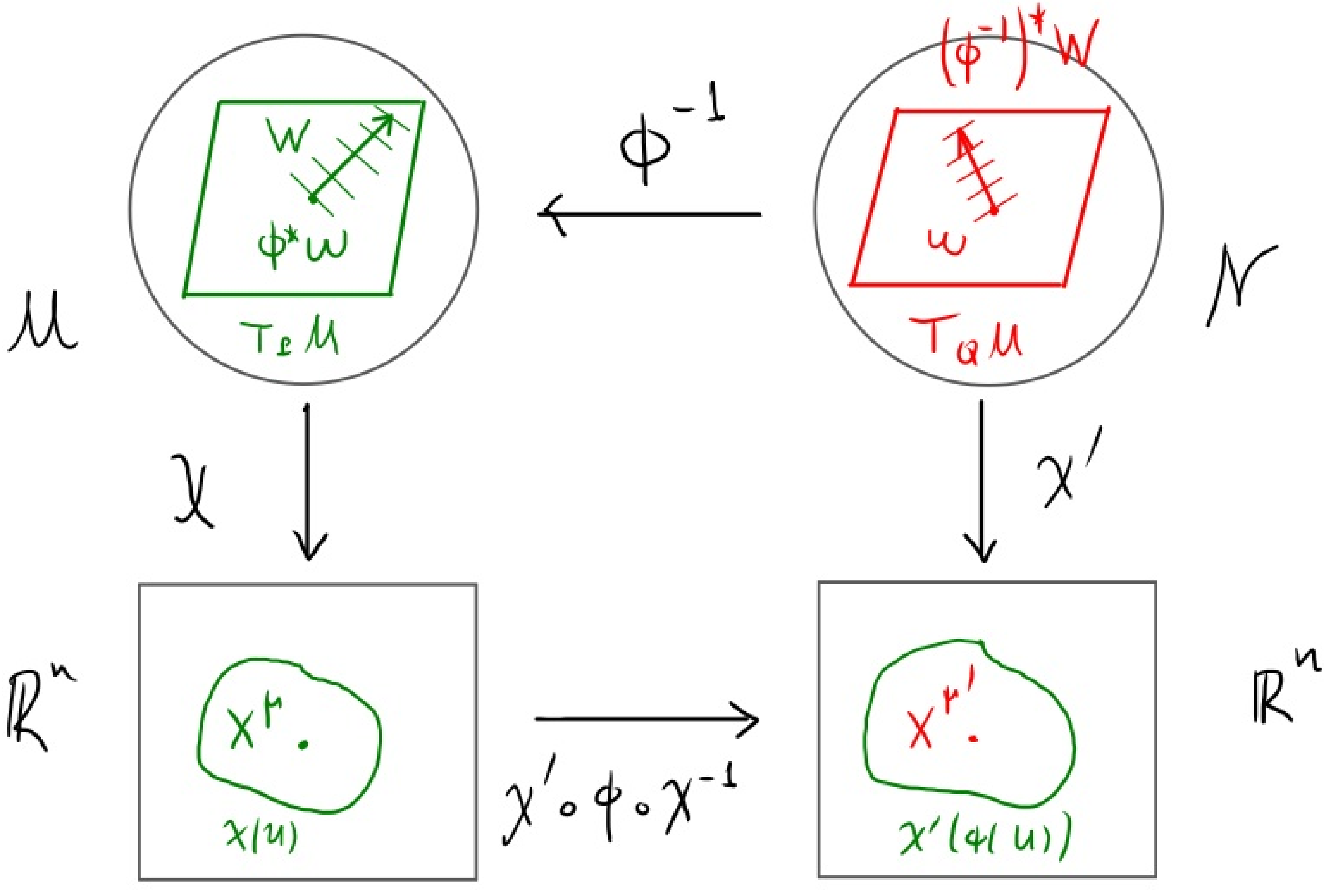
Let $\omega \in T^*_Q N$, then define

$\phi^*\omega \in T^*_P M$ by

$$\phi^*\omega(W) = \omega\left((\phi^{-1})^* W\right)$$

any vector in $T_P M$

ω is a known one form, we know how it acts on $(\phi^{-1})^* W$



Pullback of 1-form

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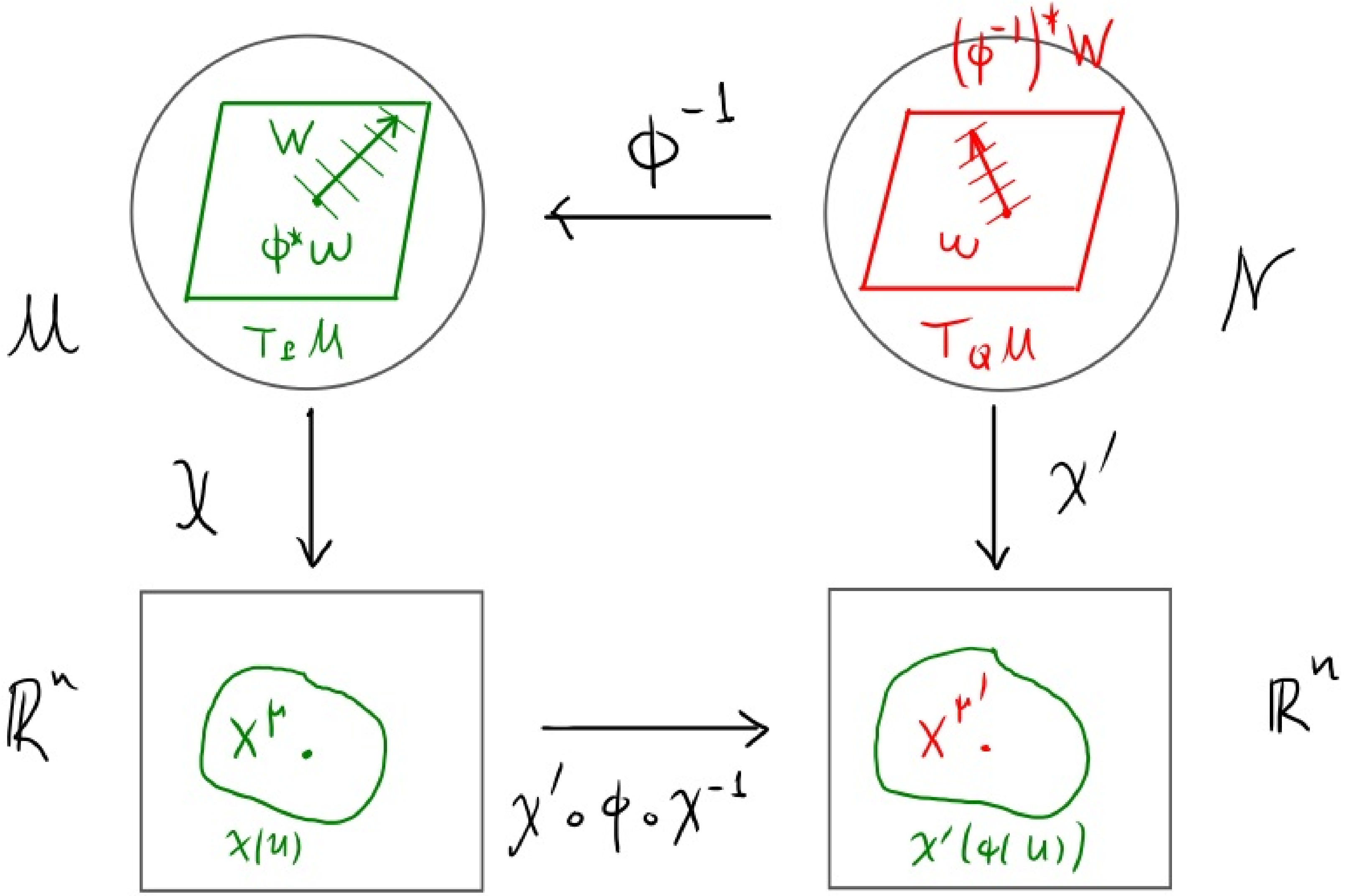

If $W = W^\mu \partial_\mu \Rightarrow (\phi^{-1})^* W = [(\phi^{-1})^* W]^{\mu'} \partial_{\mu'}$

and $[(\phi^{-1})^* W]^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} W^\mu$

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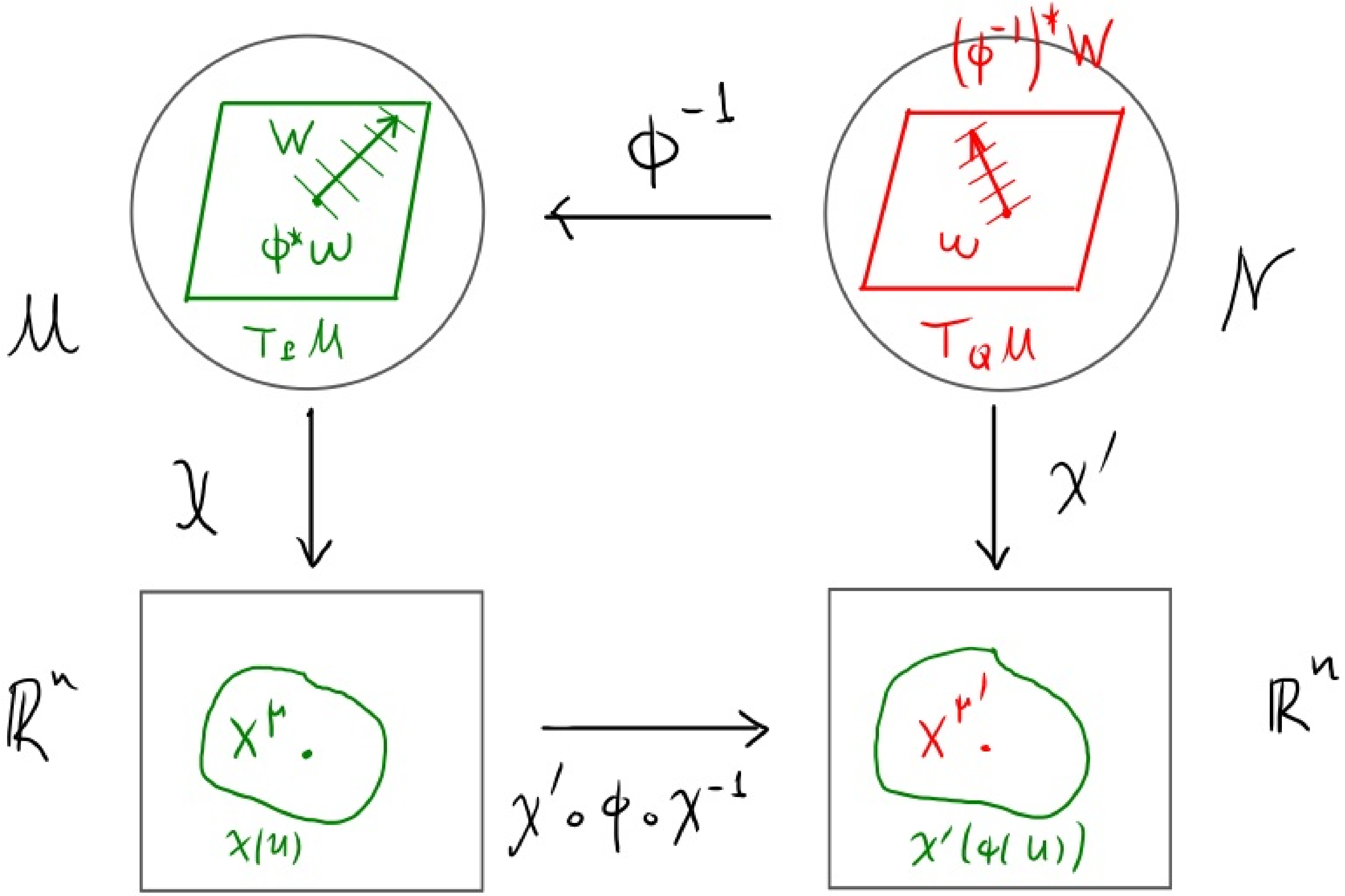

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$$\Rightarrow \omega((\phi^{-1})^* W) = \omega_{\mu'} [(\phi^{-1})^* W]^{\mu'}$$

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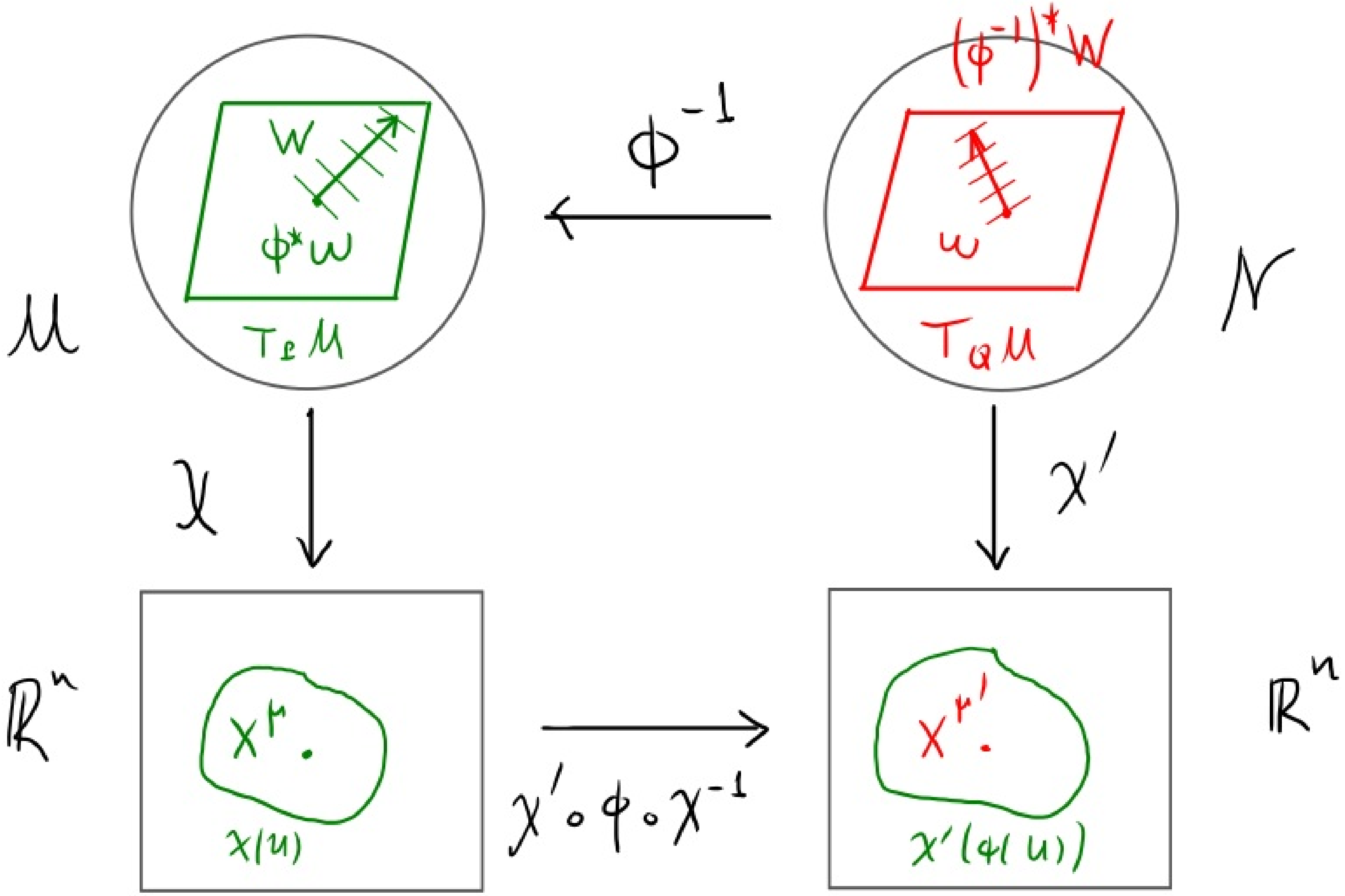

If $W = W^\mu \partial_\mu \Rightarrow (\phi^{-1})^* W = \left[(\phi^{-1})^* W\right]^{\mu'} \partial_{\mu'}$

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$$\Rightarrow \omega\left((\phi^{-1})^* W\right) = \omega_{\mu'} \left[(\phi^{-1})^* W\right]^{\mu'} = \omega_{\mu'} \left(\frac{\partial x^{\mu'}}{\partial x^\mu} W^\mu \right) = \left(\omega_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu} \right) W^\mu$$

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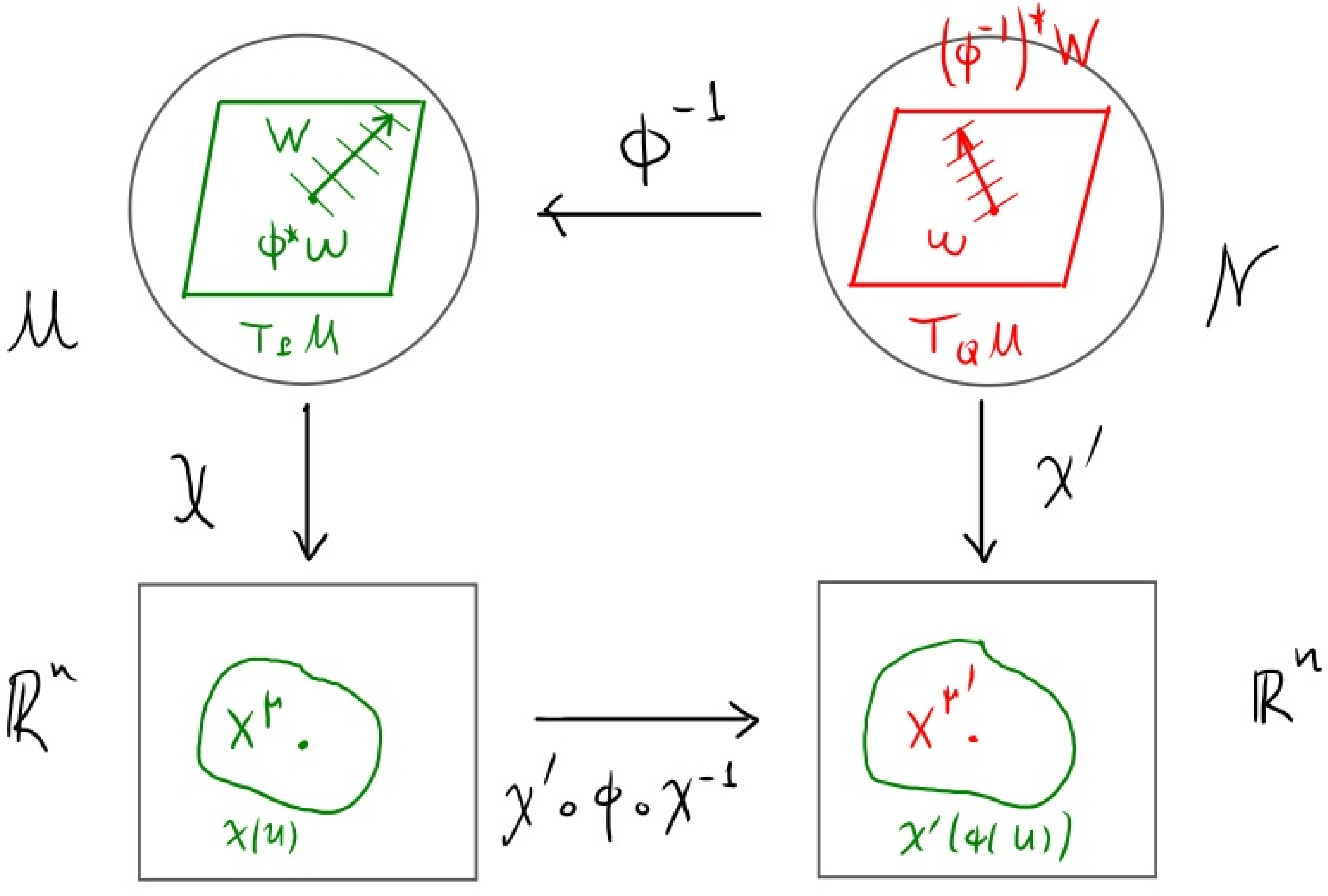

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$$\phi^* \omega (W) = (\phi^* \omega)_\mu W^\mu$$

Pullback of 1-form

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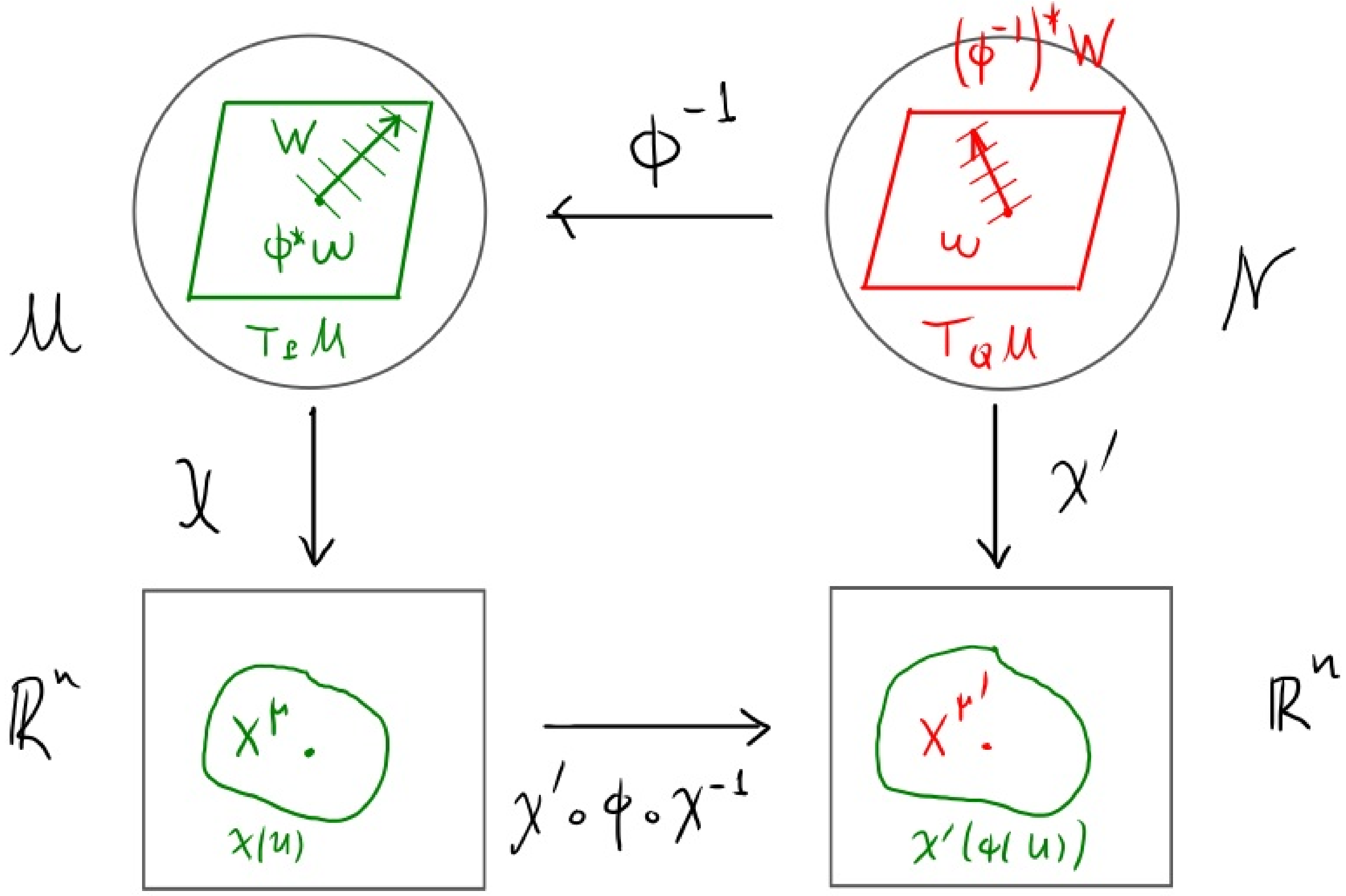


$$\Rightarrow (\phi^*\omega)_\mu = \omega_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu}$$

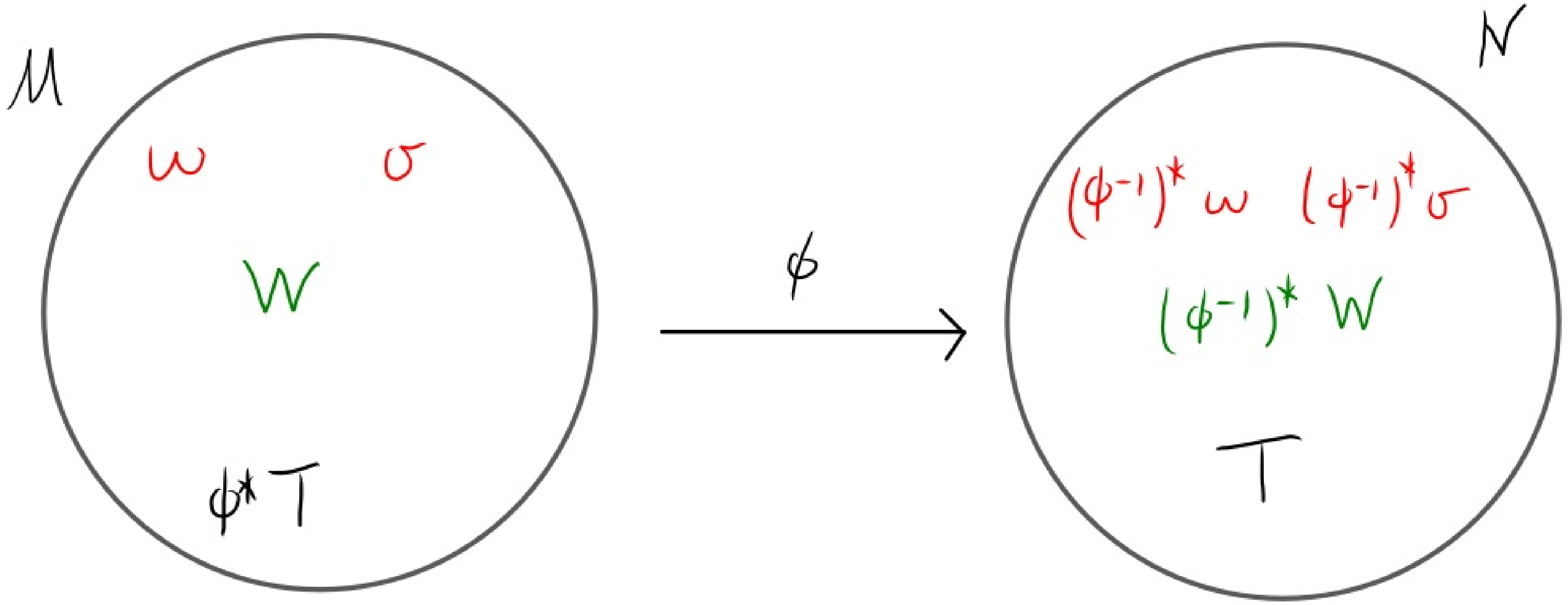
$$\Rightarrow \omega\left((\phi^{-1})^* W\right) = \omega_{\mu'}\left[(\phi^{-1})^* W\right]^{\mu'} = \omega_{\mu'}\left(\frac{\partial x^{\mu'}}{\partial x^\mu} W^\mu\right) = \left(\omega_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\mu}\right) W^\mu$$

$$\phi^*\omega(W) = (\phi^*\omega)_\mu W^\mu$$

Pullback of (2,1) tensor

Let $T \in T^{(2,1)}_Q N$, then define

$\phi^* T \in T^{(2,1)}_P M$ by

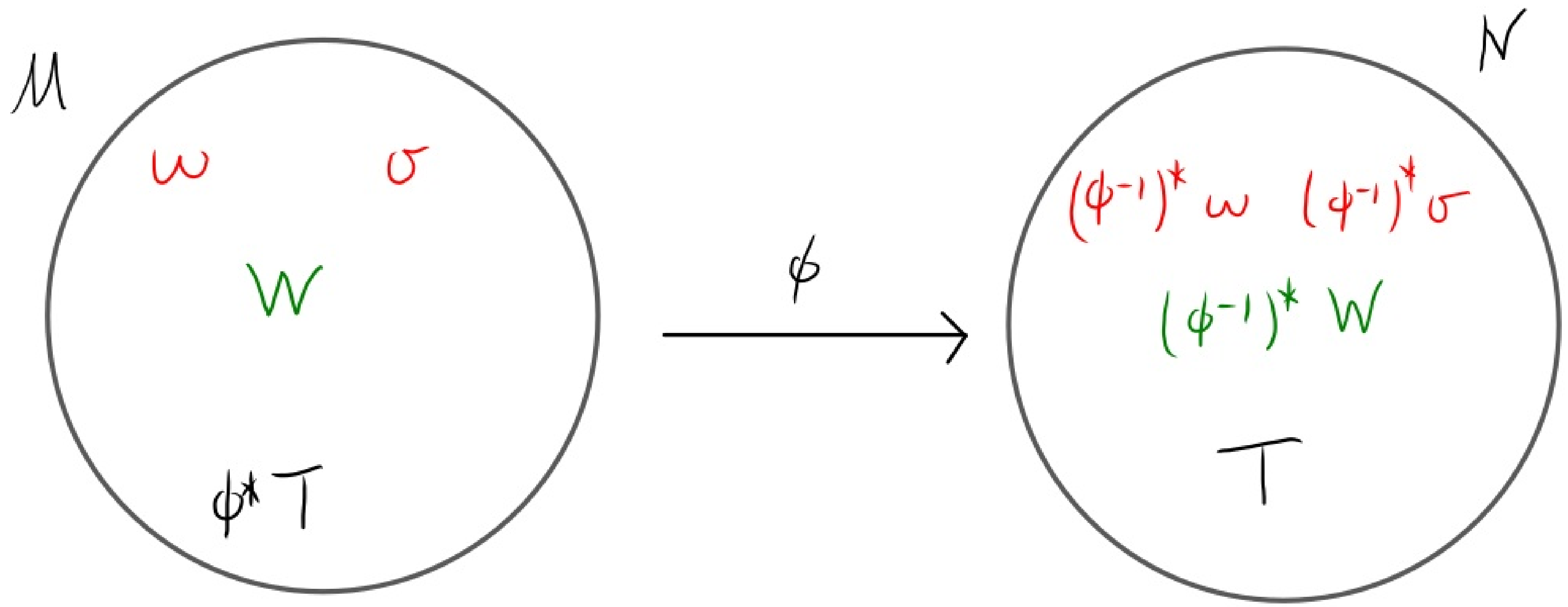


$$\phi^* T(\omega, \sigma; W) = T\left((\phi^{-1})^* \omega, (\phi^{-1})^* \sigma ; (\phi^{-1})^* W\right)$$

Pullback of (2,1) tensor

Let $T \in T_Q^{(2,1)} N$, then define

$\phi^* T \in T_P^{(2,1)} M$ by



$$\phi^* T(\omega, \sigma; W) = T\left((\phi^{-1})^* \omega, (\phi^{-1})^* \sigma ; (\phi^{-1})^* W\right)$$

$$= T^{\mu'\nu'}{}_{\lambda'} \left[(\phi^{-1})^* \omega\right]_{\mu'} \left[(\phi^{-1})^* \sigma\right]_{\nu'} \left[(\phi^{-1})^* W\right]^{\lambda'}$$

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Let $T \in T^{(2,1)}_Q N$, then define

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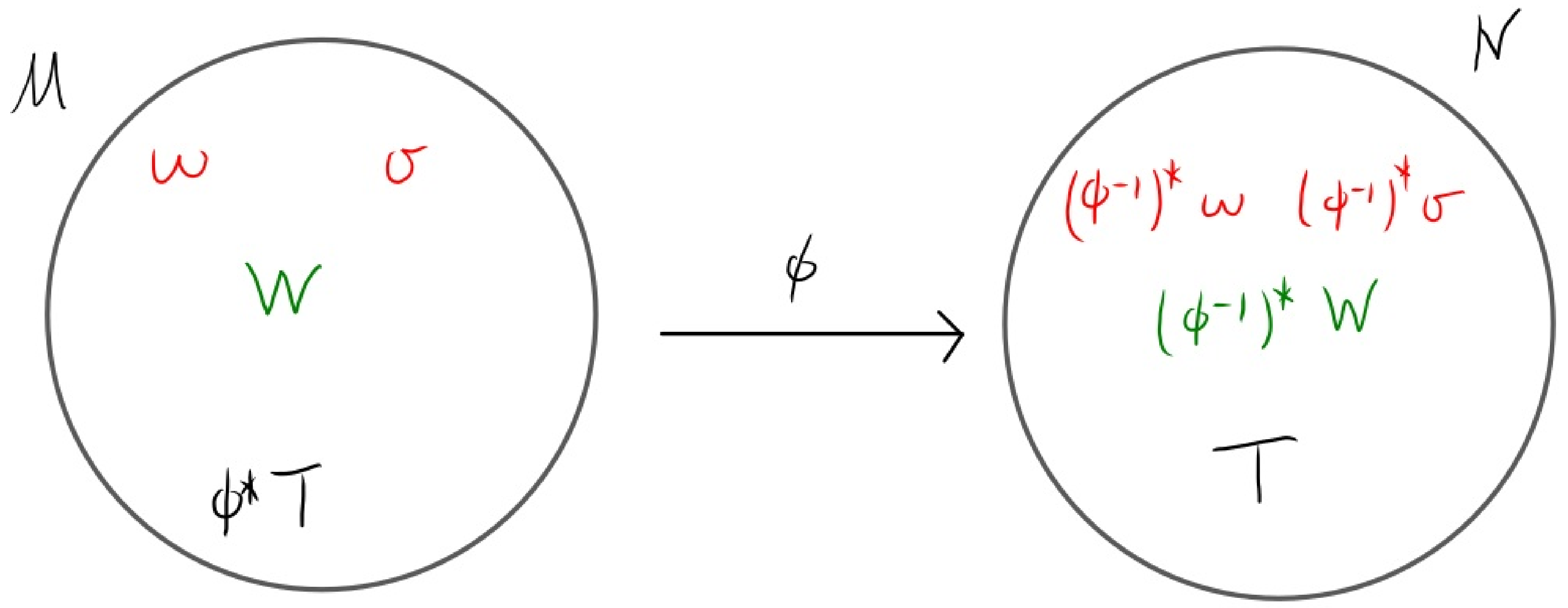

$$\phi^* T(\omega, \sigma; W) = T\left((\phi^{-1})^*\omega, (\phi^{-1})^*\sigma; (\phi^{-1})^* W\right)$$

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$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} T^{\mu'\nu'}{}_{\lambda'} \omega_{\mu} \sigma_{\nu} W^{\lambda}$$

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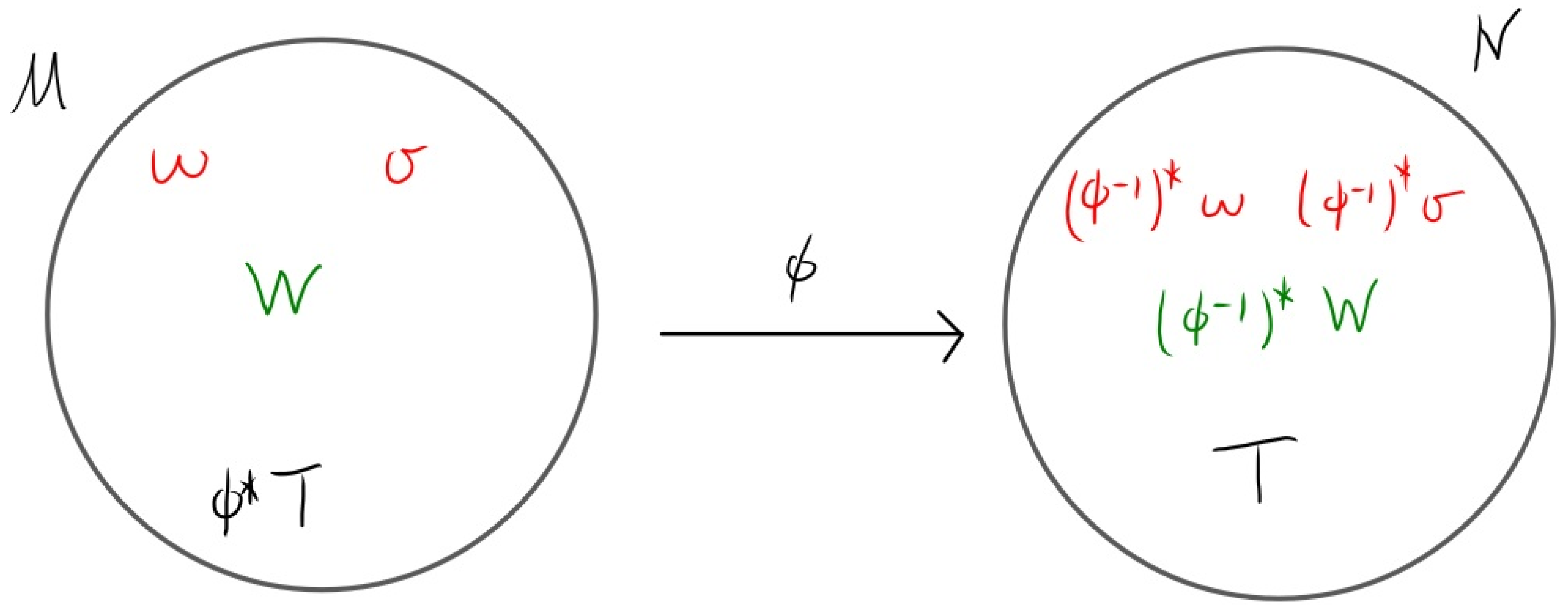


$$\phi^* T(\omega, \sigma; W) = T\left((\phi^{-1})^* \omega, (\phi^{-1})^* \sigma ; (\phi^{-1})^* W\right)$$

$$= T^{\mu'\nu'}{}_{\lambda'} \left[(\phi^{-1})^* \omega\right]_{\mu'} \left[(\phi^{-1})^* \sigma\right]_{\nu'} \left[(\phi^{-1})^* W\right]^{\lambda'}$$

$$= \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} T^{\mu'\nu'}{}_{\lambda'} \right) \omega_{\mu} \sigma_{\nu} W^{\lambda}$$

$$\phi^* T^{\mu\nu}{}_{\lambda}$$

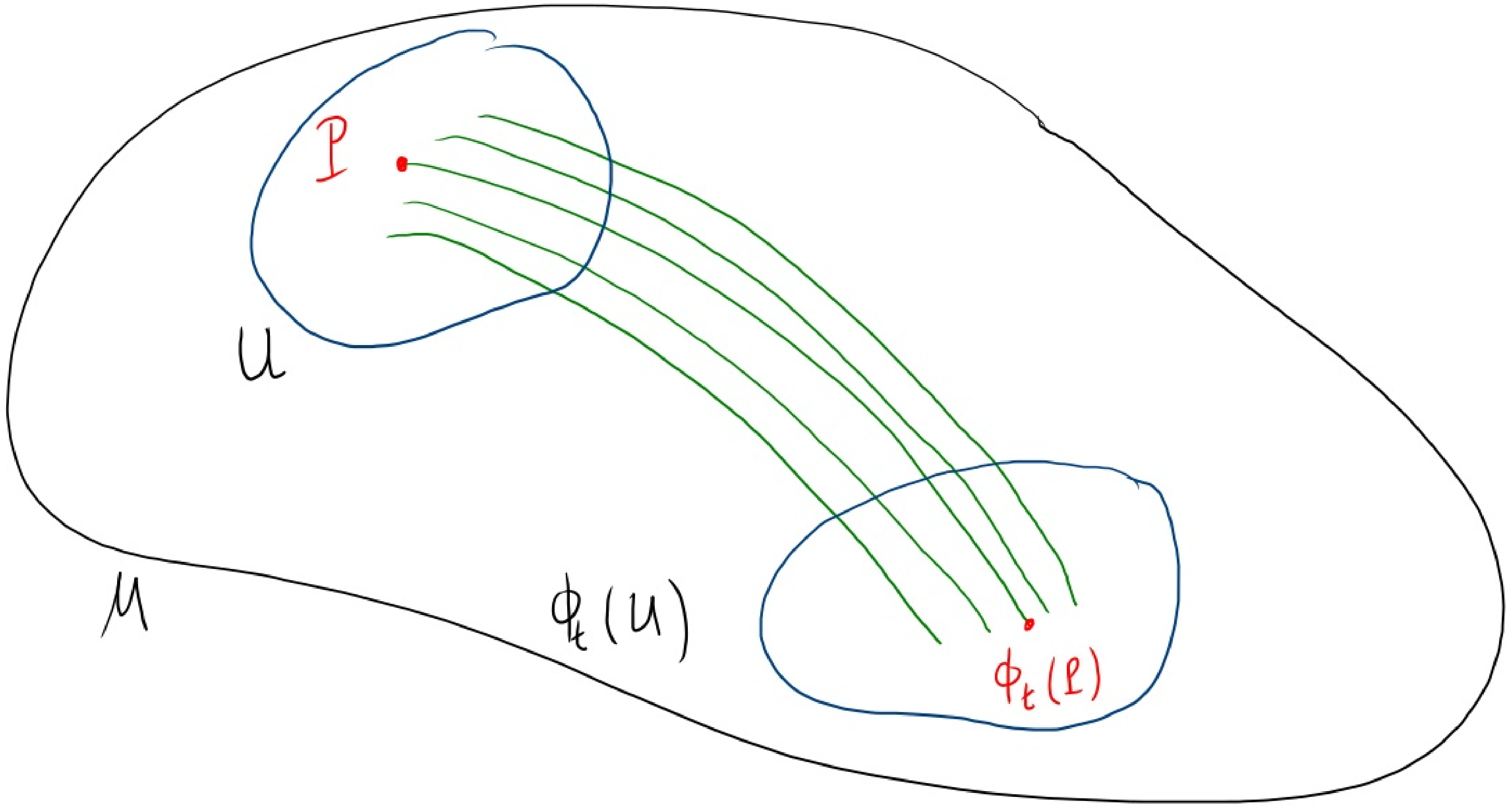
One parameter family of diffeomorphisms

$\phi_t : M \to M$, $t \in \mathbb{R}$, s.t.

$\phi_s \circ \phi_t = \phi_{s+t}$

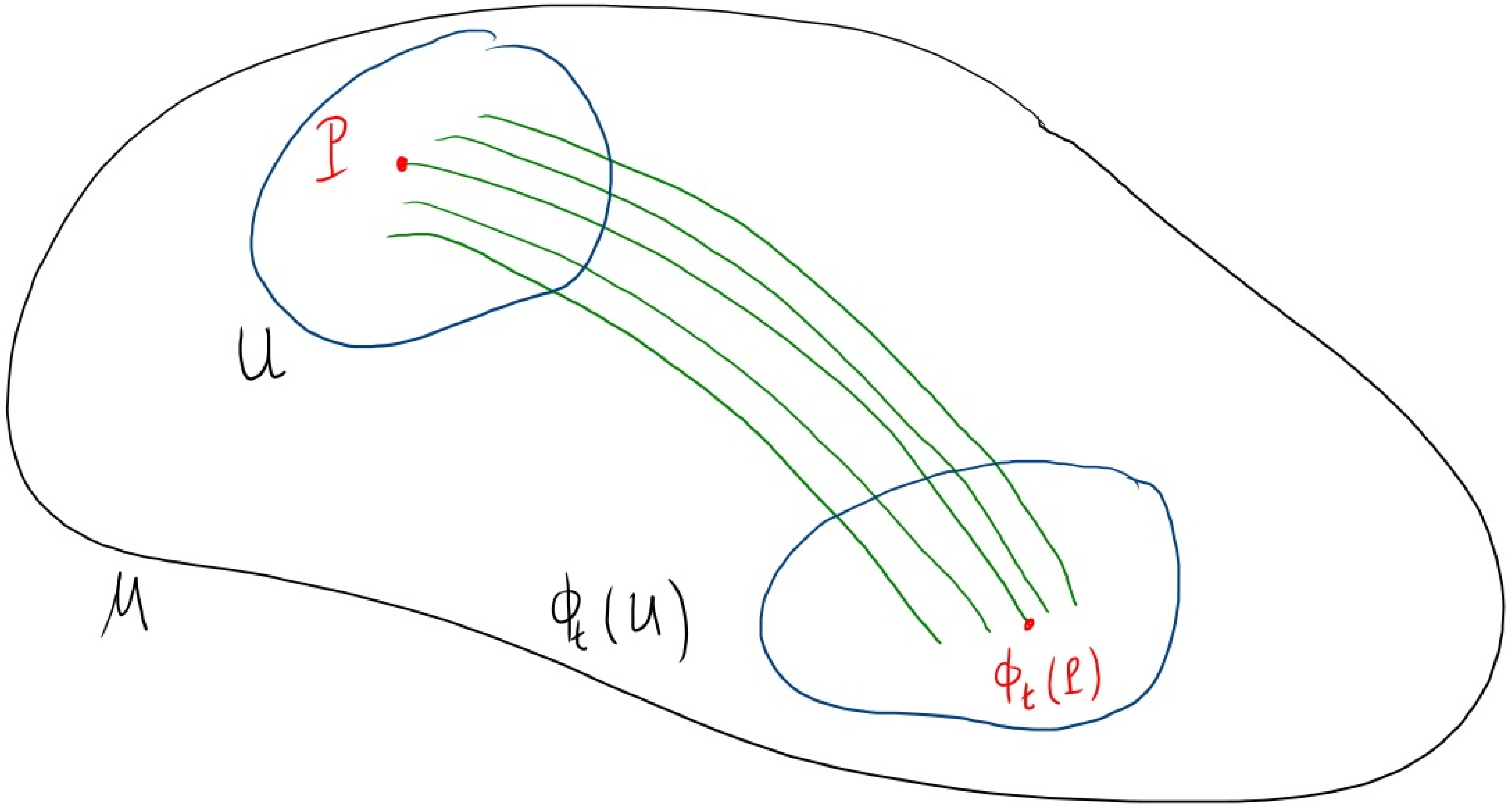
$\phi_0 = 1$

$\phi_t^{-1} = \phi_{-t}$



Differentiability: $\exists \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[f \circ \phi_{t+\epsilon}(P) - f \circ \phi_t(P) \right] \quad \forall f \in \mathcal{F}(M)$

One parameter family of diffeomorphisms



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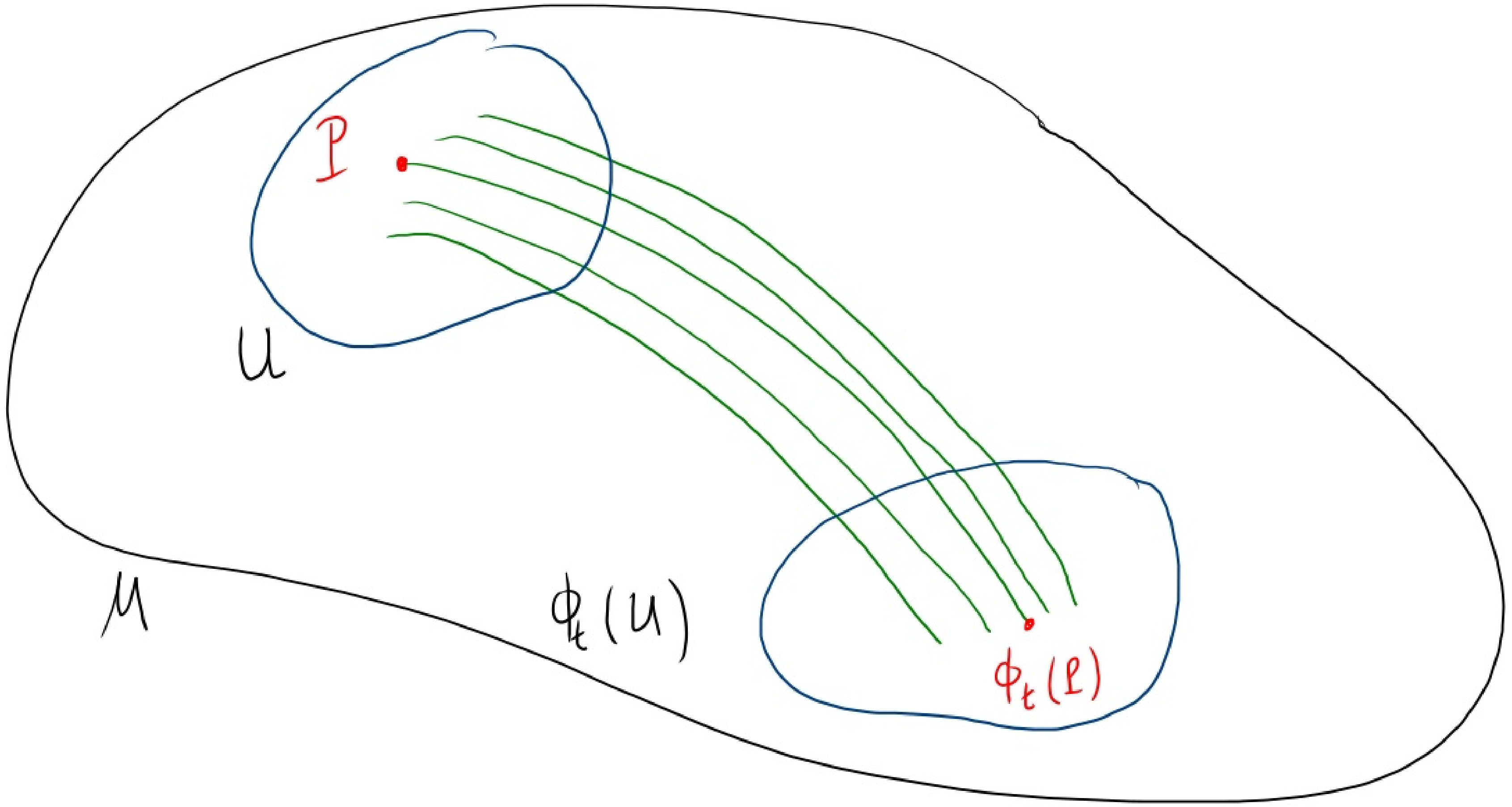
* $\gamma(t) = \phi_t(P)$ a smooth curve: the orbit of P under ϕ_t

One parameter family of diffeomorphisms



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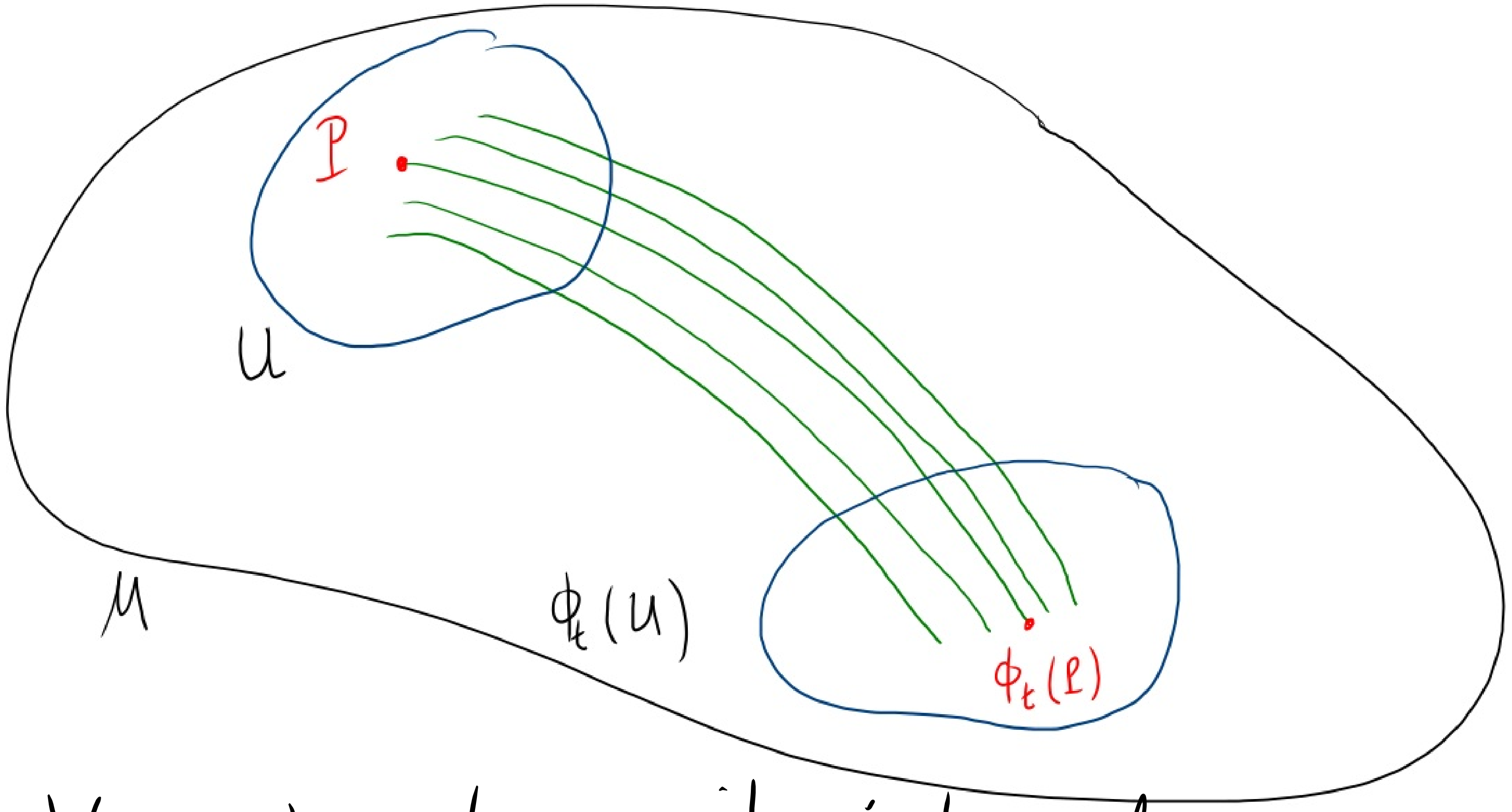
One parameter family of diffeomorphisms



* do this $\forall P \in M$
  - orbits fill M
  - they never intersect
  - define a smooth vector field V with $\phi_t(\ldots)$ its integral curves

$\Rightarrow$ a congruence of curves

---

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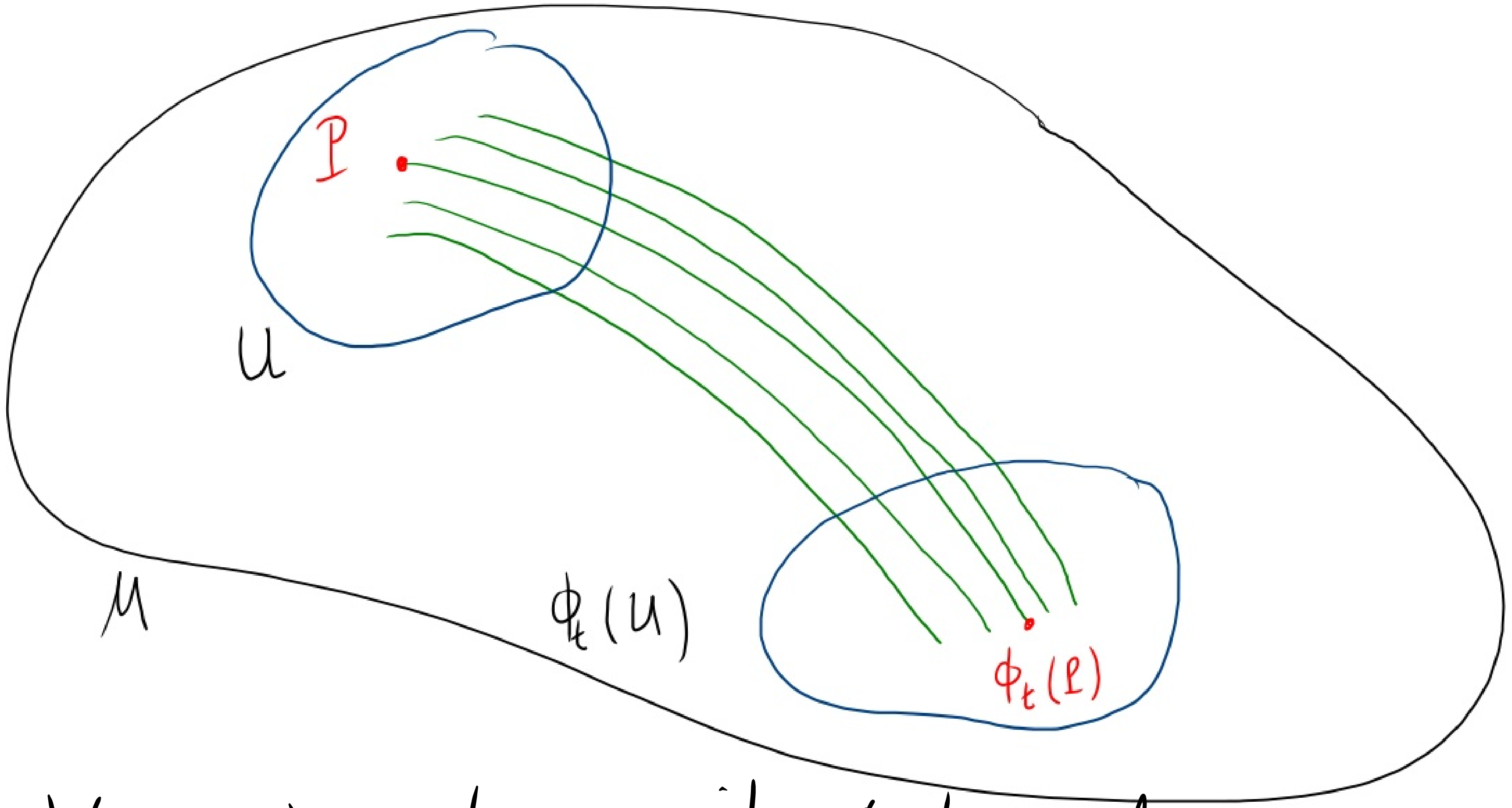
One parameter family of diffeomorphisms



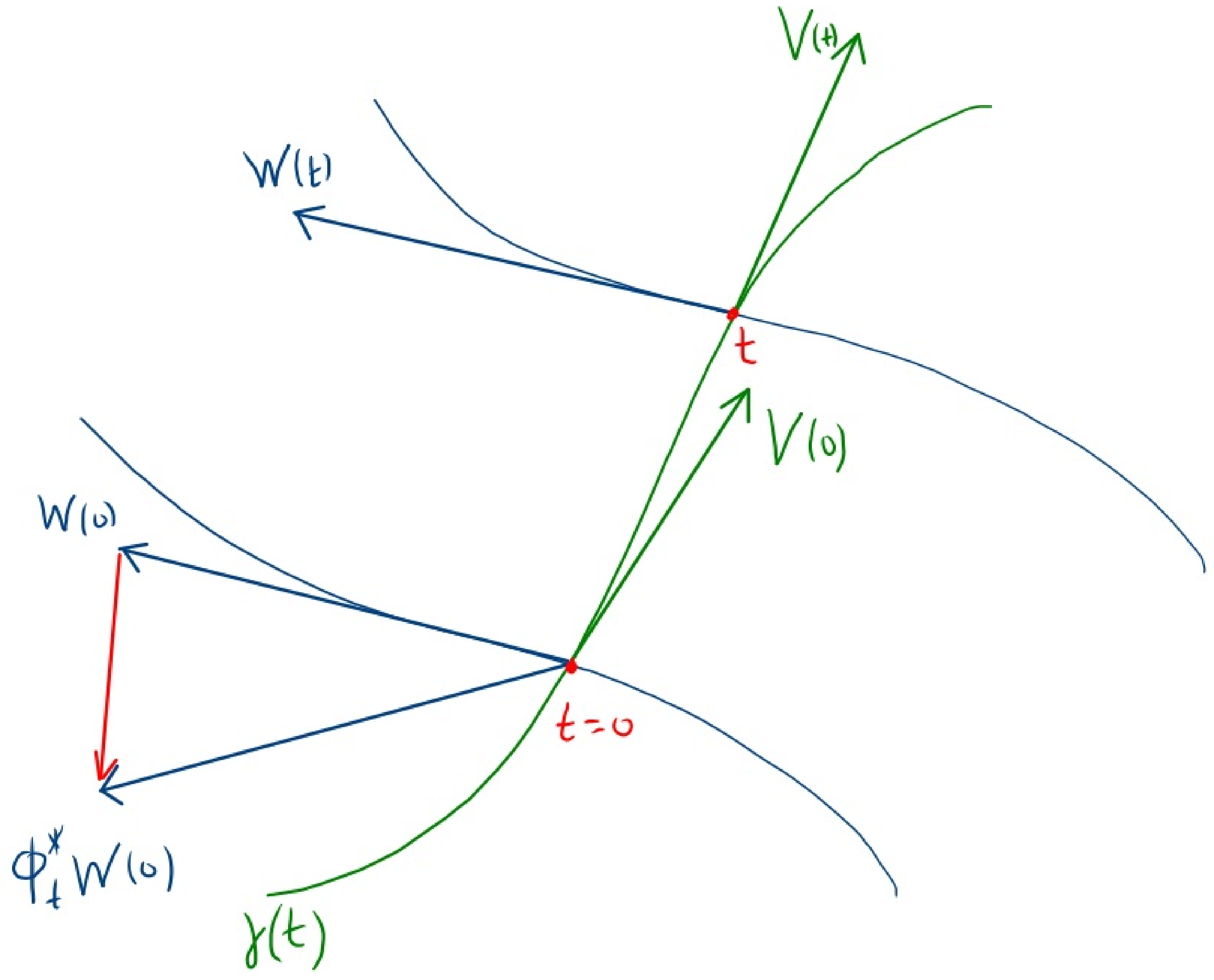
* do this $\forall P \in M$
  - orbits fill M
  - they never intersect
  - define a smooth vector field V with $\phi_t(\dots)$ its integral curves

$\Rightarrow$ a congruence of curves

* non vanishing vector fields define a ϕ_t: simply move a $P \in M$ along the integral curve that passes through $\underline{P}$ by t

Lie Derivatives

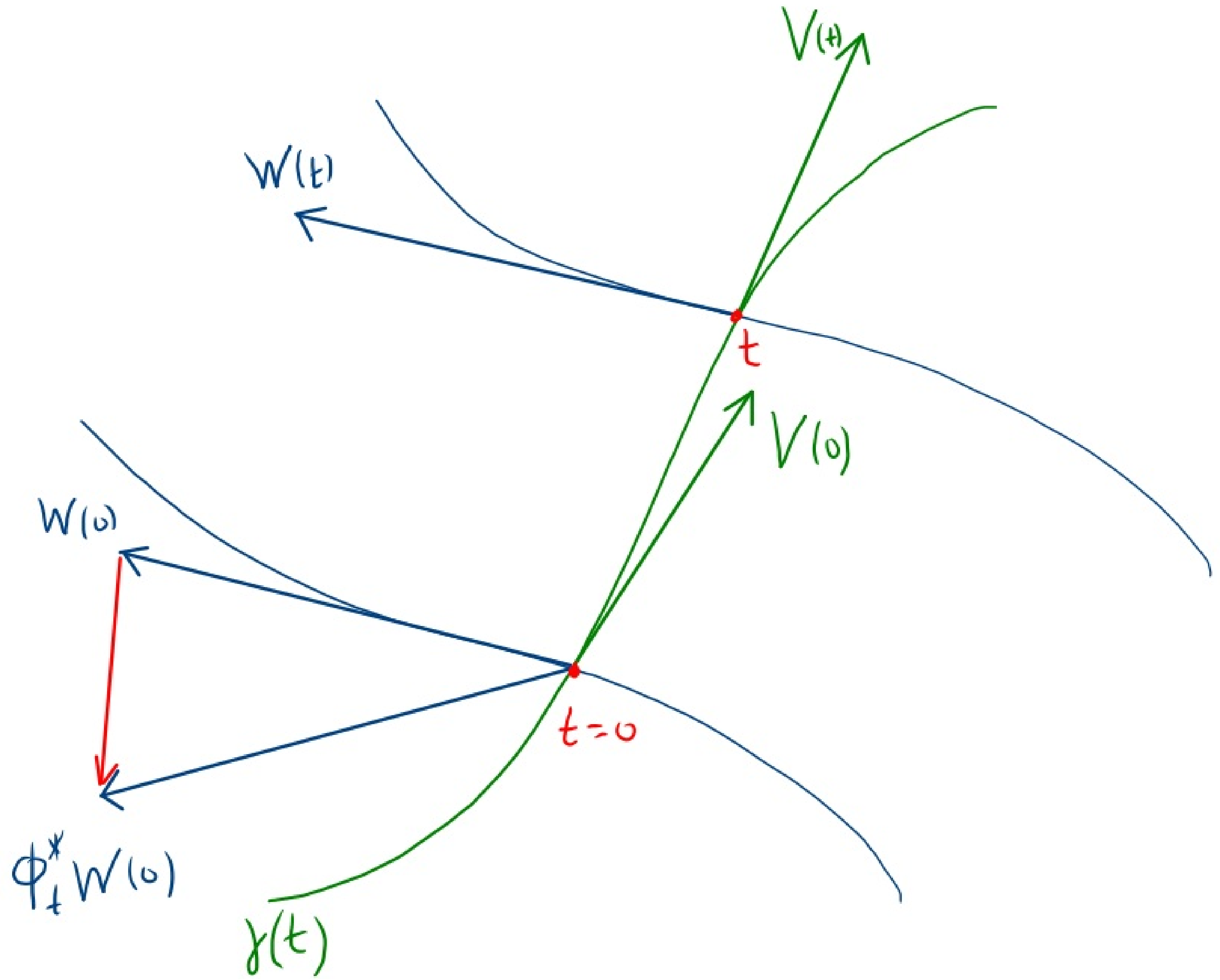
Consider two v. fields V + W; let $\gamma(t)$ an integral curve of V and the point $\gamma(0)$



Lie Derivatives

Consider two v. fields V + W; let $\gamma(t)$ an integral curve of V and the point $\gamma(0)$

- Pullback $W(t)$, the value of W @ $\gamma(t)$ to $\gamma(0)$: $\phi_t^* W(0)$

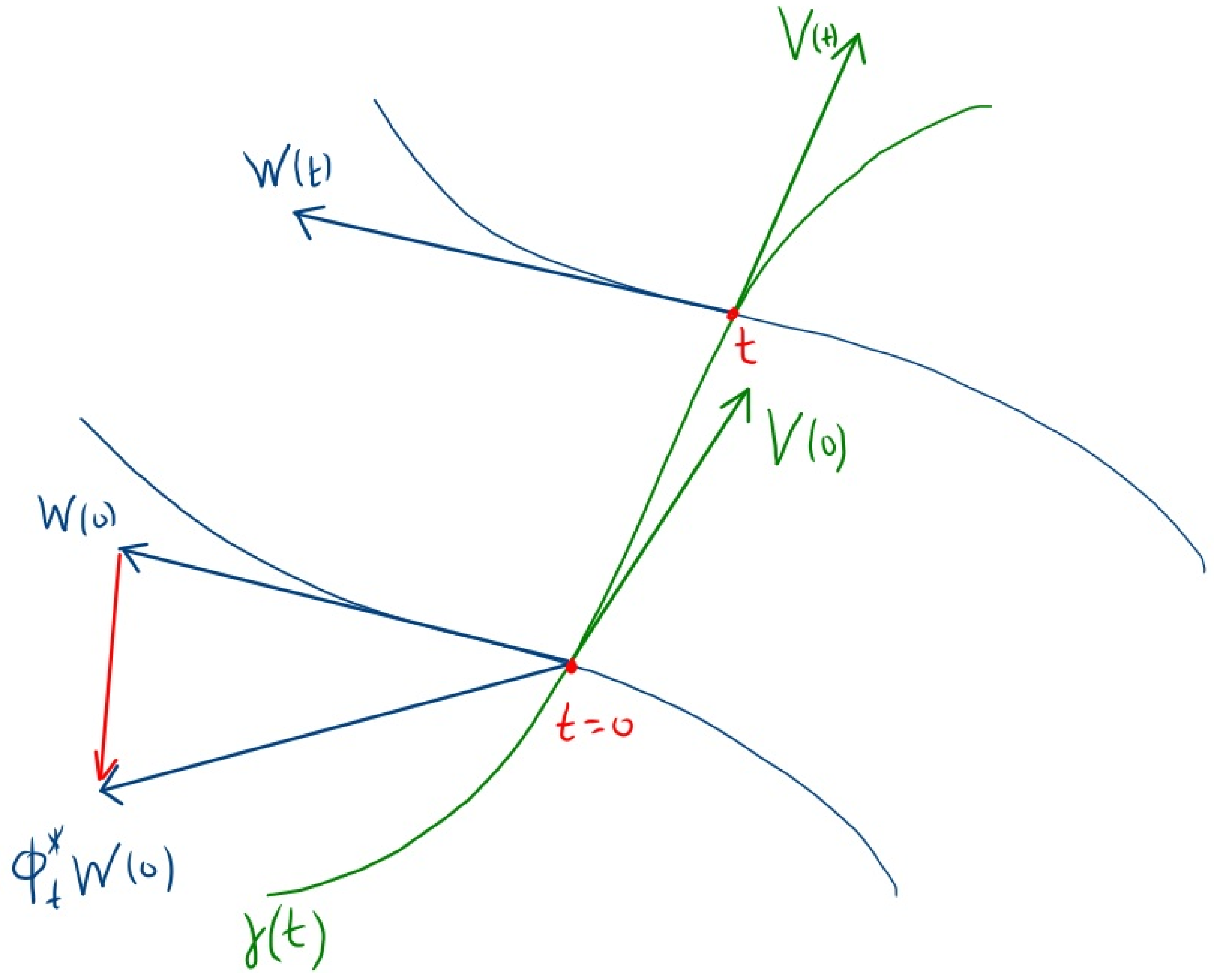


Lie Derivatives

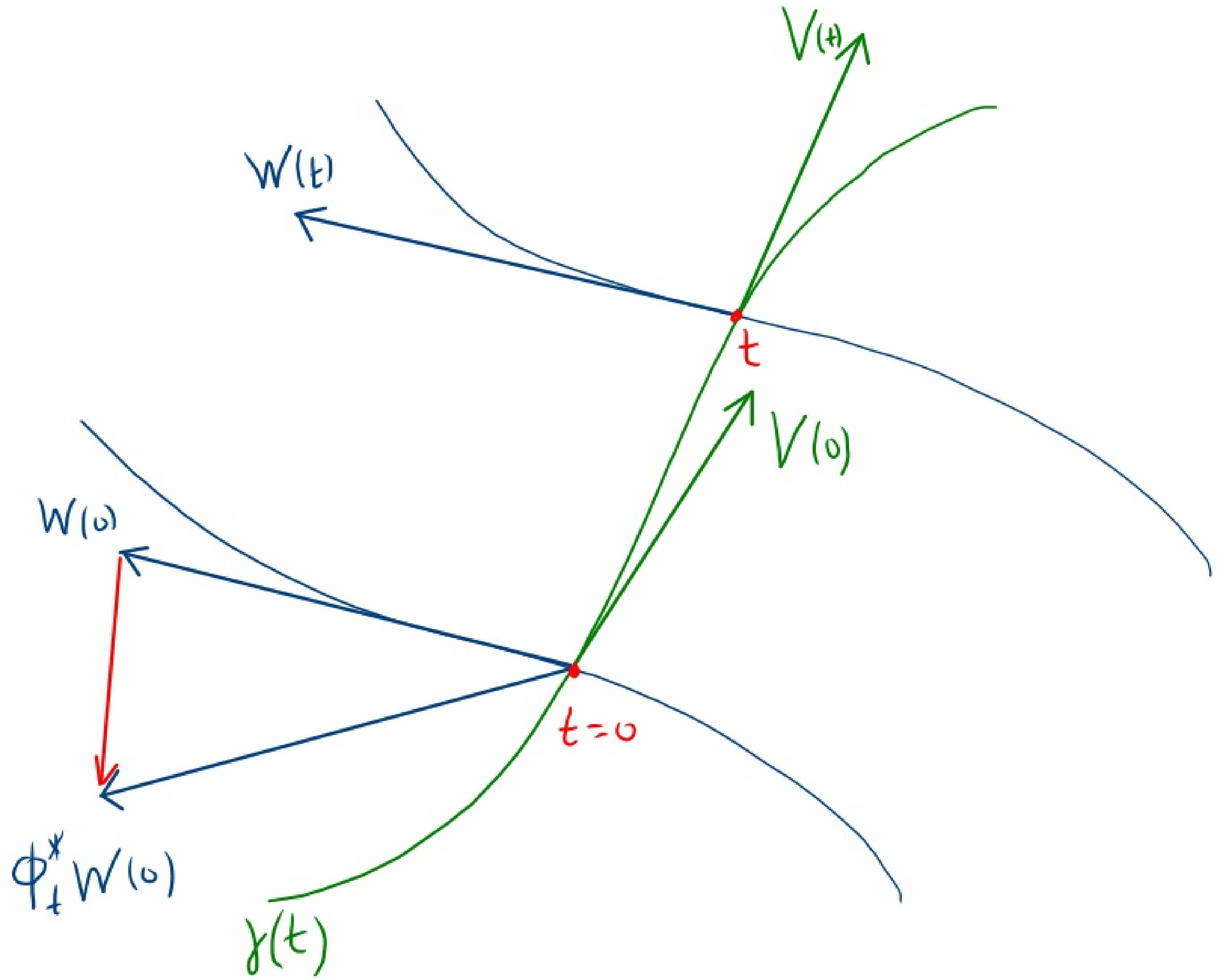
Consider two v. fields V + W; let $\gamma(t)$ an integral curve of V and the point $\gamma(0)$

- Pullback $W(t)$, the value of W @ $\gamma(t)$ to $\gamma(0)$: $\phi_t^* W(0)$
- $\phi_t^* W(0)$ a vector at $T_{\gamma(0)} M$, compute

$$\phi_t^* W(0) - W(0)$$



Lie Derivatives



Consider two v. fields $V + W$; let $\gamma(t)$ an integral curve of V and the point $\gamma(0)$

- Pullback $W(t)$, the value of W @ $\gamma(t)$ to $\gamma(0)$: $\phi_t^* W(0)$
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$$\lim_{t \to 0} \frac{1}{t} \left[\phi_t^* W(0) - W(0) \right]$$

Lie Derivatives

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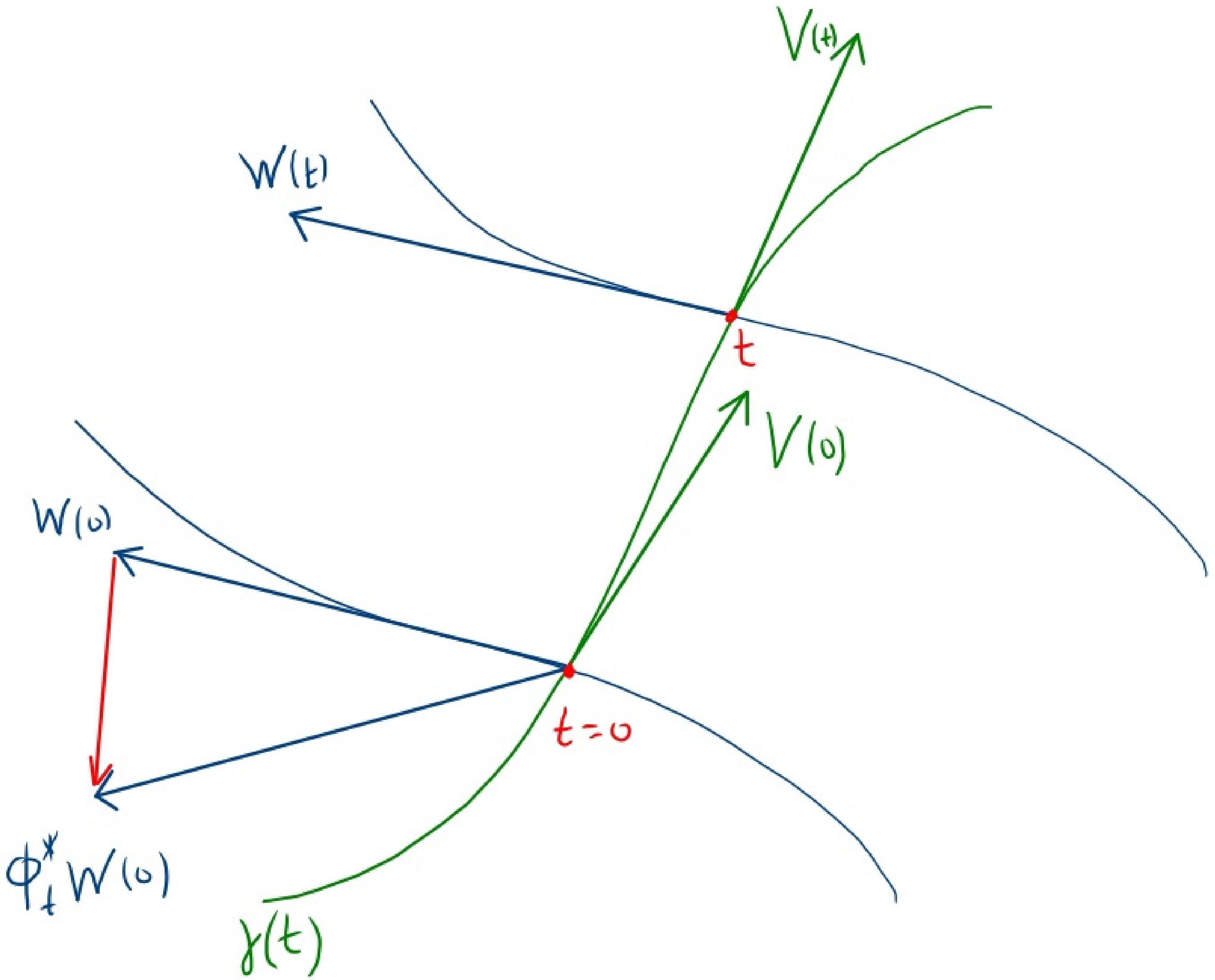

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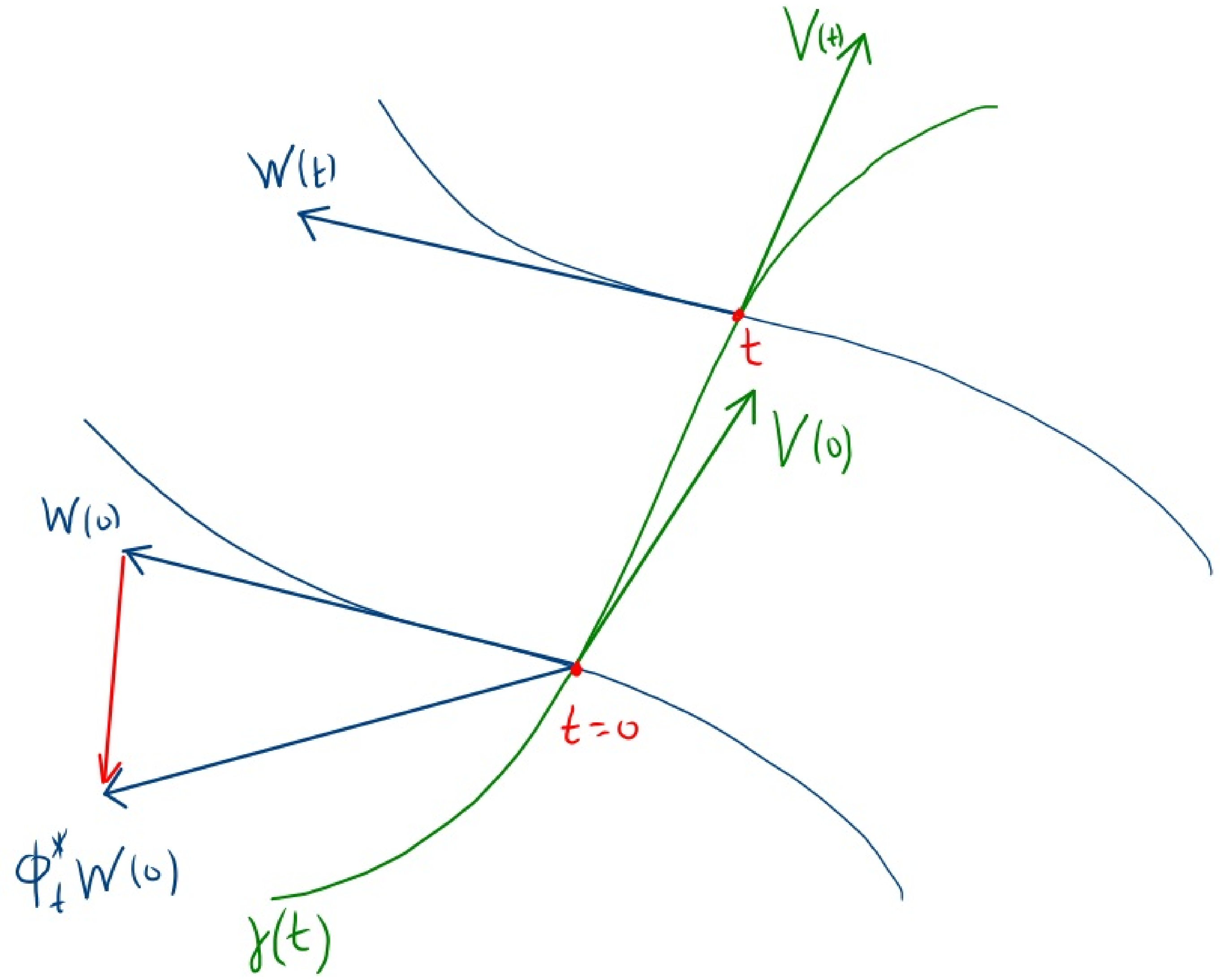
Lie Derivatives



Consider two v. fields V + W; let $\gamma(t)$ an integral curve of V and the point $\gamma(0)$

- Pullback $W(t)$, the value of W @ $\gamma(t)$ to $\gamma(0)$: $\phi_t^* W(0)$
- $\phi_t^* W(0)$ a vector at $T_{\gamma(0)}\mathcal{M}$, compute, and take

$$\mathcal{L}_V W(0) = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* W(0) - W(0) \right]$$

- Do that for any tensor field

$$\mathcal{L}_V T(0) = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* T(0) - T(0) \right]$$

$\mathcal{L}_v$ is a derivative operator:

$$\mathcal{L}_v(\alpha T + \beta S) = \alpha \mathcal{L}_v T + \beta \mathcal{L}_v S$$

$$\mathcal{L}_v(T \otimes S) = \mathcal{L}_v T \otimes S + T \otimes \mathcal{L}_v S \qquad \text{Leibniz (1)}$$

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$$\mathcal{L}_V f = V(f) = \frac{df}{dt} = V^\mu \partial_\mu f$$

$$\mathcal{L}_V(fW) = \mathcal{L}_V f \cdot W + f \mathcal{L}_V W$$ Leibniz (2)

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<u>Contractions:</u>

$$\mathcal{L}_V[\underbrace{T(\omega,\ldots;W,\ldots)}_{\text{a function}}] = \mathcal{L}_V T(\omega,\ldots;W,\ldots) + T(\mathcal{L}_V\omega,\ldots;W,\ldots) + \ldots + T(\omega,\ldots;\mathcal{L}_V W,\ldots) + \ldots$$

Leibniz (3)

$V^\mu \partial_\mu T(\omega,\ldots;W,\ldots)$

$\mathcal{L}_V$ is a derivative operator:

$$\mathcal{L}_V(\alpha T + \beta S) = \alpha \mathcal{L}_V T + \beta \mathcal{L}_V S$$

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Contractions:

$$\mathcal{L}_V[T(\omega, \ldots; W, \ldots)] = \mathcal{L}_V T(\omega, \ldots; W, \ldots) + T(\mathcal{L}_V \omega, \ldots; W, \ldots) + \ldots + T(\omega, \ldots; \mathcal{L}_V W, \ldots) + \ldots$$

e.g. $\mathcal{L}_V[g(U,W)] = \mathcal{L}_V g(U,W) + g(\mathcal{L}_V U, W) + g(U, \mathcal{L}_V W)$ Leibniz (3)

Other (useful) properties:

$$\mathcal{L}_V W = [V, W] = -\mathcal{L}_W V$$

$\hookrightarrow$ Lie Bracket: $[V, W](f) = (VW - WV)f = V(W(f)) - W(V(f))$

"commutator"

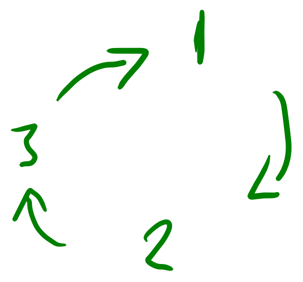
$\hookrightarrow$ a vector field!

Other (useful) properties:

$$\mathcal{L}_V W = [V, W] = -\mathcal{L}_W V$$

$$\mathcal{L}_{[V,W]} = [\mathcal{L}_V, \mathcal{L}_W]$$

$$[[\mathcal{L}_{V_1}, \mathcal{L}_{V_2}], \mathcal{L}_{V_3}] + [[\mathcal{L}_{V_3}, \mathcal{L}_{V_1}], \mathcal{L}_{V_2}] + [[\mathcal{L}_{V_2}, \mathcal{L}_{V_3}], \mathcal{L}_{V_1}] = 0$$



let's prove that $\mathcal{L}_v(\omega \otimes \chi) = \mathcal{L}_v \omega \otimes \chi + \omega \otimes \mathcal{L}_v \chi$ for 1-form fields ω, χ

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First show that

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$$\phi^*\omega\otimes\phi^*\chi(V,W) = \phi^*\omega(V)\cdot\phi^*\chi(W) = \omega((\phi^{-1})^*V)\cdot\chi((\phi^{-1})^*W)$$

Let's prove that $\mathcal{L}_V(\omega \otimes \chi) = \mathcal{L}_V \omega \otimes \chi + \omega \otimes \mathcal{L}_V \chi$ for 1-form fields ω, χ

First show that

$$\phi^*(\omega \otimes \chi) = \phi^* \omega \otimes \phi^* \chi$$

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$$\phi^* \omega \otimes \phi^* \chi (V, W) = \phi^* \omega(V) \cdot \phi^* \chi(W) = \omega((\phi^{-1})^* V) \cdot \chi((\phi^{-1})^* W)$$

$$\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi = \phi_t^* \omega \otimes \phi_t^* \chi - \omega \otimes \chi$$

Let's prove that $\mathcal{L}_V(\omega \otimes \chi) = \mathcal{L}_V \omega \otimes \chi + \omega \otimes \mathcal{L}_V \chi$ for 1-form fields ω, χ

First show that

$\phi^*(\omega \otimes \chi) = \phi^* \omega \otimes \phi^* \chi$

$\phi^*(\omega \otimes \chi)(V, W) = \omega \otimes \chi\Big((\phi^{-1})^* V, (\phi^{-1})^* W\Big) = \omega\big((\phi^{-1})^* V\big) \cdot \chi\big((\phi^{-1})^* W\big)$

$\phi^* \omega \otimes \phi^* \chi (V, W) = \phi^* \omega (V) \cdot \phi^* \chi (W) = \omega((\phi^{-1})^* V) \cdot \chi((\phi^{-1})^* W)$

$$\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi = \phi_t^* \omega \otimes \phi_t^* \chi - \omega \otimes \chi$$

$$= \phi_t^* \omega \otimes \phi_t^* \chi - \phi_t^* \omega \otimes \chi + \phi_t^* \omega \otimes \chi - \omega \otimes \chi$$

Let's prove that $\mathcal{L}_V(\omega \otimes \chi) = \mathcal{L}_V \omega \otimes \chi + \omega \otimes \mathcal{L}_V \chi$ for 1-form fields ω, χ

First show that

$$\phi^*(\omega \otimes \chi) = \phi^* \omega \otimes \phi^* \chi$$

$$\phi^*(\omega \otimes \chi)(V, W) = \omega \otimes \chi\Big((\phi^{-1})^* V, (\phi^{-1})^* W\Big) = \omega\big((\phi^{-1})^* V\big) \cdot \chi\big((\phi^{-1})^* W\big)$$

$$\phi^* \omega \otimes \phi^* \chi (V, W) = \phi^* \omega(V) \cdot \phi^* \chi(W) = \omega((\phi^{-1})^* V) \cdot \chi((\phi^{-1})^* W)$$

$$\begin{aligned}
\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi &= \phi_t^* \omega \otimes \phi_t^* \chi - \omega \otimes \chi \\
&= \phi_t^* \omega \otimes \phi_t^* \chi - \phi_t^* \omega \otimes \chi + \phi_t^* \omega \otimes \chi - \omega \otimes \chi \\
&= \phi_t^* \omega \otimes \left[\phi_t^* \chi - \chi\right] + \left[\phi_t^* \omega - \omega\right] \otimes \chi
\end{aligned}$$

Let's prove that $\mathcal{L}_V(\omega \otimes \chi) = \mathcal{L}_V\omega \otimes \chi + \omega \otimes \mathcal{L}_V \chi$ for 1-form fields ω, χ

First show that

$$\phi^*(\omega \otimes \chi) = \phi^*\omega \otimes \phi^*\chi$$

$$\phi^*(\omega \otimes \chi)(V, W) = \omega \otimes \chi\Big((\phi^{-1})^* V, (\phi^{-1})^* W\Big) = \omega\big((\phi^{-1})^* V\big) \cdot \chi\big((\phi^{-1})^* W\big)$$

$$\phi^*\omega \otimes \phi^*\chi(V, W) = \phi^*\omega(V) \cdot \phi^*\chi(W) = \omega((\phi^{-1})^* V) \cdot \chi((\phi^{-1})^* W)$$

$$\begin{aligned}
\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi &= \phi_t^*\omega \otimes \phi_t^*\chi - \omega \otimes \chi \\
&= \phi_t^*\omega \otimes \phi_t^*\chi - \phi_t^*\omega \otimes \chi + \phi_t^*\omega \otimes \chi - \omega \otimes \chi \\
&= \phi_t^*\omega \otimes \left[\phi_t^*\chi - \chi\right] + \left[\phi_t^*\omega - \omega\right] \otimes \chi \Rightarrow
\end{aligned}$$

$$\mathcal{L}_V \omega \otimes \chi = \lim_{t \to 0} \frac{1}{t}\left[\phi_t^*(\omega \otimes \chi) - \omega \otimes \chi\right] = \lim_{t \to 0} \frac{1}{t} \phi_t^*\omega \otimes \left[\phi_t^*\chi - \chi\right] + \lim_{t \to 0} \frac{1}{t}\left[\phi_t^*\omega - \omega\right] \otimes \chi$$

Let's prove that $\mathcal{L}_v(\omega\otimes\chi) = \mathcal{L}_v\omega\otimes\chi + \omega\otimes\mathcal{L}_v\chi$ for 1-form fields ω, χ

$$= \omega\otimes\mathcal{L}_v\chi + \mathcal{L}_v\omega\otimes\chi$$

$$\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi = \phi_t^*\omega\otimes\phi_t^*\chi - \omega\otimes\chi$$
$$= \phi_t^*\omega\otimes\phi_t^*\chi - \phi_t^*\omega\otimes\chi + \phi_t^*\omega\otimes\chi - \omega\otimes\chi$$
$$= \phi_t^*\omega\otimes\left[\phi_t^*\chi - \chi\right] + \left[\phi_t^*\omega - \omega\right]\otimes\chi \Rightarrow$$

$$\mathcal{L}_v\omega\otimes\chi = \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi\right] = \lim_{t\to 0}\frac{1}{t}\phi_t^*\omega\otimes\left[\phi_t^*\chi - \chi\right] + \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*\omega - \omega\right]\otimes\chi$$

Let's prove that $\mathcal{L}_v(\omega\otimes\chi) = \mathcal{L}_v\omega\otimes\chi + \omega\otimes\mathcal{L}_v\chi$ for 1-form fields ω,χ

$$= \omega \otimes \mathcal{L}_v \chi \quad + \quad \mathcal{L}_v \omega \otimes \chi$$

$$\lim_{t\to 0} \phi_t^* \omega = \omega$$

$$\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi = \phi_t^*\omega \otimes \phi_t^*\chi - \omega\otimes\chi$$

$$= \phi_t^*\omega \otimes \phi_t^*\chi - \phi_t^*\omega\otimes\chi + \phi_t^*\omega\otimes\chi - \omega\otimes\chi$$

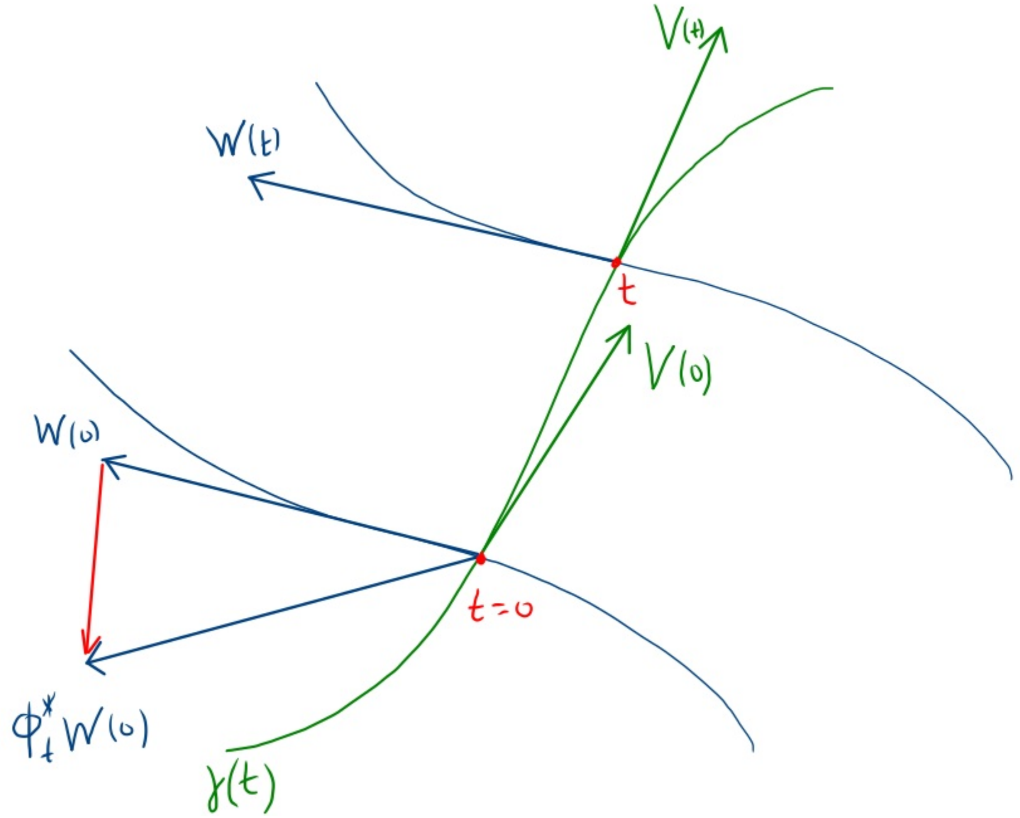
$$= \phi_t^*\omega \otimes \left[\phi_t^*\chi - \chi\right] + \left[\phi_t^*\omega - \omega\right]\otimes\chi \Rightarrow$$

$$\mathcal{L}_v\,\omega\otimes\chi = \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(\omega\otimes\chi) - \omega\otimes\chi\right] = \lim_{t\to 0}\frac{1}{t}\,\phi_t^*\omega\otimes\left[\phi_t^*\chi - \chi\right] + \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*\omega - \omega\right]\otimes\chi$$

Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

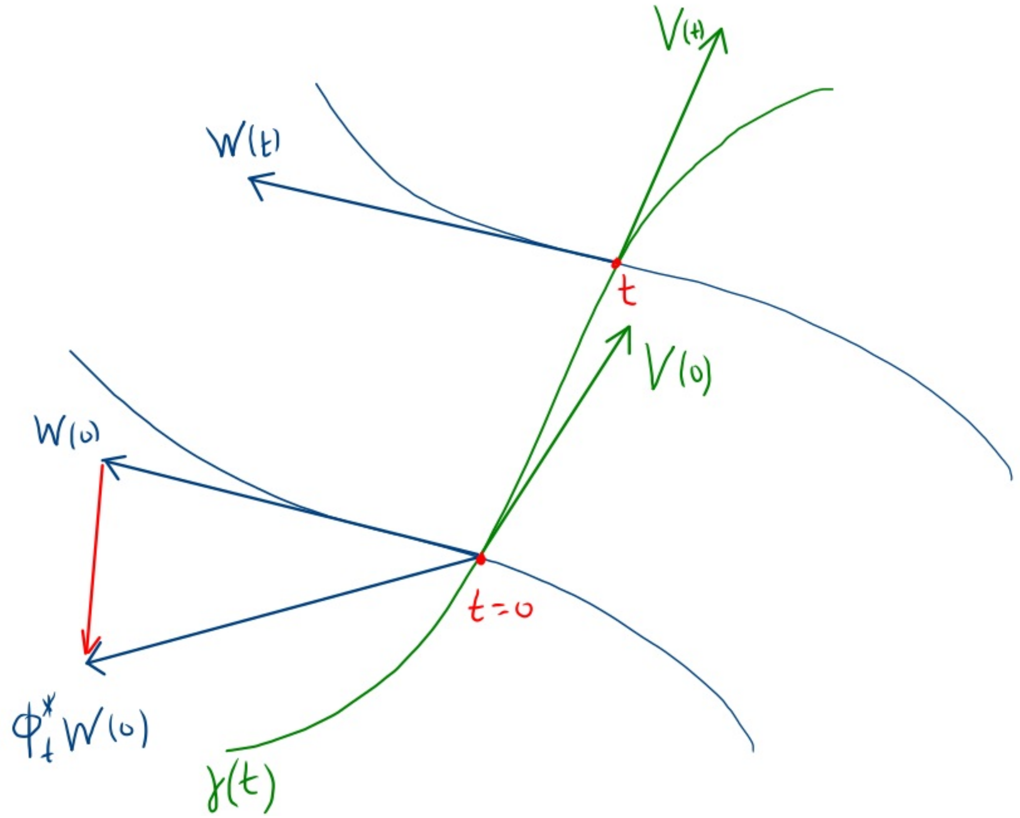


Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

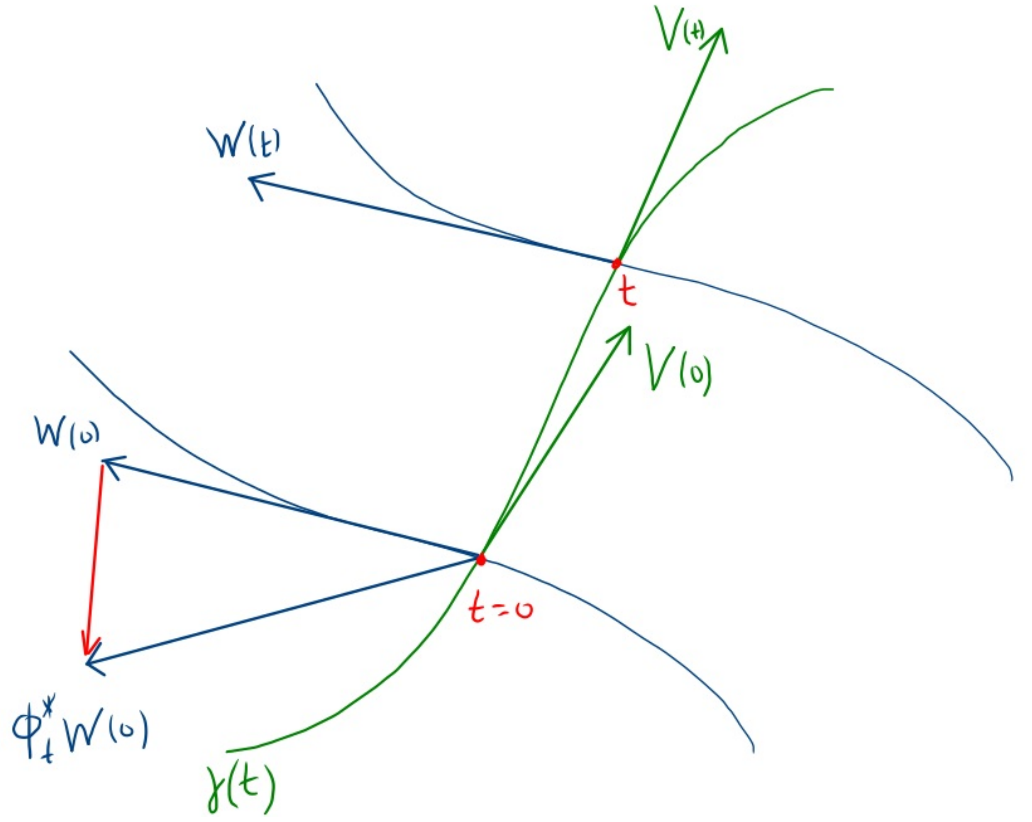
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - tV^\mu(t) + \mathcal{O}(t^2)$$



$$V^\mu = \frac{dx^\mu}{dt}$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

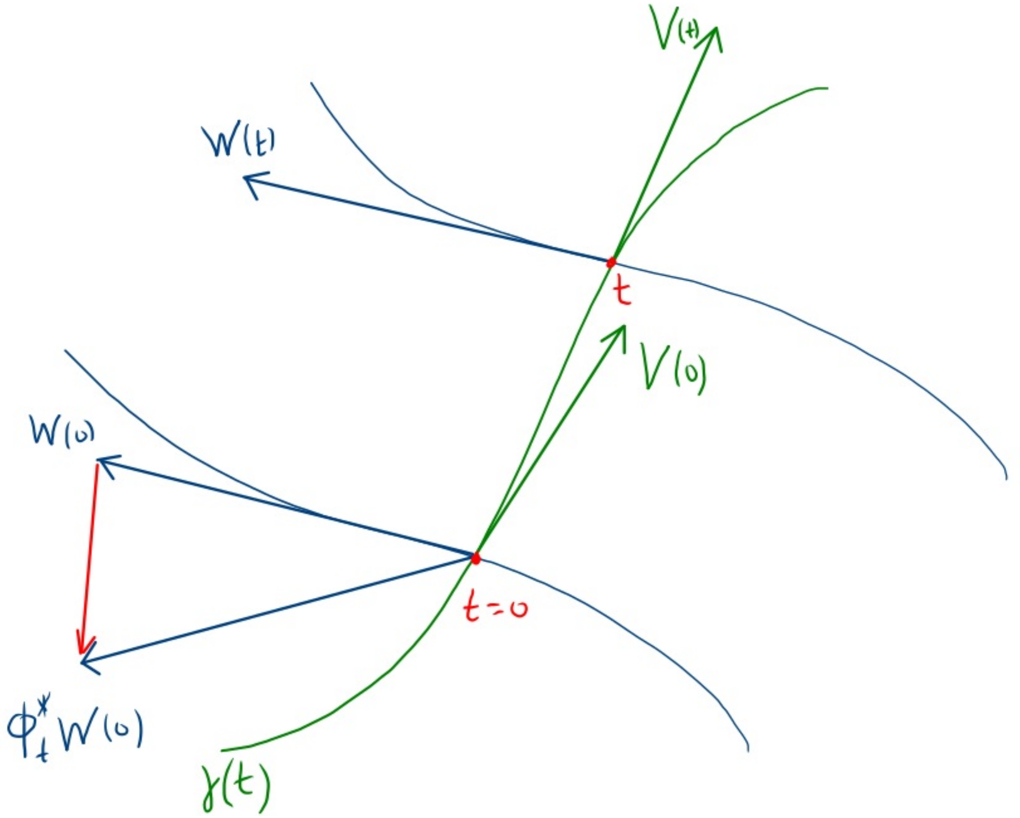
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} \, W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\,\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu(0)}{\partial x^\nu(0)} + \mathcal{O}(t^2)$$

$$\hookrightarrow \frac{\partial V^\mu}{\partial x^\nu}(t) = \frac{\partial V^\mu}{\partial x^\nu}(0) + \mathcal{O}(t) \Rightarrow t\,\frac{\partial V^\mu}{\partial x^\nu}(t) = t\,\frac{\partial V^\mu}{\partial x^\nu}(0) + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

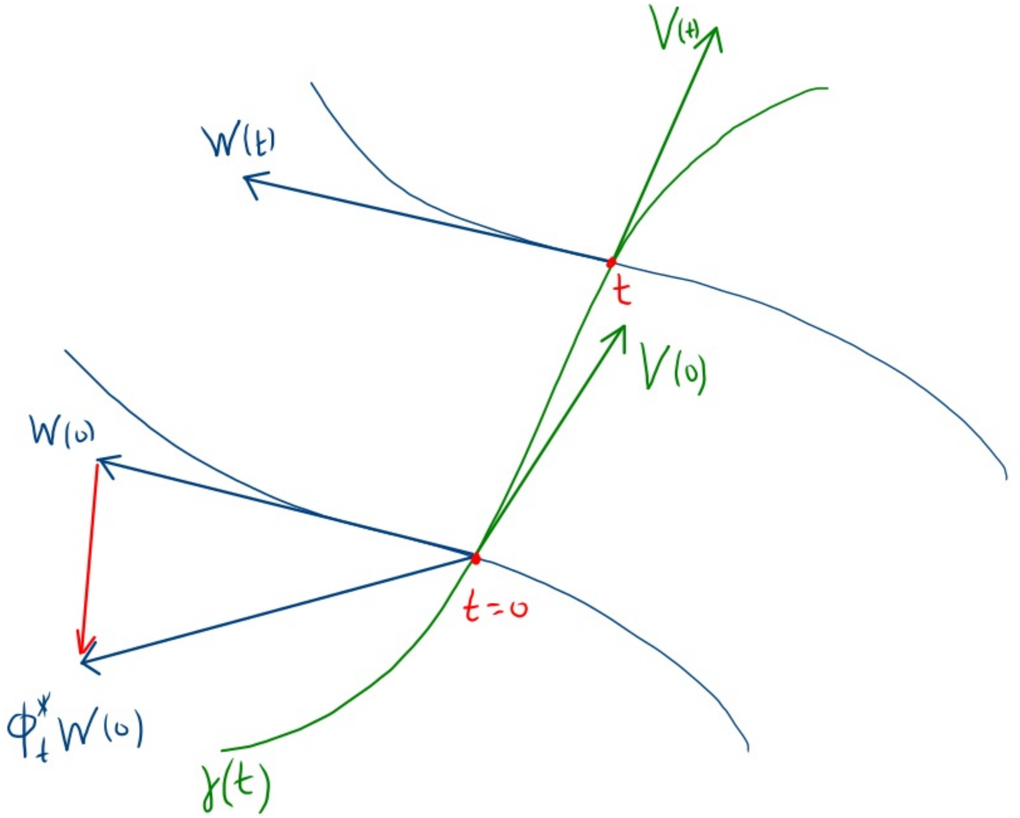
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - tV^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\frac{\partial V^\mu(0)}{\partial x^\nu(0)} + \mathcal{O}(t^2)$$

$$W^\nu(t) = W^\nu(0) + t\frac{dW^\nu(0)}{dt} + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

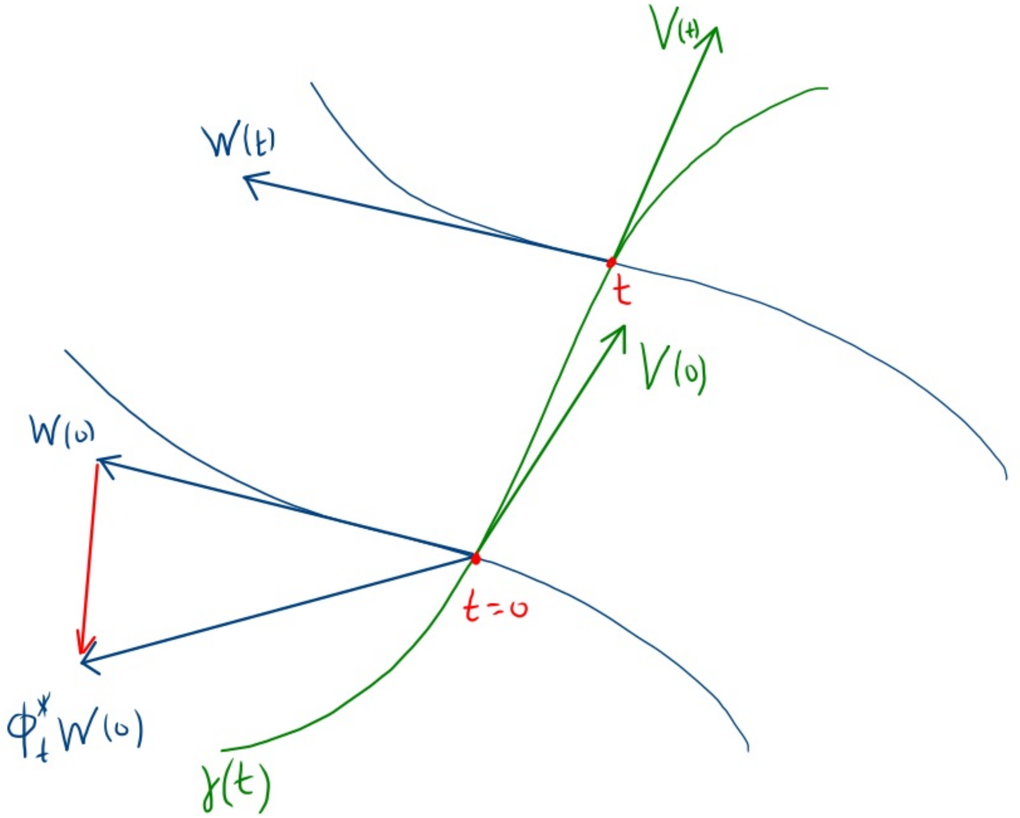
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} \, W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\,\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - t\,V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\,\frac{\partial V^\mu(0)}{\partial x^\nu(0)} + \mathcal{O}(t^2)$$

$$W^\nu(t) = W^\nu(0) + t\,\frac{dW^\nu(0)}{dt} + \mathcal{O}(t^2) = W^\nu(0) + t\,\frac{\partial W^\nu(0)}{\partial x^\lambda(0)} \underbrace{\frac{dx^\lambda}{dt}(0)}_{V^\lambda(0)} + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} \; W^\nu(t)$$

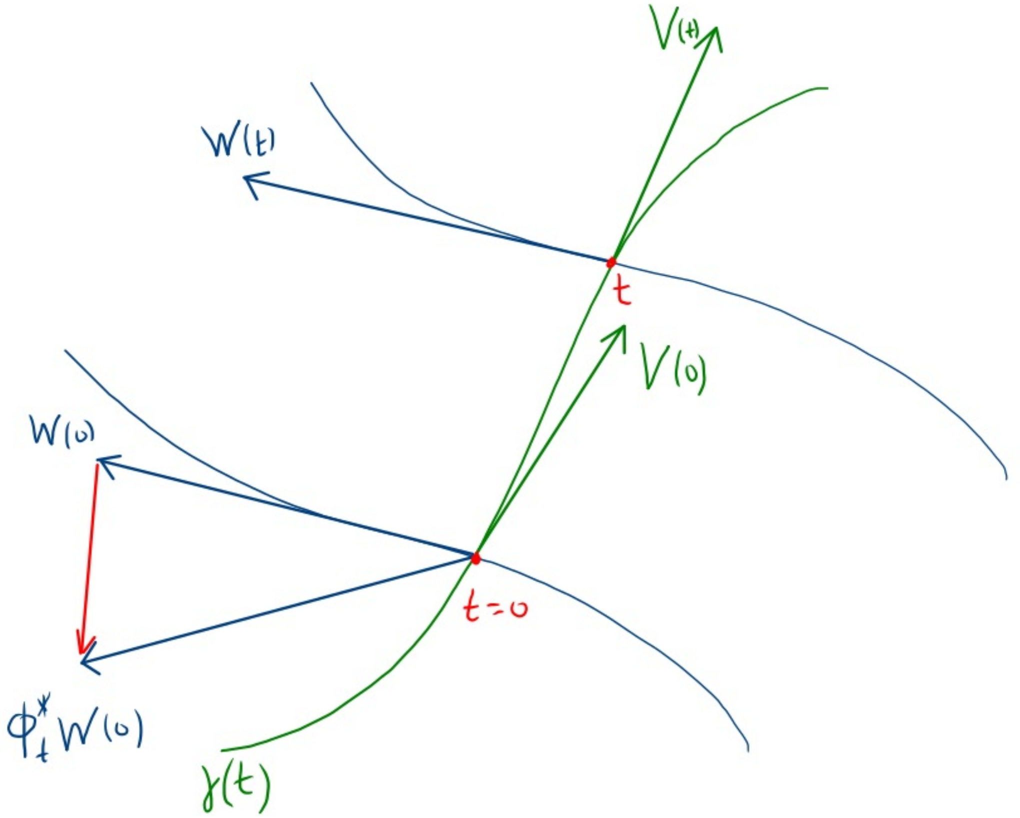
$$x^\mu(0) = x^\mu(t) - t \frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t \frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t \frac{\partial V^\mu(0)}{\partial x^\nu(0)} + \mathcal{O}(t^2)$$

$$W^\nu(t) = W^\nu(0) + t \frac{dW^\nu(0)}{dt} + \mathcal{O}(t^2) = W^\nu(0) + t \frac{\partial W^\nu(0)}{\partial x^\lambda(0)} \underbrace{\frac{dx^\lambda}{dt}(0)}_{V^\lambda(0)} + \mathcal{O}(t^2)$$

$$= W^\nu(0) + t V^\lambda(0) \frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$x^\mu(0) = x^\mu(t) - t\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2) = x^\mu(t) - tV^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta^\mu{}_\nu - t\frac{\partial V^\mu(t)}{\partial x^\nu(t)} + \mathcal{O}(t^2) = \delta^\mu{}_\nu - t\frac{\partial V^\mu(0)}{\partial x^\nu(0)} + \mathcal{O}(t^2)$$

$$W^\nu(t) = W^\nu(0) + t\frac{dW^\nu(0)}{dt} + \mathcal{O}(t^2) = W^\nu(0) + t\frac{\partial W^\nu(0)}{\partial x^\lambda(0)}\frac{dx^\lambda}{dt}(0) + \mathcal{O}(t^2)$$

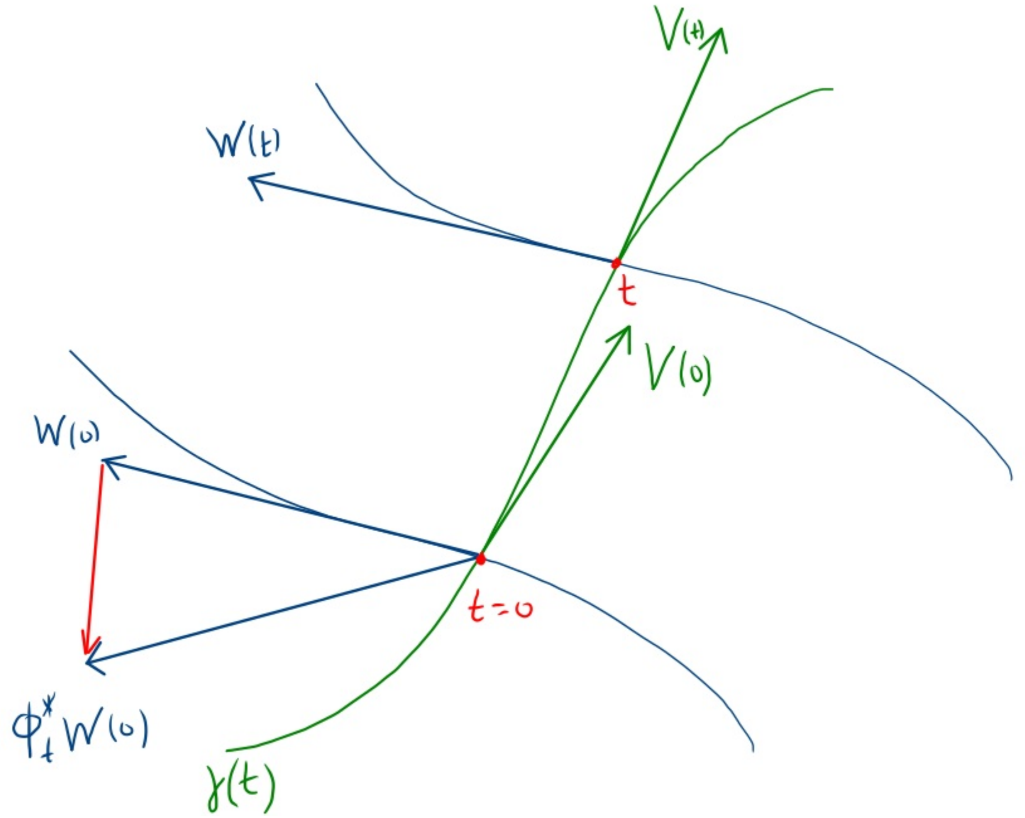
$$= W^\nu(0) + tV^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + tV^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

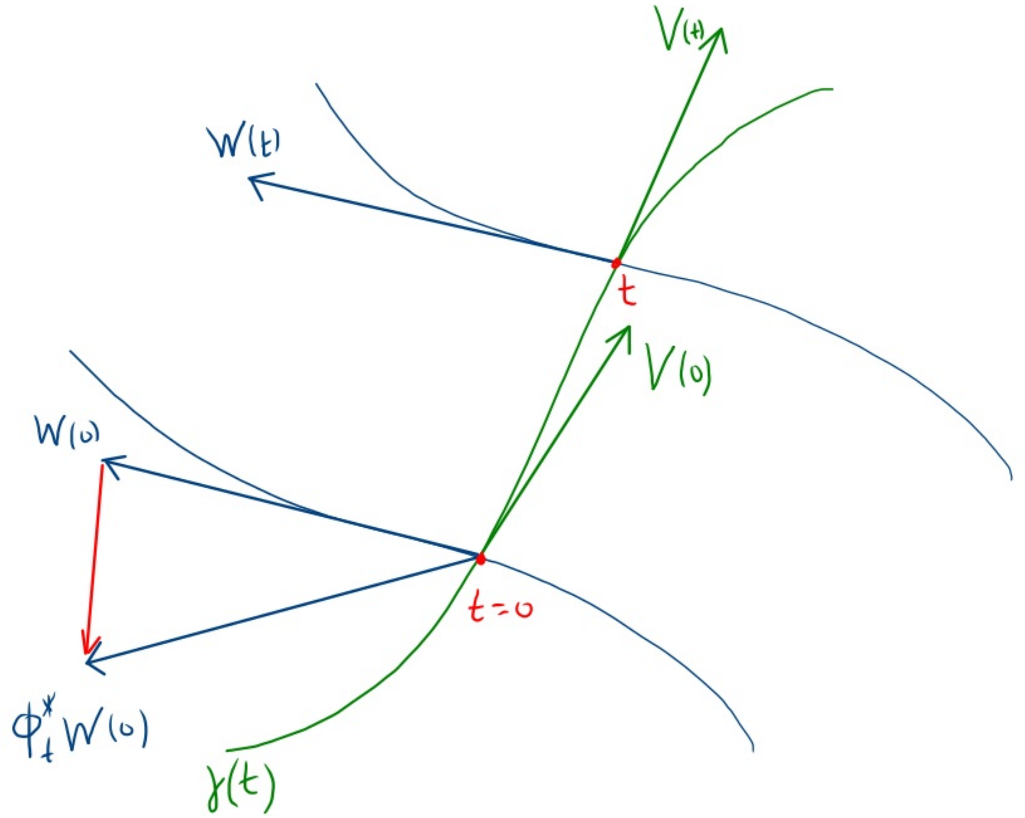
$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$



$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t\, V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} \quad W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

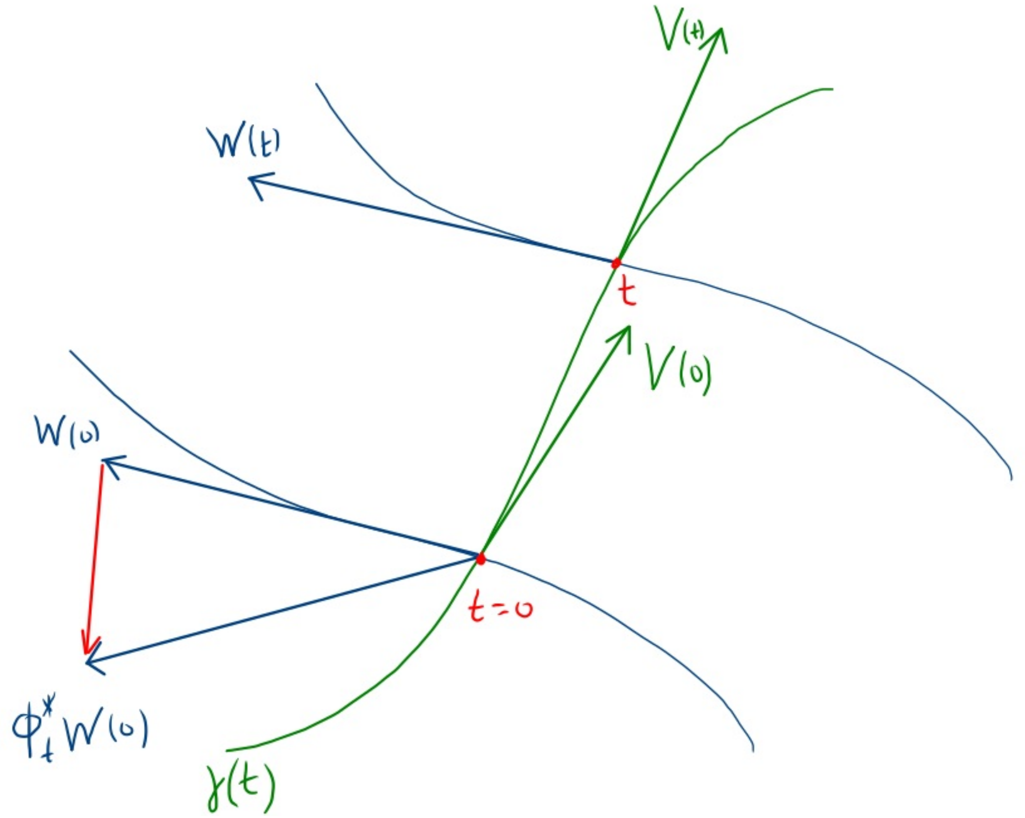
$$\Rightarrow \quad \phi_t^* W(0)^\mu - W(0)^\mu = t\left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t\ V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$



$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

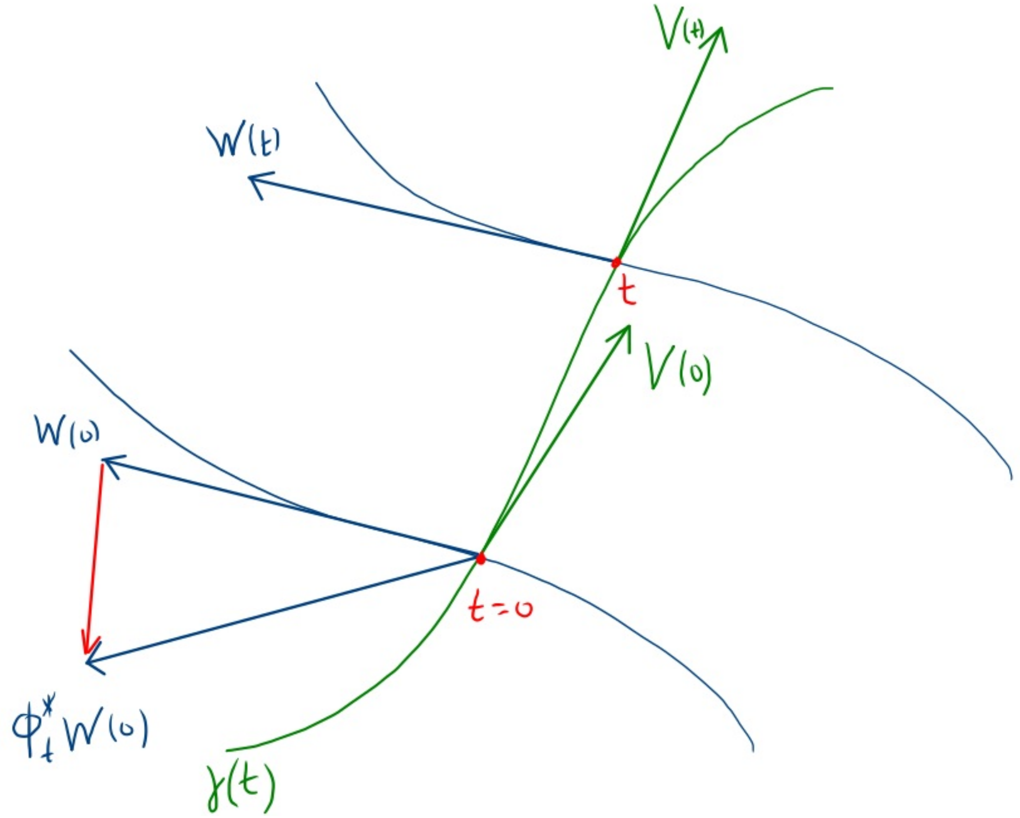
$$\Rightarrow \quad \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \frac{1}{t}\mathcal{O}(t^2)$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t\, V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$



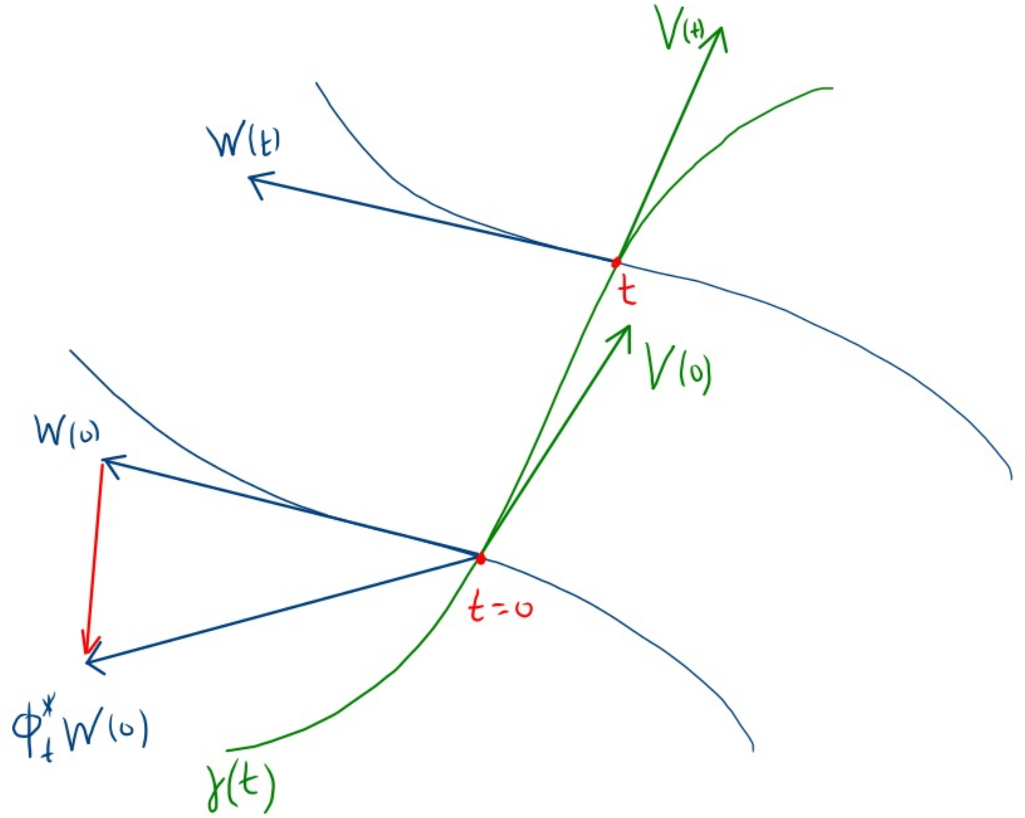
$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \quad \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \frac{1}{t}\mathcal{O}(t^2)$$

$$\lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)\Big|_0 + 0$$

$$\Rightarrow \phi_t^* W(0)^\mu = \left(\delta^\mu{}_\nu - t\frac{\partial V^\mu}{\partial x^\nu} + \mathcal{O}(t^2)\right)\left(W^\nu(0) + t\, V^\lambda(0)\frac{\partial W^\nu}{\partial x^\lambda}(0) + \mathcal{O}(t^2)\right)$$

Components of $\mathcal{L}_V W$



Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then

$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

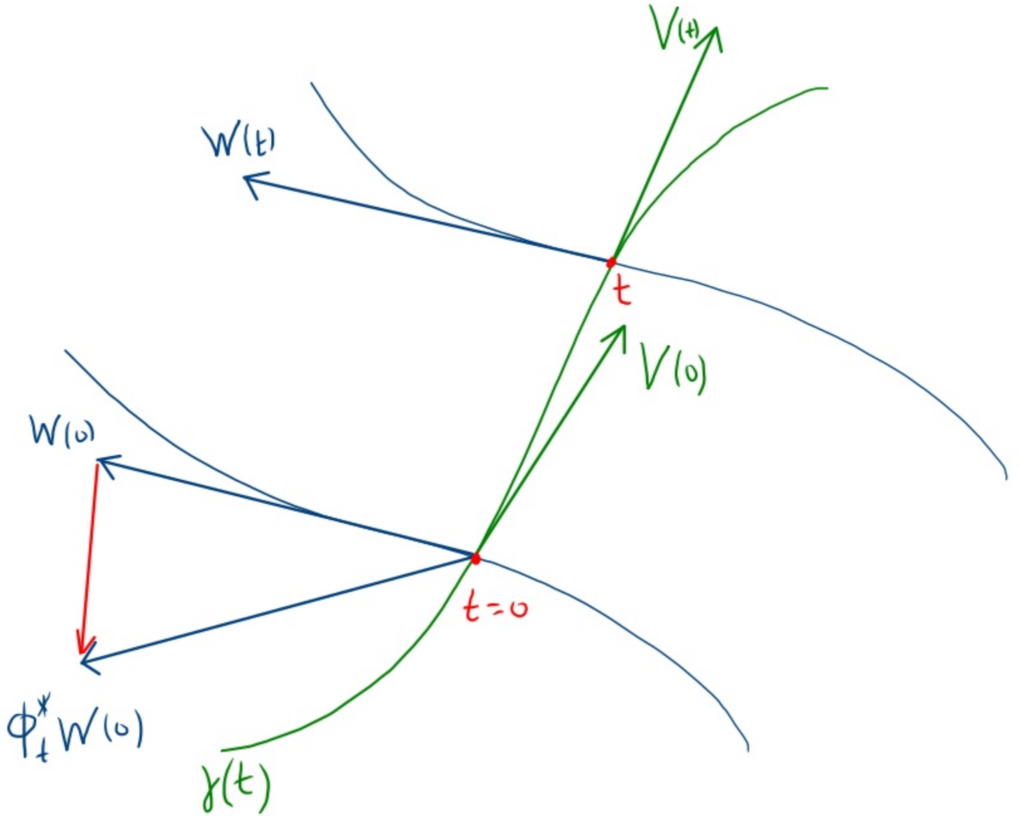
$$\Rightarrow \quad \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \frac{1}{t}\mathcal{O}(t^2)$$

$$\lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)\Big|_0 + 0$$

$$\Rightarrow \quad \mathcal{L}_V W = V^\nu \partial_\nu W - W^\nu \partial_\nu V^\mu$$

Components of $\mathcal{L}_V W$

Consider $\{x^\mu\} \to \{\partial_\mu\}$ - basis, then



$$\phi_t^* W(0)^\mu = \frac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$$

$$= W^\mu(0) + t\left[V^\lambda \partial_\lambda W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \mathcal{O}(t^2)$$

$$\Rightarrow \quad \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left[V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right]\Big|_0 + \frac{1}{t}\mathcal{O}(t^2)$$

$$\lim_{t\to 0} \frac{1}{t}\left[\phi_t^* W(0)^\mu - W(0)^\mu\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)\Big|_0 + 0$$

$$\Rightarrow \quad \mathcal{L}_V W = V^\nu \partial_\nu W - W^\nu \partial_\nu V^\mu$$

Exercise: Show that $[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$

Compute $\mathcal{L}_v f$

$$\mathcal{L}_v f = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* f(0) - f(0) \right]$$

Compute $\mathcal{L}_v f$

$$\mathcal{L}_v f = \lim_{t\to 0} \frac{1}{t}\left[\phi_t^* f(0) - f(0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[f\circ\phi_t(0) - f(0)\right]$$

Compute $\mathcal{L}_v f$

$$\mathcal{L}_v f = \lim_{t \to 0} \frac{1}{t} \left[\phi_t^* f(0) - f(0) \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \left[f \circ \phi_t(0) - f(0) \right]$$

$$= \lim_{t \to 0} \frac{1}{t} \left[f(t) - f(0) \right] = \frac{df}{dt}(0)$$

Compute $\mathcal{L}_v \omega$

$$\mathcal{L}_v(\underbrace{\omega(W)}_{\text{a function, we know}}) = \underbrace{\mathcal{L}_v \omega}_{\text{we don't know}}(W) + \omega(\underbrace{\mathcal{L}_v W}_{\text{we know}})$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = \underbrace{V^\mu \partial_\mu}_{\frac{d}{dt} \text{ of a function}}(\omega(W))$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu\partial_\mu(\omega(W)) = V^\mu\partial_\mu(\underbrace{\omega_\nu W^\nu}_{\text{contraction}})$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(W)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(W)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$$\omega(\mathcal{L}_V W) = \omega_\mu (\mathcal{L}_V W)^\mu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(W)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \omega_\nu \partial_\mu W^\nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$$\omega(\mathcal{L}_V W) = \omega_\mu (\mathcal{L}_V W)^\mu = \omega_\mu (V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu)$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(W)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \cancel{\omega_\nu \partial_\mu W^\nu}$$

$$\parallel \qquad \mu \leftrightarrow \nu$$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$$+$$

$$\omega(\mathcal{L}_V W) = \omega_\mu (\mathcal{L}_V W)^\mu = \omega_\mu(\cancel{V^\nu \partial_\nu W^\mu} - W^\nu \partial_\nu V^\mu)$$

$$V^\mu \partial_\mu \omega_\nu W^\nu = (\mathcal{L}_V \omega)_\nu W^\nu - \omega_\mu \partial_\nu V^\mu W^\nu$$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega(W) + \omega(\mathcal{L}_V W)$$

$$\mathcal{L}_V(\omega(W)) = V^\mu \partial_\mu(\omega(W)) = V^\mu \partial_\mu(\omega_\nu W^\nu) = V^\mu \partial_\mu \omega_\nu W^\nu + V^\mu \cancel{\omega_\nu \partial_\mu} W^\nu$$

$\mu \leftrightarrow \nu$

$$\mathcal{L}_V \omega(W) = (\mathcal{L}_V \omega)_\nu W^\nu$$

$+$

$$\omega(\mathcal{L}_V W) = \omega_\mu (\mathcal{L}_V W)^\mu = \omega_\mu (\cancel{V^\nu \partial_\nu W^\mu} - W^\nu \partial_\nu V^\mu)$$

$$V^\mu \partial_\mu \omega_\nu \cancel{W^\nu} = (\mathcal{L}_V \omega)_\nu \cancel{W^\nu} - \omega_\mu \partial_\nu V^\mu \cancel{W^\nu} \Rightarrow$$

$$(\mathcal{L}_V \omega)_\nu = V^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu V^\mu$$

$$\mathcal{L}_V W = V \partial W - W \partial V$$

$$\mathcal{L}_V \omega = V \partial \omega + \omega \partial V$$

$$\mathcal{L}_V W^\mu = V \partial W^\mu - W \partial V^\mu$$

$$(\mathcal{L}_V \omega)_\mu = V \partial \omega_\mu + \omega \partial_\mu V$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

$$(\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$(\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

↳ <u>any</u> torsion free covariant derivative

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$(\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

$$(\mathcal{L}_V T)^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} = V^\mu \partial_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda}$$

$$- T^{\rho \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} \partial_\rho V^{\mu_1} - \dots - T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_\lambda} \partial_\rho V^{\mu_k}$$

$$+ T^{\mu_1 \dots \mu_k}{}_{\rho \dots \nu_\lambda} \partial_{\nu_1} V^\rho + \dots + T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho} \partial_{\nu_l} V^\rho$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = V^\nu \nabla_\nu W^\mu - W^\nu \nabla_\nu V^\mu$$

$$(\mathcal{L}_V \omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu = V^\nu \nabla_\nu \omega_\mu + \omega_\nu \nabla_\mu V^\nu$$

$$(\mathcal{L}_V T)^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} = V^\mu \nabla_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda}$$

$$- T^{\rho \dots \mu_k}{}_{\nu_1 \dots \nu_\lambda} \nabla_\rho V^{\mu_1} - \dots - T^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_\lambda} \nabla_\rho V^{\mu_k}$$

$$+ T^{\mu_1 \dots \mu_k}{}_{\rho \dots \nu_\lambda} \nabla_{\nu_1} V^\rho + \dots + T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho} \nabla_{\nu_\ell} V^\rho$$

Example: $(0,2)$ tensor (like a metric)

$$\mathcal{L}_V\underbrace{[g(X,Y)]}_{\text{a function, we know}} = \underbrace{\mathcal{L}_V g}_{\text{we don't know}}(X,Y) + g(\underbrace{\mathcal{L}_V X}_{\text{we know}}, Y) + g(X, \underbrace{\mathcal{L}_V Y}_{\text{we know}})$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho)$$

Example: (0, 2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

Example: $(0,2)$ tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g\,(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

Example: $(0,2)$ tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

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$$\mathcal{L}_V g\,(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho}(V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu)\, Y^\rho$$

Example: (0, 2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu g_{\nu\rho}\, \partial_\mu X^\nu\, Y^\rho + V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho$$

$$\mathcal{L}_V g\,(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (V^\mu \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu)\, Y^\rho$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (V^\mu \partial_\mu Y^\rho - Y^\mu \partial_\mu V^\rho)$$

Example: $(0,2)$ tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g(\mathcal{L}_V X, Y) + g(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + \cancel{V^\mu g_{\nu\rho} \partial_\mu X^\nu Y^\rho} + \cancel{V^\mu g_{\nu\rho} X^\nu \partial_\mu Y^\rho}$$

$$\shortparallel$$

$$\mathcal{L}_V g\,(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$+$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (\cancel{V^\mu \partial_\mu X^\nu} - X^\mu \partial_\mu V^\nu) Y^\rho$$

$$+$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (\cancel{V^\mu \partial_\mu Y^\rho} - Y^\mu \partial_\mu V^\rho)$$

$$V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho - g_{\nu\rho} \partial_\mu V^\nu X^\mu Y^\rho - g_{\nu\rho} \partial_\mu V^\rho X^\nu Y^\mu$$

Example: (0,2) tensor (like a metric)

$$\mathcal{L}_V[g(X,Y)] = \mathcal{L}_V g\,(X,Y) + g\,(\mathcal{L}_V X, Y) + g\,(X, \mathcal{L}_V Y)$$

$$\mathcal{L}_V[g(X,Y)] = V^\mu \partial_\mu (g_{\nu\rho} X^\nu Y^\rho) = V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho + V^\mu \cancel{g_{\nu\rho}} \partial_\mu X^\nu Y^\rho + V^\mu g_{\nu\rho} \cancel{X^\nu} \partial_\mu Y^\rho$$

$$\parallel$$

$$\mathcal{L}_V g\,(X,Y) = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho$$

$$+$$

$$g(\mathcal{L}_V X, Y) = g_{\nu\rho} (\mathcal{L}_V X)^\nu Y^\rho = g_{\nu\rho} (\cancel{V^\mu} \partial_\mu X^\nu - X^\mu \partial_\mu V^\nu) Y^\rho$$

$$+$$

$$g(X, \mathcal{L}_V Y) = g_{\nu\rho} X^\nu (\mathcal{L}_V Y)^\rho = g_{\nu\rho} X^\nu (V^\mu \cancel{\partial_\mu} Y^\rho - Y^\mu \partial_\mu V^\rho)$$

$$V^\mu \partial_\mu g_{\nu\rho} X^\nu Y^\rho = (\mathcal{L}_V g)_{\nu\rho} X^\nu Y^\rho - g_{\mu\rho} \partial_\nu V^\mu X^\nu Y^\rho - g_{\nu\mu} \partial_\rho V^\mu X^\nu Y^\rho$$

$$\nu \leftrightarrow \mu \qquad \mu \leftrightarrow \rho$$

Example: (0,2) tensor (like a metric)

$$(\mathcal{L}_V g)_{\nu\rho} = V^\mu \partial_\mu g_{\nu\rho} + g_{\underline{\mu}\rho} \partial_{\underline{\nu}} V^\mu + g_{\nu\underline{\mu}} \partial_{\underline{\rho}} V^\mu$$

$$V^\mu \partial_\mu g_{\nu\rho} \cancel{X^\nu Y^\rho} = (\mathcal{L}_V g)_{\nu\rho} \cancel{X^\nu Y^\rho} - g_{\mu\rho} \partial_\nu V^\mu \cancel{X^\nu Y^\rho} - g_{\nu\mu} \partial_\rho V^\mu \cancel{X^\nu Y^\rho}$$

$$\nu \leftrightarrow \mu \qquad \mu \leftrightarrow \rho$$