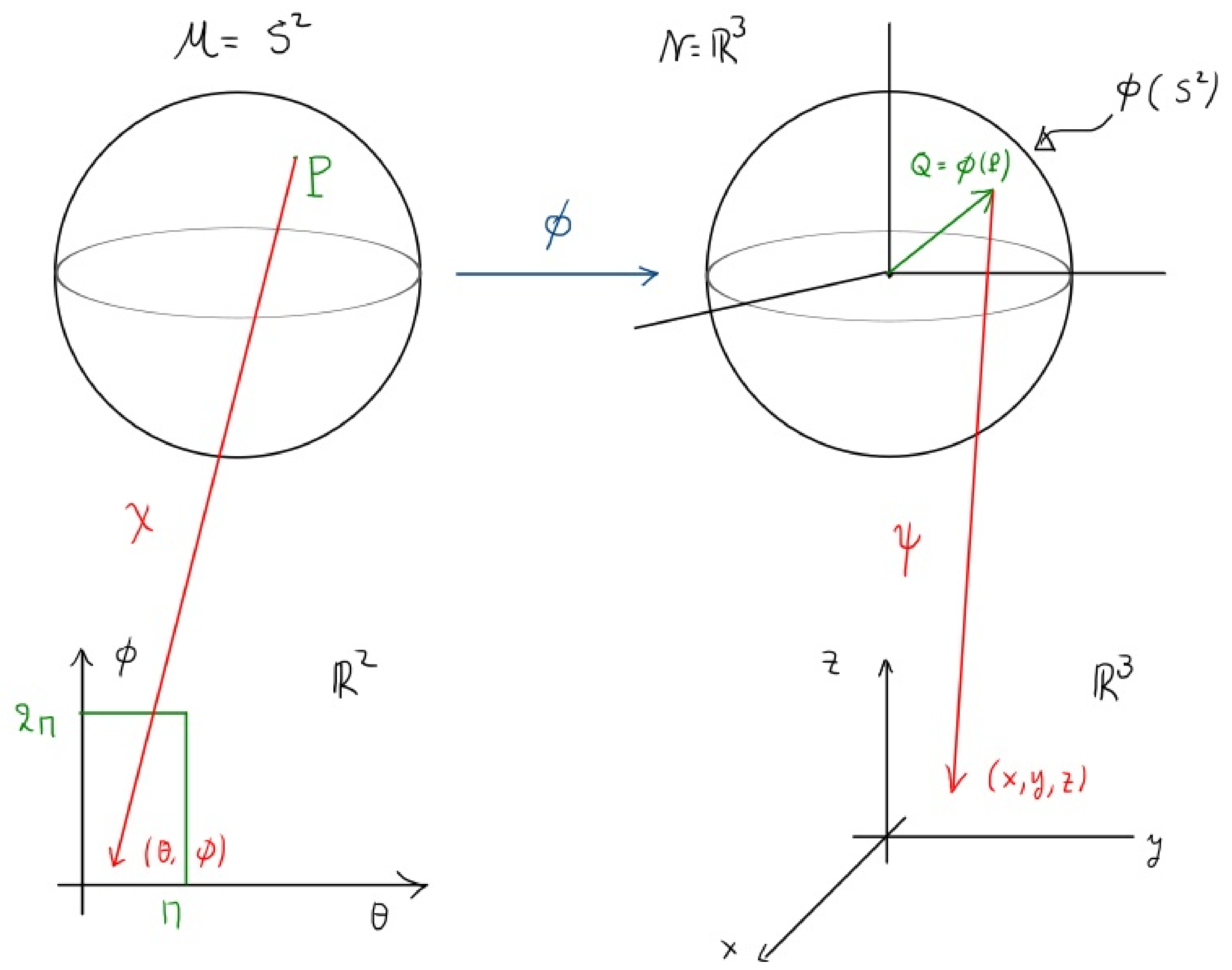
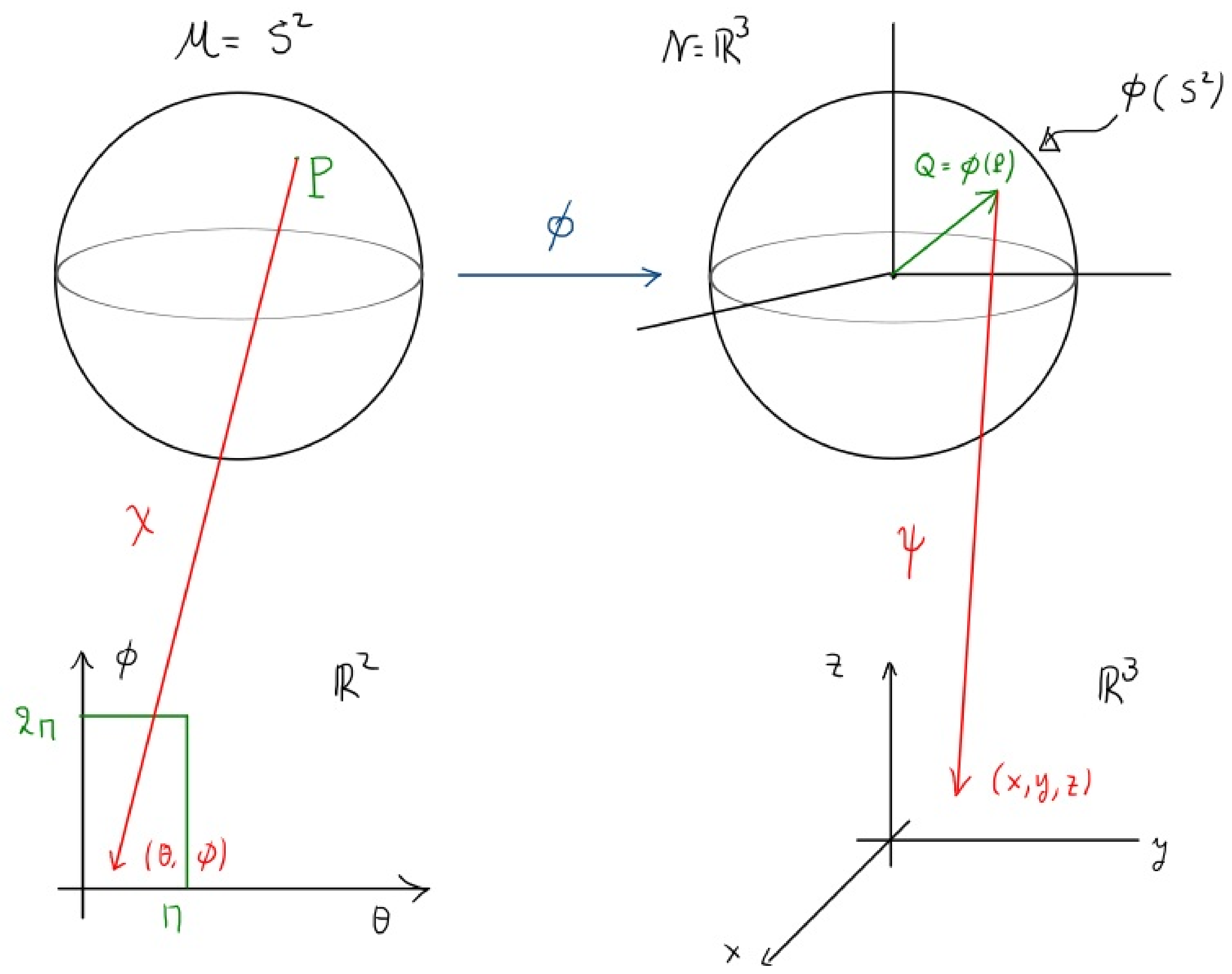


Embed S^2 in $\mathbb{R}^3$

$x^\mu = (r, \theta)$ $\qquad$ $y^\alpha = (x, y, z)$

$y^\alpha(x^\mu)$:
$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$



Embed S^2 in $\mathbb{R}^3$

$x^\mu = (r, \theta)$ $\qquad y^\alpha = (x, y, z)$

$$y^\alpha(x^\mu): \quad \begin{aligned} x &= \sin\theta \cos\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\theta \end{aligned}$$

Metric $g_{\alpha\beta}$: $\qquad ds^2 = dx^2 + dy^2 + dz^2$

$$(g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \, g_{\alpha\beta}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \, g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha \, (\phi^*)_\nu{}^\beta \, g_{\alpha\beta}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha (\phi^*)_\nu{}^\beta g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^{*T})^\beta{}_\nu$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha (\phi^*)_\nu{}^\beta g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^{*T})^\beta{}_\nu$$

$$= \left[\phi^* \cdot g \cdot (\phi^*)^T\right]_{\mu\nu}$$

$$2\times 2 = 2\times 3 \quad 3\times 3 \quad 3\times 2$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \, g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha \, (\phi^*)_\nu{}^\beta \, g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha \, g_{\alpha\beta} \, (\phi^{*T})^\beta{}_\nu$$

$$= \left[\phi^* \cdot g \cdot (\phi^*)^T\right]_{\mu\nu}$$

$$\phi^* = \left(\phi^*_\mu{}^\alpha\right) = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha (\phi^*)_\nu{}^\beta g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha g_{\alpha\beta} (\phi^{*T})^\beta{}_\nu$$

$$= [\phi^* \cdot g \cdot (\phi^*)^T]_{\mu\nu}$$

$$\phi^* = (\phi^*_\mu{}^\alpha) = \left(\frac{\partial y^\alpha}{\partial x^\mu} \right)$$

$$= \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$(\phi^* g)_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} \, g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha \, (\phi^*)_\nu{}^\beta \, g_{\alpha\beta}$$

$$= (\phi^*)_\mu{}^\alpha \, g_{\alpha\beta} \, (\phi^{*T})^\beta{}_\nu$$

$$= \left[\phi^* \cdot g \cdot (\phi^*)^T \right]_{\mu\nu}$$

$$\phi^* = \left(\phi^*_\mu{}^\alpha \right) = \left(\frac{\partial y^\alpha}{\partial x^\mu} \right)$$

$$= \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$

$$\phi^* = \left(\phi^{*}{}_{\mu}{}^{\alpha} \right) = \left(\frac{\partial y^\alpha}{\partial x^\mu} \right)$$

$$= \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\theta \sin\phi & \sin\theta \cos\phi & 0 \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\phi^* = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$(\phi^*) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot \mathbb{1}_{3\times 3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\theta$$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$(\phi^*) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot \mathbb{1}_{3\times 3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\theta$$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$(\phi^*) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} c\theta^2 c\phi^2 + c\theta^2 s\phi^2 + s\theta^2 & -c\theta s\theta c\phi s\phi + c\theta s\theta s\phi c\phi \\ -s\theta c\theta s\phi c\phi + s\theta c\theta s\phi c\phi & s\theta^2 c\phi^2 + s\theta^2 s\phi^2 \end{pmatrix}$$

$c\theta \to \cos\theta \qquad c\phi \to \cos\phi$

$s\theta \to \sin\theta \qquad s\phi \to \sin\phi$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\theta$$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$(\phi^*) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underbrace{c\theta^2 \overbrace{c\phi^2 + c\theta^2 s\phi^2}^{1} + s\theta^2}_{=1} & -c\theta\,\cancel{s\theta}\,c\phi s\phi + c\theta\,\cancel{s\theta}\,s\phi c\phi \\ -s\theta\,\cancel{c\theta}\,s\phi c\phi + s\theta\,\cancel{c\theta}\,s\phi c\phi & s\theta^2 \underbrace{c\phi^2 + s\theta^2 s\phi^2}_{1} \end{pmatrix}$$

$c\theta \to \cos\theta \qquad c\phi \to \cos\phi$

$s\theta \to \sin\theta \qquad s\phi \to \sin\phi$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$

$$(\phi^*)^T = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$(\phi^*) = \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times 3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underbrace{c\theta^2 \overbrace{c\phi^2 + c\theta^2 s\phi^2}^{1} + s\theta^2}_{=1} & -c\theta \cancel{s\theta} c\phi s\phi + c\theta \cancel{s\theta} s\phi c\phi \\ -s\theta \cancel{c\theta} s\phi c\phi + s\theta \cancel{c\theta} s\phi c\phi & \underbrace{s\theta^2 c\phi^2 + s\theta^2 s\phi^2}_{1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$
$$= d\Omega^2$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times 3} \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\begin{pmatrix} \underbrace{c\theta^2 \overbrace{c\phi^2 + c\theta^2 s\phi^2}^{=1} + s\theta^2}_{=1} & -c\theta \cancel{s\theta} c\phi s\phi + c\theta \cancel{s\theta} s\phi c\phi \\ -s\theta \cancel{c\theta} s\phi c\phi + s\theta \cancel{c\theta} s\phi c\phi & s\theta^2 \underbrace{c\phi^2 + s\theta^2 s\phi^2}_{=1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Now consider a vector on S^2:

$V = (V^\theta, V^\phi)$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$
$y = \sin\theta \sin\phi$
$z = \cos\theta$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Now consider a vector on S^2:

$V = (V^\theta, V^\phi)$ then $\phi_* V = (W^x, W^y, W^z)$

and $W^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu = (\phi_*)^\alpha{}_\mu V^\mu$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$
$y = \sin\theta \sin\phi$
$z = \cos\theta$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Now consider a vector on S^2:

$V = (V^\theta, V^\phi)$ then $\phi_* V = (W^x, W^y, W^z)$

and $W^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu = (\phi_*)^\alpha{}_\mu V^\mu$

$$W^x = \frac{\partial x}{\partial \theta} V^\theta + \frac{\partial x}{\partial \phi} V^\phi = \cos\theta \cos\phi \, V^\theta - \sin\theta \sin\phi \, V^\phi$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$$x = \sin\theta\cos\phi$$
$$y = \sin\theta\sin\phi$$
$$z = \cos\theta$$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

Now consider a vector on S^2:

$V = (V^\theta, V^\phi)$ then $\phi_* V = (W^x, W^y, W^z)$

and $W^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu = (\phi_*)^\alpha{}_\mu V^\mu$

$$W^x = \frac{\partial x}{\partial\theta} V^\theta + \frac{\partial x}{\partial\phi} V^\phi = \cos\theta\cos\phi\, V^\theta - \sin\theta\sin\phi\, V^\phi$$

$$W^y = \frac{\partial y}{\partial\theta} V^\theta + \frac{\partial y}{\partial\phi} V^\phi = \cos\theta\sin\phi\, V^\theta + \sin\theta\cos\phi\, V^\phi$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$
$y = \sin\theta \sin\phi$
$z = \cos\theta$

$$(\phi^* g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$d\tilde{s}^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

Now consider a vector on S^2:

$V = (V^\theta, V^\phi)$ then $\phi_* V = (W^x, W^y, W^z)$

and $W^\alpha = \dfrac{\partial y^\alpha}{\partial x^\mu} V^\mu = (\phi_*)^\alpha{}_\mu V^\mu$

$$W^x = \frac{\partial x}{\partial\theta} V^\theta + \frac{\partial x}{\partial\phi} V^\phi = \cos\theta \cos\phi \, V^\theta - \sin\theta \sin\phi \, V^\phi$$

$$W^y = \frac{\partial y}{\partial\theta} V^\theta + \frac{\partial y}{\partial\phi} V^\phi = \cos\theta \sin\phi \, V^\theta + \sin\theta \cos\phi \, V^\phi$$

$$W^z = \frac{\partial z}{\partial\theta} V^\theta + \frac{\partial z}{\partial\phi} V^\phi = -\sin\theta \, V^\theta + 0$$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$

$W^x = \cos\theta\cos\phi\, V^\theta - \sin\theta\sin\phi\, V^\phi$

$W^y = \cos\theta\sin\phi\, V^\theta + \sin\theta\cos\phi\, V^\phi$

$W^z = -\sin\theta \quad V^\theta$

Coordinate vectors: $\partial_\theta = (1,0)$ $\quad \partial_\phi = (0,1)$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$z = \cos\theta$

$W^x = \cos\theta\cos\phi\, V^\theta - \sin\theta\sin\phi\, V^\phi$

$W^y = \cos\theta\sin\phi\, V^\theta + \sin\theta\cos\phi\, V^\phi$

$W^z = -\sin\theta \quad V^\theta$

Coordinate vectors: $\partial_\theta = (1,0) \quad \partial_\phi = (0,1)$

$(\phi_* \partial_\theta)^x = \cos\theta\cos\phi$

$(\phi_* \partial_\theta)^y = \cos\theta\sin\phi$

$(\phi_* \partial_\theta)^z = -\sin\theta$

$(\partial_\theta)^\theta = 1$

$(\partial_\theta)^\phi = 0$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$

$W^x = \cos\theta\cos\phi\, V^\theta - \sin\theta\sin\phi\, V^\phi$

$W^y = \cos\theta\sin\phi\, V^\theta + \sin\theta\cos\phi\, V^\phi$

$W^z = -\sin\theta \quad V^\theta$

Coordinate vectors: $\partial_\theta = (1,0) \quad \partial_\phi = (0,1)$

$(\phi_* \partial_\theta)^x = \cos\theta\cos\phi$

$(\phi_* \partial_\theta)^y = \cos\theta\sin\phi$

$(\phi_* \partial_\theta)^z = -\sin\theta$

$(\partial_\theta)^\theta = 1$

$(\partial_\theta)^\phi = 0$

$(\phi_* \partial_\phi)^x = -\sin\theta\sin\phi$

$(\phi_* \partial_\phi)^y = \sin\theta\cos\phi$

$(\phi_* \partial_\phi)^z = 0$

$(\partial_\phi)^\theta = 0$

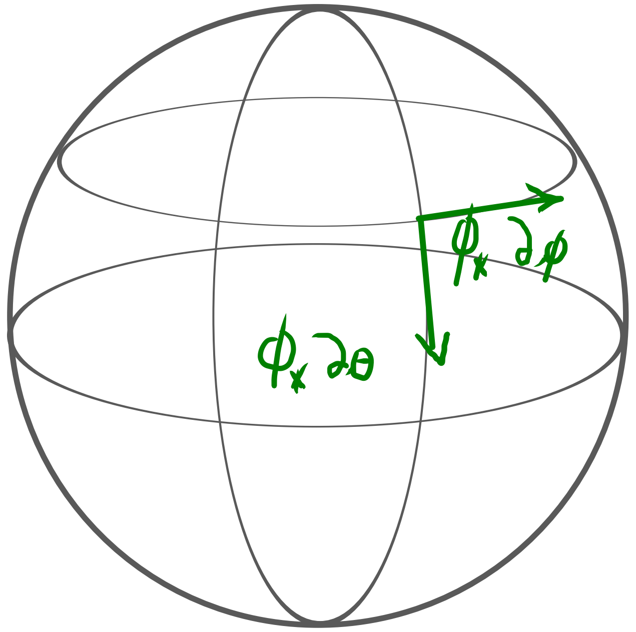
$(\partial_\phi)^\phi = 1$

Compute $(\phi^* g)_{\mu\nu}$: induced metric on S^2

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$



$\vec{e}_\theta = \phi_* \partial_\theta \qquad \vec{e}_\phi = \phi_* \partial_\phi$

Coordinate vectors: $\partial_\theta = (1, 0) \quad \partial_\phi = (0, 1)$

$(\phi_* \partial_\theta)^x = \cos\theta \cos\phi$

$(\phi_* \partial_\theta)^y = \cos\theta \sin\phi$

$(\phi_* \partial_\theta)^z = -\sin\theta$

$(\partial_\theta)^\theta = 1$

$(\partial_\theta)^\phi = 0$

$(\phi_* \partial_\phi)^x = -\sin\theta \sin\phi$

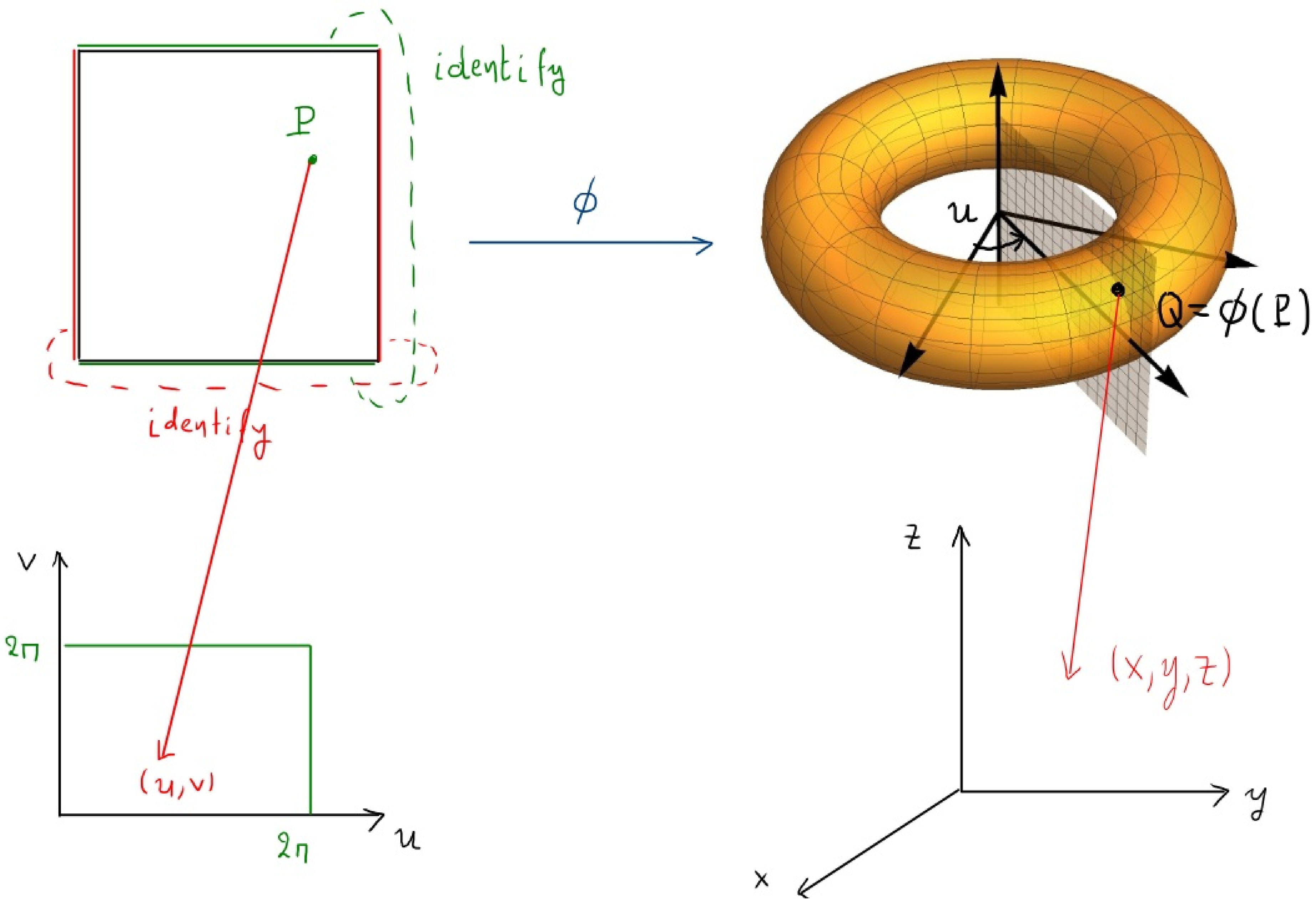
$(\phi_* \partial_\phi)^y = \sin\theta \cos\phi$

$(\phi_* \partial_\phi)^z = 0$

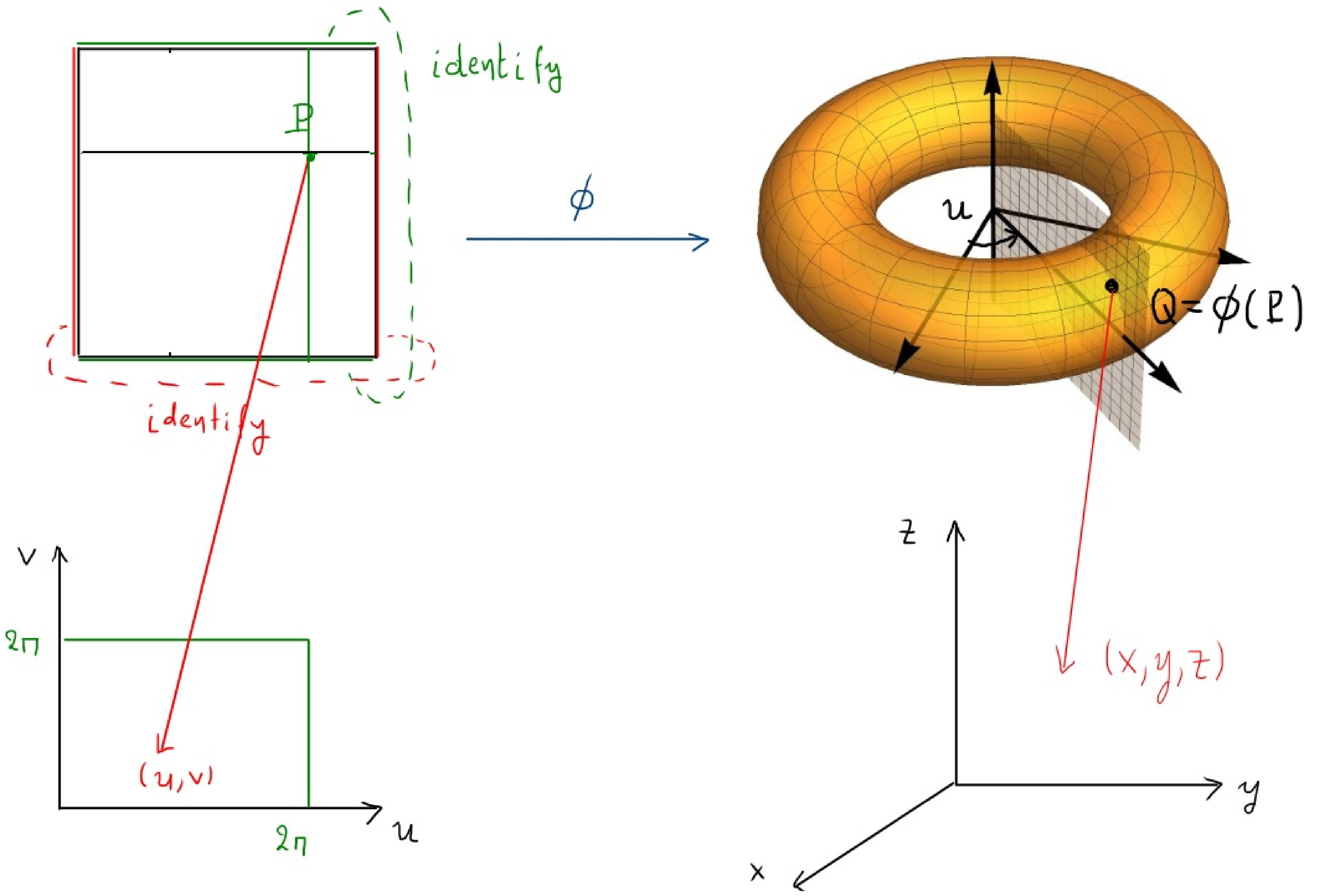
$(\partial_\phi)^\theta = 0$

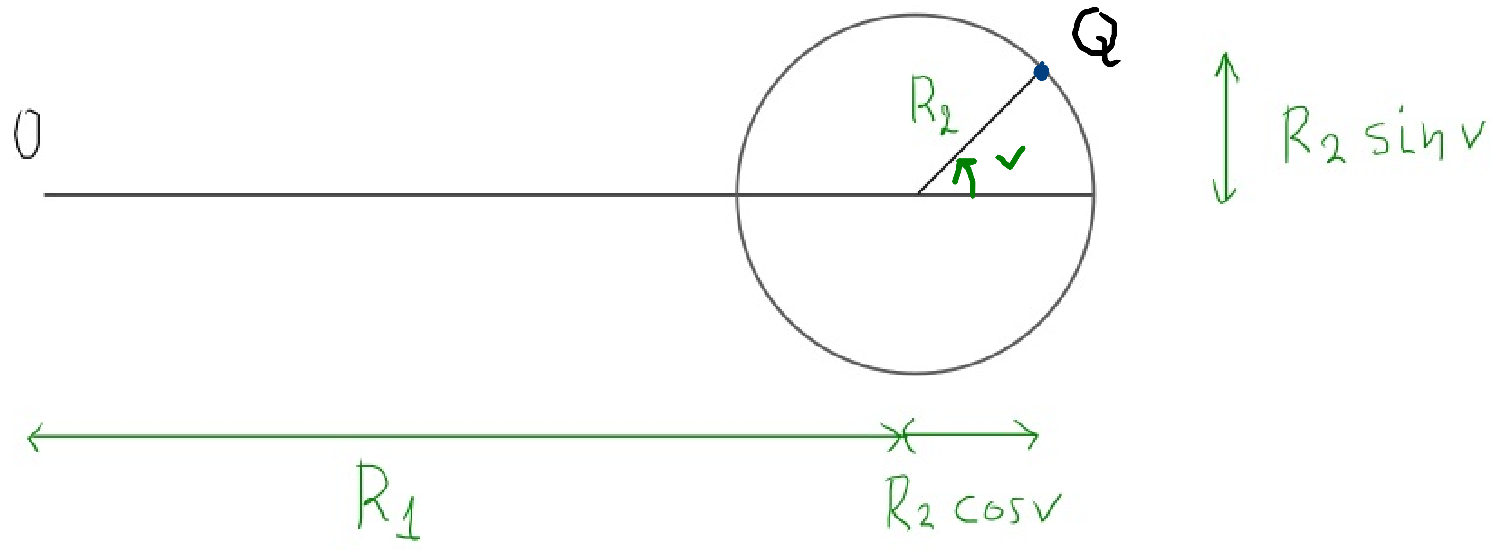
$(\partial_\phi)^\phi = 1$

Torus $T^2 \to \mathbb{R}^3$

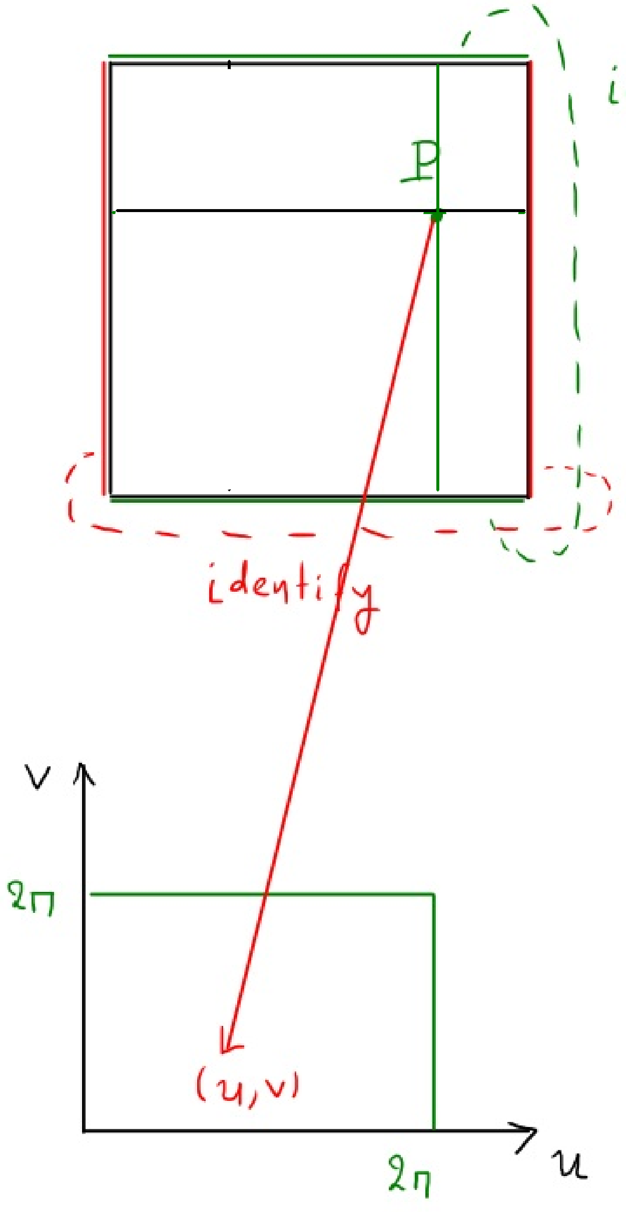


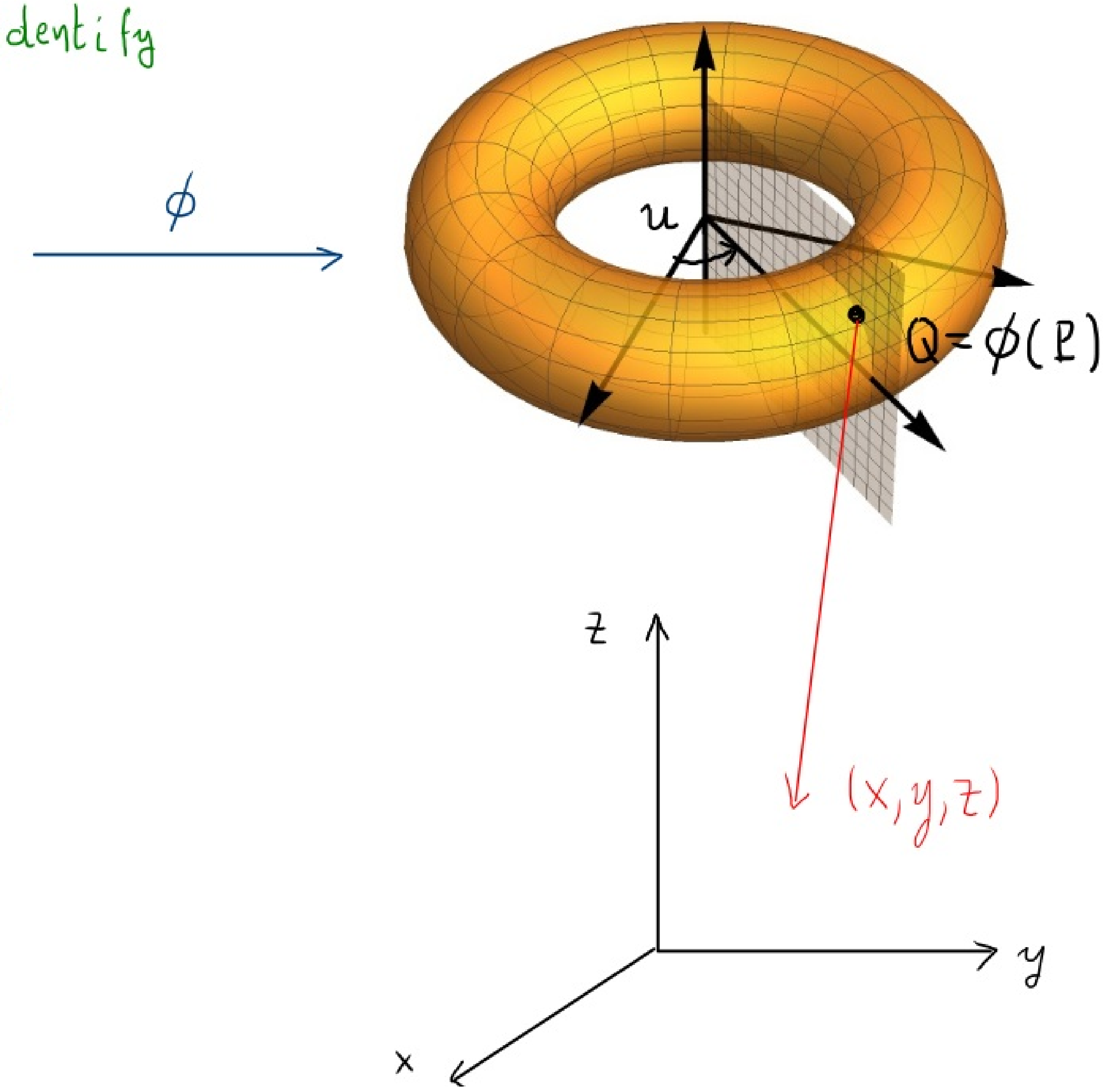
Torus $T^2 \to \mathbb{R}^3$

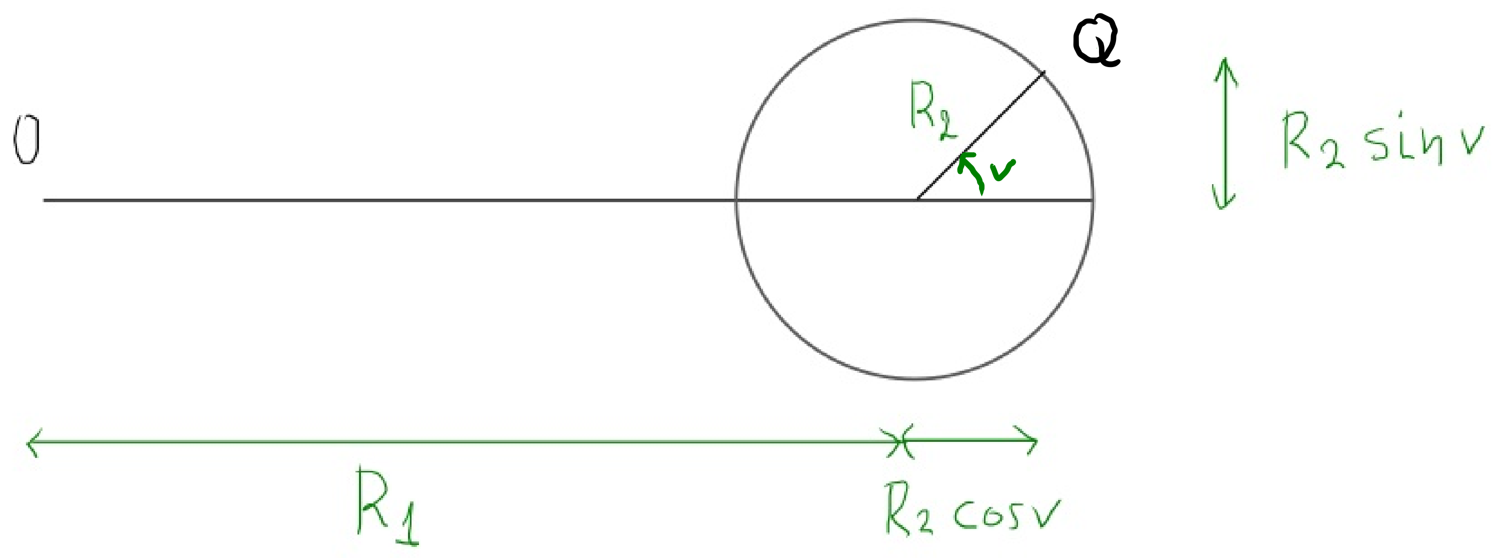




Torus $T^2 \to \mathbb{R}^3$







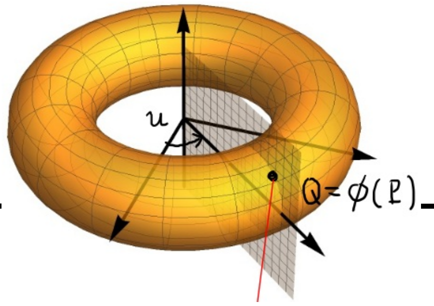
$$x = (R_1 + R_2 \cos v) \cos u$$

$$y = (R_1 + R_2 \cos v) \sin u$$

$$z = R_2 \sin v$$

$$0 \leq u < 2\pi \qquad 0 \leq v < 2\pi$$

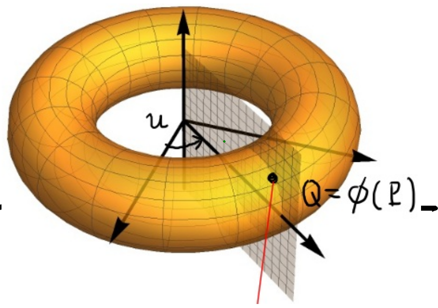
Torus $T^2 \to \mathbb{R}^3$



$\mathcal{M} = T^2 \qquad \mathcal{N} = \mathbb{R}^3$

$x^\mu = (u, v) \qquad y^\alpha = (x, y, z)$

Torus $T^2 \to \mathbb{R}^3$

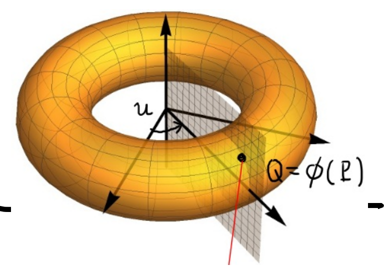


$\mathcal{M} = T^2 \qquad \mathcal{N} = \mathbb{R}^3$

$x^\mu = (u, v) \qquad y^\alpha = (x, y, z)$

$$x = R_1 \cos u + R_2 \cos u \cos v$$
$$y = R_1 \sin u + R_2 \sin u \cos v$$
$$z = R_2 \sin v$$

Torus $T^2 \to \mathbb{R}^3$



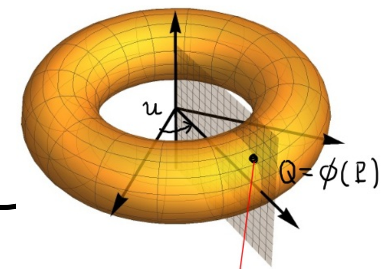
$\mathcal{M} = T^2 \qquad N = \mathbb{R}^3$

$x^\mu = (u, v) \qquad y^\alpha = (x, y, z)$

$$x = R_1 \cos u + R_2 \cos u \cos v$$
$$y = R_1 \sin u + R_2 \sin u \cos v$$
$$z = R_2 \sin v$$

$$\phi^* = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\[2ex] \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

Torus $T^2 \to \mathbb{R}^3$



$\mathcal{M} = T^2 \qquad \mathcal{N} = \mathbb{R}^3$

$x^\mu = (u, v) \qquad y^\alpha = (x, y, z)$

$x = R_1 \cos u + R_2 \cos u \cos v$

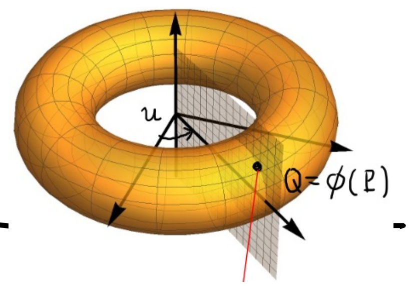
$y = R_1 \sin u + R_2 \sin u \cos v$

$z = R_2 \sin v$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

$$= \begin{pmatrix} -R_1 \sin u - R_2 \sin u \cos v & R_1 \cos u + R_2 \cos u \cos v & 0 \\ -R_2 \cos u \sin v & -R_2 \sin u \sin v & R_2 \cos v \end{pmatrix}$$

Torus $T^2 \to \mathbb{R}^3$



$\mathcal{M} = T^2 \qquad \mathcal{N} = \mathbb{R}^3$

$x^\mu = (u, v) \qquad y^\alpha = (x, y, z)$

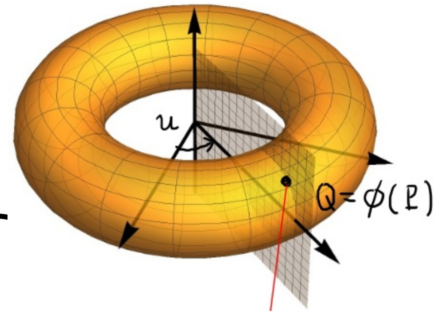
$$x = R_1 \cos u + R_2 \cos u \cos v$$
$$y = R_1 \sin u + R_2 \sin u \cos v$$
$$z = R_2 \sin v$$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

$$= \begin{pmatrix} -R_1 \sin u - R_2 \sin u \cos v & R_1 \cos u + R_2 \cos u \cos v & 0 \\ -R_2 \cos u \sin v & -R_2 \sin u \sin v & R_2 \cos v \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Torus $T^2 \to \mathbb{R}^3$



$\mathcal{M} = T^2 \qquad \mathcal{N} = \mathbb{R}^3$

$x^\mu = (u,v) \qquad y^\alpha = (x,y,z)$

$x = R_1 \cos u + R_2 \cos u \cos v$

$y = R_1 \sin u + R_2 \sin u \cos v$

$z = R_2 \sin v$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

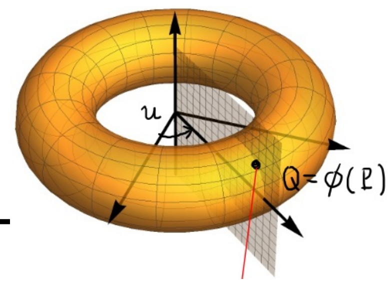
$$= \begin{pmatrix} -R_1 \sin u - R_2 \sin u \cos v & R_1 \cos u + R_2 \cos u \cos v & 0 \\ -R_2 \cos u \sin v & -R_2 \sin u \sin v & R_2 \cos v \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times 3} \cdot (\phi^*)^T = \phi^* (\phi^*)^T$$

$2\times 2 = 2\times 3 \quad 3\times 3 \quad 3\times 2$

Torus $T^2 \to \mathbb{R}^3$



$$\phi^* g = \begin{pmatrix} (R_1 + R_2 \cos v)^2 & 0 \\ 0 & R_2^2 \end{pmatrix}$$

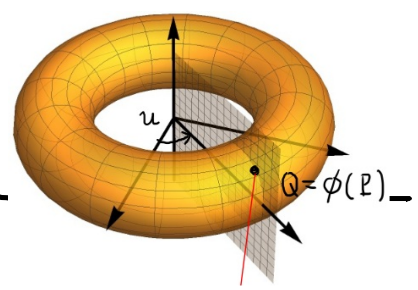
$$= \begin{pmatrix} -R_1 \sin u - R_2 \sin u \cos v & R_1 \cos u + R_2 \cos u \cos v & 0 \\ -R_2 \cos u \sin v & -R_2 \sin u \sin v & R_2 \cos v \end{pmatrix}$$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times 3} \cdot (\phi^*)^T = \phi^* (\phi^*)^T$$

Torus $T^2 \to \mathbb{R}^3$



$$\phi^* g = \begin{pmatrix} (R_1 + R_2 \cos v)^2 & 0 \\ 0 & R_2^2 \end{pmatrix}$$

$$d\tilde{s}^2 = (R_1 + R_2 \cos v)^2 \, du^2 + R_2^2 \, dv^2$$

not a flat metric! $\quad R = \dfrac{2\cos v}{R_2^2 \cos v + R_1 R_2}$

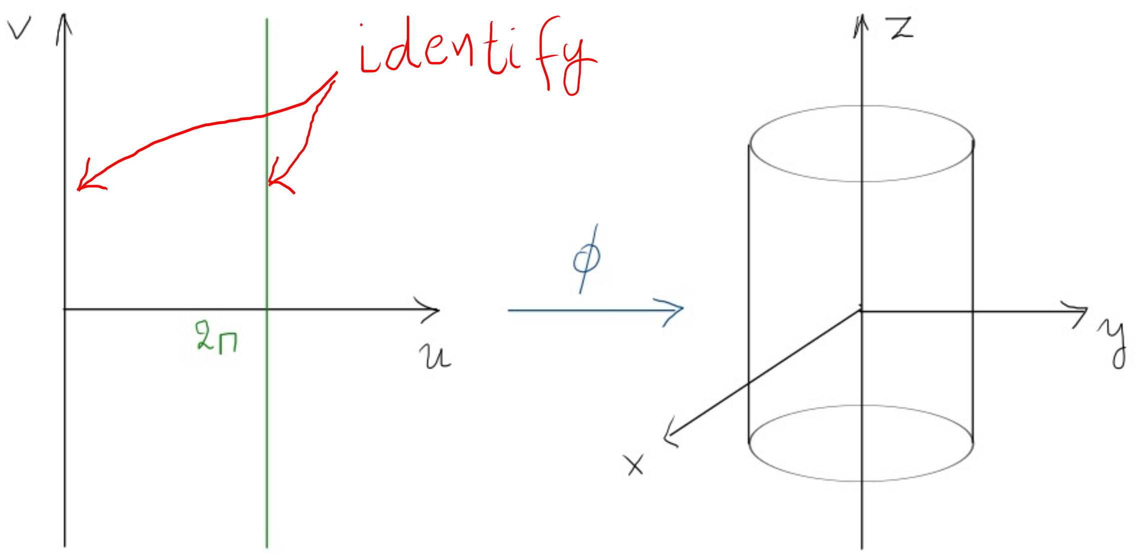
$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (g_{\alpha\beta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot 1_{3\times 3} \cdot (\phi^*)^T = \phi^* (\phi^*)^T$$

use Mathematica to calculate!

Cylinder $\to \mathbb{R}^3$

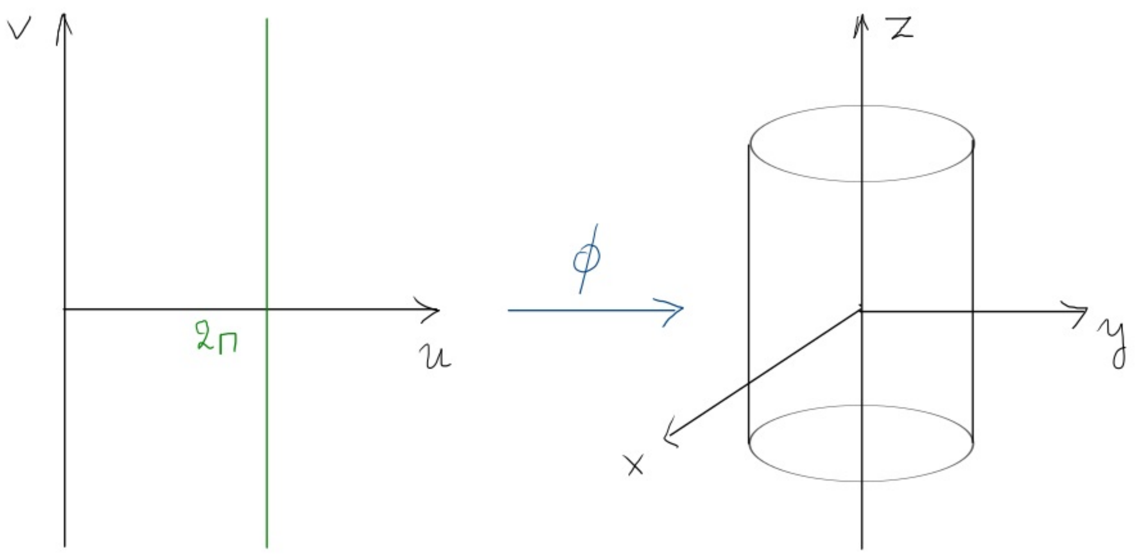


$x = R\cos u \qquad 0 \leq u < 2\pi$

$y = R\sin u \qquad -\infty < v < +\infty$

$z = v$

Cylinder $\to \mathbb{R}^3$



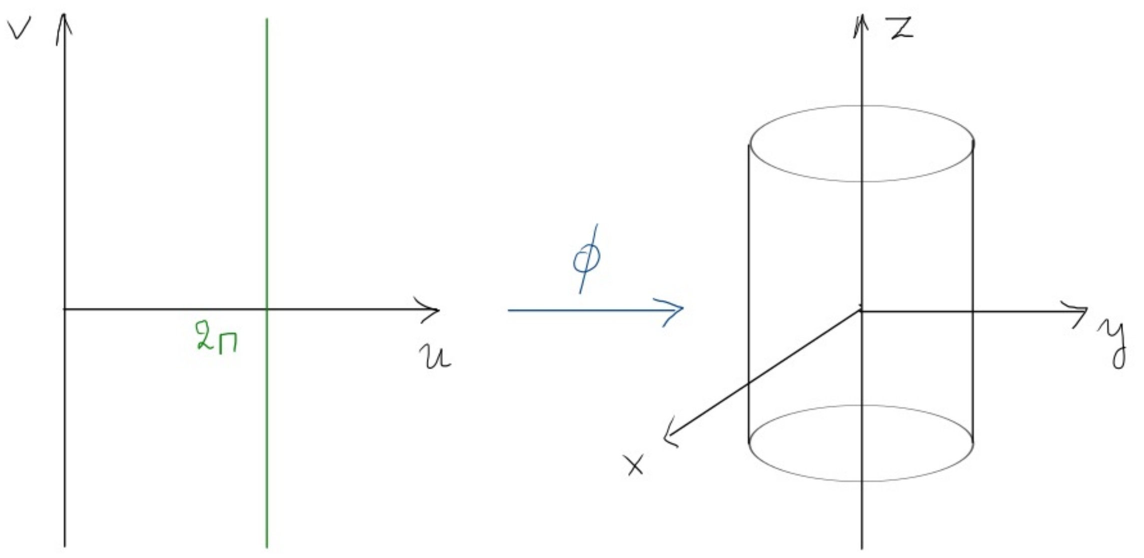
$x = R\cos u \qquad 0 \le u < 2\pi$

$y = R\sin u \qquad -\infty < v < +\infty$

$z = v$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Cylinder $\to \mathbb{R}^3$



$$x = R\cos u \qquad 0 \le u < 2\pi$$

$$y = R\sin u \qquad -\infty < v < +\infty$$

$$z = v$$

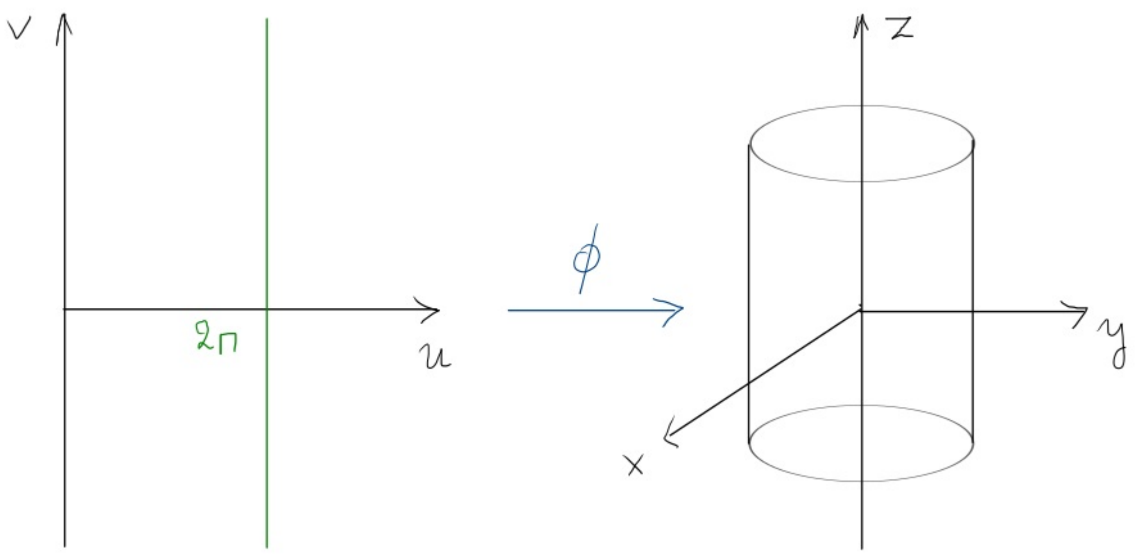
$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

$$\begin{pmatrix} -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overset{3\times 3}{\phi^* g} = \overset{2\times 3}{\phi^*} \cdot \overset{3\times 3}{g} \cdot \overset{3\times 2}{(\phi^*)^T} = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Cylinder $\to \mathbb{R}^3$



$$d\tilde{s}^2 = R^2\, du^2 + dv^2$$

$$\phi^* = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix} =$$

$$\begin{pmatrix} -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi^* g = \phi^* \cdot g \cdot (\phi^*)^T = \phi^* \cdot (\phi^*)^T$$

$$= \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}$$

will use Mathematica

de Sitter $dS_4 \to \mathbb{R}^5$

$N = dS_4 \quad M = \mathbb{R}^5$

$x^\mu = (t, \chi, \theta, \phi)$

$y^\alpha = (u, w, x, y, z)$

$ds_5^2 = -du^2 + dw^2 + dx^2 + dy^2 + dz^2$

(Minkowski metric)

de Sitter $dS_4 \to \mathbb{R}^5$

$\mathcal{N} = dS_4 \quad \mathcal{M} = \mathbb{R}^5$

$x^\mu = (t, \chi, \theta, \phi)$

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$ds_5^2 = -du^2 + dw^2 + dx^2 + dy^2 + dz^2$

(Minkowski metric)

Embedded surface is hyperboloid:

$-u^2 + w^2 + x^2 + y^2 + z^2 = R^2$

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(Minkowski metric)

Embedded surface is hyperboloid:

$-u^2 + w^2 + x^2 + y^2 + z^2 = R^2$

$u = R \sinh\left(\frac{t}{R}\right)$

$w = R \cosh\left(\frac{t}{R}\right) \cos\chi$

$x = R \cosh\left(\frac{t}{R}\right) \sin\chi \cos\theta$

$y = R \cosh\left(\frac{t}{R}\right) \sin\chi \sin\theta \cos\phi$

$z = R \cosh\left(\frac{t}{R}\right) \sin\chi \sin\theta \sin\phi$

$-\infty < t < +\infty$

$0 \leq \chi < 2\pi$

$0 \leq \theta < \pi$

$0 \leq \phi < 2\pi$

de Sitter $dS_4 \to \mathbb{R}^5$

ch → cosh c → cos s → sin

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & R^2 ch^2\left(\frac{t}{R}\right) & 0 & 0 \\ 0 & 0 & R^2 ch^2\left(\frac{t}{R}\right) s\chi^2 & 0 \\ 0 & 0 & 0 & R^2 ch^2\left(\frac{t}{R}\right) s\chi^2 s\theta^2 \end{pmatrix}$$

$$\underset{4\times 4}{\phi^* g} = \underset{4\times 5}{\phi^*} \cdot \underset{5\times 5}{g} \cdot \underset{5\times 4}{(\phi^*)^T}$$

careful! not unit matrix now...

$$u = R \sinh\left(\frac{t}{R}\right)$$

$$w = R \cosh\left(\frac{t}{R}\right) \cos\chi$$

$$x = R \cosh\left(\frac{t}{R}\right) \sin\chi \cos\theta$$

$$y = R \cosh\left(\frac{t}{R}\right) \sin\chi \sin\theta \cos\phi$$

$$z = R \cosh\left(\frac{t}{R}\right) \sin\chi \sin\theta \sin\phi$$

Compute $\phi^* = \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)$

use Mathematica!

Anti de Sitter: $AdS_4 \to \mathbb{R}^5$

$\mathcal{N} = AdS_4 \quad \mathcal{M} = \mathbb{R}^5$

$x^\mu = (t, \chi, \theta, \phi)$

$y^\alpha = (u, w, x, y, z)$

$0 \leq t < 2\pi$: periodic!

Anti de Sitter: $AdS_4 \to \mathbb{R}^5$

$N = AdS_4 \qquad M = \mathbb{R}^5$

$x^\mu = (t, x, \theta, \phi)$

$y^\alpha = (u, w, x, y, z)$

$0 \leq t < 2\pi$: periodic!

embed hyperboloid:

$-u^2 - w^2 + x^2 + y^2 + z^2 = -R^2$

Anti de Sitter: $AdS_4 \to \mathbb{R}^5$

$\mathcal{N} = AdS_4 \qquad \mathcal{M} = \mathbb{R}^5$

$x^\mu = (t, r, \theta, \phi)$

$y^\alpha = (u, w, x, y, z)$

$0 \leq t < 2\pi$: periodic!

embed hyperboloid:

$-u^2 - w^2 + x^2 + y^2 + z^2 = -R^2$

metric in $\mathbb{R}^5$:

$ds_5^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$

Anti de Sitter: $AdS_4 \to \mathbb{R}^5$

$N = AdS_4 \qquad M = \mathbb{R}^5$

$x^\mu = (t, \chi, \theta, \phi)$

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$0 \leq t < 2\pi$: periodic!

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metric in $\mathbb{R}^5$:

$$ds_5^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$$

$$u = R \sin t \cosh\chi \qquad 0 \leq t < 2\pi$$

$$w = R \cos t \cosh\chi \qquad -\infty < \chi < +\infty$$

$$x = R \sinh\chi \cos\theta \qquad 0 \leq \theta < \pi$$

$$y = R \sinh\chi \sin\theta \cos\phi \qquad 0 \leq \phi < 2\pi$$

$$z = R \sinh\chi \sin\theta \sin\phi$$

Anti de Sitter: $AdS_4 \to \mathbb{R}^5$

$\mathcal{N} = AdS_4 \quad \mathcal{M} = \mathbb{R}^5$

$x^\mu = (t, \chi, \theta, \phi)$

$y^\alpha = (u, w, x, y, z)$

$0 \leq t < 2\pi$: periodic!

embed hyperboloid:

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metric in $\mathbb{R}^5$:

$ds_5^2 = -du^2 - dw^2 + dx^2 + dy^2 + dz^2$

compute $\phi^* = \left(\frac{\partial y^\alpha}{\partial x^\mu} \right)$ $\rightsquigarrow$ use Mathematica!

$$\underset{4\times 4}{\phi^* g} = \underset{4\times 5}{\phi^*} \cdot \underset{5\times 5}{g} \cdot \underset{5\times 4}{(\phi^*)^T} =$$

$$\begin{pmatrix} -R^2 ch^2\chi & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & R^2 sh^2\chi & 0 \\ 0 & 0 & 0 & R^2 sh^2\chi \, s\theta^2 \end{pmatrix}$$