(1) Compute $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ in the free falling astronaut problem when $r<2M$ and show that the results are the same as for the $r>2M$ case.

(2) Consider the Killing vector field $\xi = \partial_t$ of the Schwarzschild geometry. Show that at the horizon $r=2M$

$$D_\xi \xi^\mu = \xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu ,$$

and compute κ. κ is the "surface gravity" of the black hole (Use the Eddington-Finkelstein coordinates for the computation)

($\Gamma^\mu{}_{\nu\rho}$ for E-F coordinates can be found in the Mathematica notebook @ Lecture 10)

- A stationary observer is hovering at $(t,r,\theta,\phi)=(t,R,\frac{\pi}{2},0)$, over a Schwarzschild black hole. Her 4-velocity is u^μ, and her acceleration $a^\mu = u^\nu \nabla_\nu u^\mu = \frac{Du^\mu}{d\tau}$. If $\xi = \partial_t$ is the timelike killing vector field, then:

- Compute a^μ and a_μ in the (t,r,θ,ϕ) and the (v,r,θ,ϕ) coordinate systems (E-F: Eddington-Finkelstein coordinates)
- Compute the scalar $a^2 = a_\mu a^\mu$ and show that a^μ is a spacelike vector for $r>2M$
- Compute $V=(-\xi_\mu \xi^\mu)^{1/2}$, and show that $a \cdot V = K = \left(\begin{array}{c}\text{surface}\\ \text{gravity}\end{array}\right)$
- Compute $(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)$ using E-F coordinates, and show that, at $r=2M$, $K = -\frac{1}{2}(\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu)$
- Show that $\eta^\mu = \xi^\nu \nabla_\nu \xi^\mu$ is space like for $r>2M$. Is this in contradiction

with $\xi^\nu \nabla_\nu \xi^\mu = k\, \xi^\mu$, since for $r > 2M$ ξ^μ is timelike?

3) A particle of rest mass m is initially at rest relative to a stationary observer O_1 at $r = \infty$ from a Schwarzschild black hole. The particle is left to fall freely towards the center of the black hole. Another stationary observer O_2 is hovering above the black hole at $(t, r, \theta, \phi) = (t, R, \frac{\pi}{2}, 0)$,

$R = \text{const} > 2M$.

- What is the speed of the particle as measured by O_2, when the particle goes through her lab?
- The particle is converted to radiation at O_2, which is

sent back to $r=\infty$. What is the total energy of that radiation, as measured by stationary detectors at $r=\infty$?

- Suppose that when the particle reaches O_2, it is brought to rest in the O_2's lab, and the excess energy is converted to radiation, which is sent back to $r=\infty$. What is the energy of the radiation as measured by stationary detectors at $r=\infty$?

(4) Rindler Coordinates: Consider the (t, r, θ, φ) coordinates of the Schwarzschild metric, and make the transformation to the $(t, \xi, \theta, \varphi)$ coordinates, where

$$r - 2M = \frac{\xi^2}{8M}.$$

Show that the metric in the new coordinates is given by

$$ds^2 = -\frac{k^2\xi^2}{k^2\xi^2+1}\,dt^2 + (k^2\xi^2+1)\,d\xi^2 + \frac{1}{4k^2}\,(k^2\xi^2+1)^2\,d\Omega^2,$$ where

$$k = \frac{1}{4M}, \quad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$$

- Show that for $r \approx 2M$

$$ds^2 \approx -\kappa^2 \xi^2 dt^2 + d\xi^2 + \frac{1}{4\kappa^2} d\Omega^2$$

- Consider the coordinate transformation

$$T = \xi \sinh(\kappa t)$$

$$X = \xi \cosh(\kappa t)$$

and show that the approximate metric above is flat

- Orbital speed: Consider a hovering stationary observer O_1 at $(t, r, \theta, \phi) = (t, R, \frac{\pi}{2}, 0)$, $R = \text{const} > 2M$, over a Schwarzschild black hole.
A particle falls freely, moving on a $r = R$ circular trajectory, and goes through the lab of O_1.
Calculate the speed $v(R)$ of the particle, as measured by O_1.