EMT = energy-momentum tensor

Consider the "physical" or "metric" EMT $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}$

S: the action of matter fields.

① Compute $T_{\mu\nu}$ for $S = \int \sqrt{-g} \left[-\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - V(\phi) \right]$ $\phi \in \mathbb{R}$

$S_2 = \int \sqrt{-g} \left[+ \frac{1}{2} \phi \nabla_\mu \nabla^\mu \phi - V(\phi) \right]$ $\phi \in \mathbb{R}$

$S_{EM} = \int \sqrt{g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$ $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

② For S_{EM}, show that

$T^{00} = \frac{1}{2} \left(\mathcal{E}^2 + B^2 \right)$ $T^{0i} = \left(\vec{\mathcal{E}} \times \vec{B} \right)_i$ $T^{ij} = \left(-\mathcal{E}_i \mathcal{E}_j + \frac{1}{2} \delta_{ij} \mathcal{E}^2 \right) + \left(-B_i B_j + \frac{1}{2} \delta_{ij} B^2 \right)$

③ Assuming Maxwell's equations ("on shell") $\partial_\mu F^{\mu\rho} = 0$ ($J^\mu = 0$)
$\partial_{[\mu} F_{\nu\rho]} = 0$, show that for S_{EM} $\partial_\mu T^{\mu\nu} = 0$

④ Show that diffeomorphism invariance of the Hilbert-Einstein action

$$S^{EH} = \int \sqrt{-g}\, R$$

implies $\nabla^\mu G_{\mu\nu} = 0$ for the Einstein tensor $G_{\mu\nu}$
(ignore all boundary terms in variations.

⑤ Consider the action $S = \int \sqrt{-g}\, \mathcal{L}$, $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \beta\, R^{\mu\nu} F_{\mu\rho} F_\nu{}^\rho$

(α) Compute the equations of motion for the EM field ("Maxwell's eqs")

(β) Compute the modification to Einstein's equation by β. (i.e. the new $T_{\mu\nu}$)

The following problem is not going to be tested. But it is very deserving your effort (see solutions)

Consider a field theory given by $\mathcal{L}(\phi_n, \partial_\mu \phi_n)$ for the fields $\phi_n(x)$, $n = 1, 2, \dots$. The canonical EMT is

$$T_{(c)}{}^{\mu}{}_{\nu} = -\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta_\nu \phi_n + \delta^\mu{}_\nu \mathcal{L},$$

wher $\delta_\nu \phi_n$ are the changes of the fields under infinitesimal translations $x^\mu \to x^\mu + \xi^\mu$ $\quad \phi_n \to \phi_n + \xi^\nu \delta_\nu \phi_n$

(a) Compute $T_{(c)}{}^{\mu}{}_{\nu} = -\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \delta^\mu{}_\nu \mathcal{L}$, $\quad \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$, $\phi \in \mathbb{R}$

(β) Compute $T_{(c)}{}^{\mu}{}_{\nu} = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial_\nu A_\rho + \delta^\mu{}_\nu \mathcal{L}$, $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

This corresponds to translations $A_\rho \to A_\rho + \xi^\nu \partial_\nu A_\rho$

(γ) Show that

$$T^\mu{}_\nu = T_{(c)}{}^\mu{}_\nu - F^{\mu\lambda} \partial_\lambda A_\nu \neq T_{(c)}{}^\mu{}_\nu$$

(δ) Show that for $\delta_\nu A_\rho = F_{\nu\rho}$ $\quad T^\mu{}_\nu = T_{(c)}{}^\mu{}_\nu$

Can you justify this choice?