Monte Carlo Studies of the Dimensionally Reduced 4d $SU(N)$ Super Yang-Mills Theory

J. Ambjørn [a], K.N. Anagnostopoulos [b], W. Bietenholz [c], T. Hotta [d] and J. Nishimura [a] *

[a] Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
[b] Dept. of Physics, Univ. of Crete, P.O. Box 2208, GR-71003 Heraklion, Greece
[c] Institut für Physik, Humboldt Universität, Invalidenstr. 110, D-10115 Germany
[d] Institute of Physics, Univ. of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan

We simulate a supersymmetric matrix model obtained from dimensional reduction of 4d SU(N) super Yang-Mills theory. The model is well defined for finite N and it is found that the large N limit obtained by keeping g^2N fixed gives rise to well defined operators which represent string amplitudes. The space-time structure which arises dynamically from the eigenvalues of the bosonic matrices is discussed, as well as the effect of supersymmetry on the dynamical properties of the model. Eguchi-Kawai equivalence of this model to ordinary gauge theory does hold within a finite range of scale. We report on new simulations of the bosonic model for N up to 768 that confirm this property, which comes as a surprise since no quenching or twist is introduced.

1. Introduction

Recent excitement in string theory stems from the fact that known string theories are thought to be perturbative expansions of an 11 dimensional theory called M-theory. The former are believed to be related by dualities and once we construct a non–perturbative definition of one of them, then we can also describe the vacua of any of the other theories. Two models, the IKKT [1] and BFSS models [2], have been proposed as possible definitions of M–theory. Both models are thought to be closely related [3]. For analytic work in this context, see Ref. [4]. The IKKT model (or IIB matrix model) [1] is a candidate for a constructive definition of non–perturbative type IIB string theory. If the model possesses a unique vacuum this should describe the space-time in which we live. Space–time arises dynamically in this model and one can in principle predict its dimensionality and low energy geometry. It has even been argued that the gauge group and matter content of our world can arise from the solution of this model [5].

The IKKT model is a reduction of the 10d $SU(N)$ super Yang–Mills theory to a point, i.e. we restrict the path integral to be only over constant field configurations. The model that we investigate in this talk is a 4d counterpart of the IKKT model. Although it is much simpler than the original model we hope to capture the essential dynamical features of the 10d model through full scale numerical simulations which will yield information about non–perturbative properties in $d = 4$. Simulations in four dimensions are possible because the model does not suffer from the sign problem, unlike its higher dimensional cousins. Several important issues can be studied in depth like the well definiteness of the model at finite N, the large N limit, the space–time structure and the role of supersymmetry. We can also address the important dynamical issue of the equivalence of the matrix model to the original large–N gauge theory in the sense of Eguchi and Kawai [6]. We report on large scale simulations of the supersymmetric and the bosonic model – obtained by omitting the fermions in the action [7]. By using a carefully constructed hybrid-R algorithm [8] the computational effort in the SUSY case increases only as N^5 and we are thus able to simulate systems with size up to $N = 48$. The bosonic model is simulated as in Ref. [9] up to $N = 768$. The results for such large N have not been published yet.

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2. The model

The IIB matrix model is given by

$$\begin{aligned} \mathbf{Z} &= \int dA\; e^{-S_b} \int d\bar{\psi} d\psi\; e^{-S_f} \\ S_b &= -\frac{1}{4g^2}\,\mathrm{Tr}[A_\mu, A_\nu]^2 \\ S_f &= -\frac{1}{g^2}\,\mathrm{Tr}(\bar{\psi}_\alpha \Gamma^\mu_{\alpha\beta}[A_\mu, \psi_\beta]) \end{aligned} \tag{1}$$

where A_μ, $\bar{\psi}_\alpha$, ψ_α ($\mu = 1 \ldots d$, $\alpha = 1 \ldots 2^{\frac{d}{2}-1}$) are complex, traceless $N \times N$ matrices. In our case, $d = 4$, the A_μ (only) are Hermitian and we use $\vec{\Gamma} = i\vec{\sigma}$, $\Gamma_4 = \mathbb{1}$.

The model has $SO(d)$ rotational symmetry, which is the euclidean version of the Lorentz invariance of the original model before reduction. The $SU(N)$ gauge invariance of the non–reduced model becomes

$$\begin{aligned} A_\mu &\rightarrow V A_\mu V^\dagger \\ \psi_\alpha &\rightarrow V \psi_\alpha V^\dagger\,, \quad V \in SU(N) \\ \bar{\psi}_\alpha &\rightarrow V \bar{\psi}_\alpha V^\dagger\,. \end{aligned} \tag{2}$$

The $\mathcal{N} = 1$ supersymmetry of the non–reduced model takes the form

$$\begin{aligned} \delta^{(1)} A_\mu &= i\bar{\epsilon}_1 \Gamma_\mu \psi \\ \delta^{(1)} \psi &= \frac{i}{2} \Gamma^{\mu\nu} [A_\mu, A_\nu] \epsilon_1 \end{aligned} \tag{3}$$

whereas *after reduction* the model acquires a second supersymmetry

$$\begin{aligned} \delta^{(2)} A_\mu &= 0 \\ \delta^{(2)} \psi &= \epsilon_2\,. \end{aligned} \tag{4}$$

The supercharges can be combined to $\tilde{Q}_1 = Q^{(1)} + Q^{(2)}$ and $\tilde{Q}_2 = i(Q^{(1)} - Q^{(2)})$ which obey the commutation relation

$$[\bar{\epsilon}_1 \tilde{Q}_i, \bar{\epsilon}_2 \tilde{Q}_j] = -2\bar{\epsilon}_1 \Gamma_\mu \epsilon_2 p_\mu \delta_{ij}\,, \tag{5}$$

where p_μ is the generator of the transformation $\delta^t A_\mu = c_\mu \mathbf{1}_N$. The latter is a symmetry of the action which also appears after reduction. Eq. (5) suggests that the eigenvalues of the bosonic matrices A_μ can be interpreted as space–time points. In this context, p_μ is the space–time translation operator.

By adopting the above interpretation of the model there are several reasons to believe that the IKKT model ($d = 10$) is related to type IIB superstrings. First, by using the semiclassical correspondence [1] one obtains the action of the Green-Schwarz type IIB superstring in the so–called Schild gauge [10]. Second, classical solutions corresponding to D-strings are constructed in Ref. [1]. An arbitrary number of D-strings and anti D-strings can be described as blocks of matrices which interact via the off diagonal blocks which can change their number and size. Thus the model can be interpreted to contain a second quantized theory of D-strings. Third the authors of Ref. [11] have obtained the string field theory supercharges and Hamiltonian of the type IIB string in the light cone gauge from the Schwinger–Dyson equations (which describe joining and splitting of *fundamental* strings) that the Wilson loops obey by using only the $\mathcal{N} = 2$ supersymmetry and scaling arguments. The low energy physics that one obtains from the model remains a mystery. The authors in Ref. [5] have proposed that the space–time metric is encoded in the density correlations of the eigenvalues and that diffeomorphism invariance stems from the invariance of the model under permutations of the eigenvalues. They also suggest that the gauge group is obtained from the clustering of eigenvalues in clusters of size n. Then the low energy theory acquires $SU(n)$ local space-time gauge symmetry.

The first question about our 4d model is whether it is well-defined as it stands. Since the integration domain of dA is non-compact, divergences are conceivable. However, our results [7] confirm the original results of Ref. [12] for SUSY — and they agree with very recent analytic results for the bosonic case [13] — that this model *is* well-defined for large enough N; there is no need to impose an IR cutoff. This implies that the only parameter g is simply a *scale parameter* that the theory determines dynamically. It can be absorbed by introducing dimensionless quantities

$$X_\mu = A_\mu / g^{1/2}\;; \quad \Psi_\alpha = \psi_\alpha / g^{3/4}\,. \tag{6}$$

3. Numerical Simulations

For our simulation we start by integrating out the fermionic variables which can be done explicitly [7]. The result is given by $\det \mathcal{M}$, $\mathcal{M}$ being a $2(N^2-1) \times 2(N^2-1)$ complex matrix which depends on A_μ. Hence the system we want to simulate can be written in terms of bosonic variables as

$$Z = \int \mathrm{d}A \; \mathrm{e}^{-S_b} \det \mathcal{M} \; . \tag{7}$$

A crucial point for the present work is that the determinant $\det \mathcal{M}$ is actually real positive. This is shown explicitly in Ref. [7]. Due to this property, we can introduce a $2(N^2-1) \times 2(N^2-1)$ Hermitian positive matrix $\mathcal{D} = \mathcal{M}^\dagger \mathcal{M}$, so that $\det \mathcal{M} = \sqrt{\det \mathcal{D}}$, and the effective action of the system takes the form

$$S_{\mathrm{eff}} = S_b - \frac{1}{2} \ln \det \mathcal{D} \; . \tag{8}$$

We apply the Hybrid R algorithm [8] to simulate this system. In the framework of this algorithm, each update of a configuration is made by solving a Hamiltonian equation for a fixed "time" τ. The algorithm is plagued by a systematic error due to the discretization of τ that we used to solve the equation numerically. Special care is taken so that the systematic error is of order $\Delta\tau^2$, up to logarithmic corrections [7]. We performed simulations at three different values of the time step $\Delta\tau$. Except in Fig. 2, we find that the results do not depend much on $\Delta\tau$ (below a certain threshold), so we just present the results for the value $\Delta\tau = 0.002$, which appears to be sufficiently small. Extra care is taken so that the computational effort increases only as N^5 (in the bosonic case the corresponding effort increases only as N^3). Therefore for the supersymmetric case we were able to obtain 3060, 1508, 1296, 436 configurations for $N = 16, 24, 32, 48$ respectively. For the bosonic case, we used 1000 configurations for each N. The $N \leq 32$ simulations were performed on a linux farm at NBI and the $N = 48$ on the Fujitsu VPP500 at High Energy Accelerator Research Organization (KEK), the Fujitsu VPP700E at The Institute of Physical and Chemical Research (RIKEN), and the NEC SX4 at Research Center for Nuclear Physics (RCNP) of Osaka University supercomputers.

4. The space structure

In the IIB matrix model, the space coordinates arise dynamically from the eigenvalues of the matrices A_μ [1]. In general the latter cannot be diagonalized simultaneously, which implies that we deal with a non-classical space. We measure its uncertainty by

$$\Delta^2 = \frac{1}{N} \Big[\mathrm{Tr}(A_\mu^2) - \underset{U \in SU(N)}{max} \sum_i \{ (U A_\mu U^\dagger)_{ii} \}^2 \Big] \; ,$$

and the "maximizing" matrix U is also used for introducing the coordinates of N points,

$$x_{i,\mu} = (U_{max} A_\mu U_{max}^\dagger)_{ii} \qquad (i = 1 \ldots N). \tag{9}$$

What we are really interested in is their pairwise separation $r(x_i, x_j) = |x_i - x_j|$, and we show the distribution $\rho(r)$ in Fig. 1. We observe $\rho \approx 0$ at short distances ($r/\sqrt{g} \lesssim 1.5$), hence a UV cutoff is generated *dynamically*. We also see that increasing N favors larger values of r. To quantify this effect we measure the "extent of space"

$$R_{new} = \int_0^\infty r \rho(r) \; dr \; . \tag{10}$$

Fig. 2 shows R_{new} and Δ as functions of N (at $g = 1$). R_{new} is finite in contrast to the quantity $R = \sqrt{\langle \frac{1}{N} \mathrm{Tr} \; (A_\mu^2) \rangle} \sim \sqrt{\int_o^\infty dr \; r^2 \; \rho(r)}$ which diverges logarithmically as $\Delta\tau \to 0$. This is consistent with the prediction by Ref. [12], $\rho(r) \sim r^{-3}$ for large r. The inclusion of fermions enhances R_{new} and suppresses Δ, keeping their product approximately constant and $\sim g N^{1/2}$. We will see that the latter product remains finite in the large N scaling limit. It is a kind of uncertainty principle for space–time fluctuations. The lines in Fig. 2 show that both quantities follow the same power law, $R_{new}, \; \Delta \propto N^{1/4}$, in SUSY and in the bosonic case. In particular in the bosonic case $R_{new} = 1.56(1) g^{1/2} N^{1/4}$, $\Delta = 0.907(3) g^{1/2} N^{1/4}$ so that $\Delta^{1/2} \approx 0.58 R_{new}$. In the SUSY case $R_{new} = 3.30(1) g^{1/2} N^{1/4}$, $\Delta = 0.730(3) g^{1/2} N^{1/4}$ so that $\Delta^{1/2} \approx 0.22 R_{new}$. In SUSY this behavior is consistent with the branched polymer picture: there one would relate the number of points as $N \sim (R_{new}/\ell)^{d_H}$, where ℓ is some minimal bond,

which corresponds to the above UV cutoff. The Hausdorff dimension $d_H = 4$ then reveals consistency with our result. In the bosonic case the (same) exponent has a qualitatively different explanation. It originates from a logarithmic attractive potential between the eigenvalues of the matrices found in the one loop approximation of the model [9].

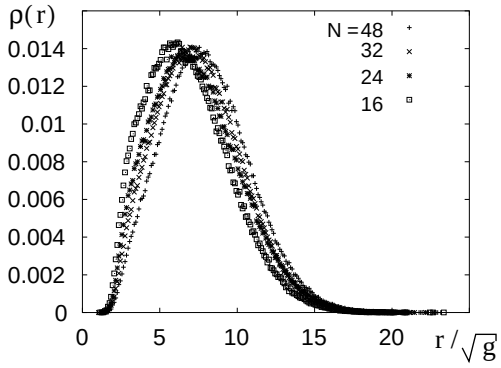

Figure 1. *The distribution of distances between space-points in the SUSY case at various N.*

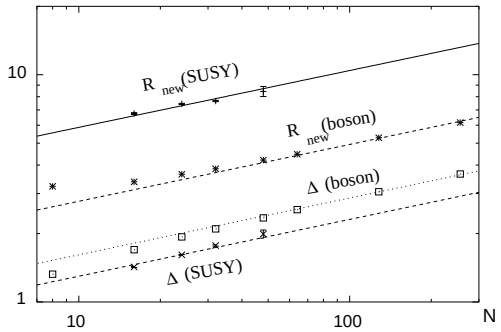

Figure 2. *The "extent of space" R_{new} and the space uncertainty Δ as functions of N at $g = 1$.*

5. Polyakov and Wilson loops

We define the Polyakov loop P and the Wilson loop W — which is conjectured to correspond to the string creation operator — as

$$P(p) = \frac{1}{N}\mathrm{Tr}\Big(e^{ipA_1}\Big), \quad (11)$$

$$W(p) = \frac{1}{N}\mathrm{Tr}\Big(e^{ipA_1}e^{ipA_2}e^{-ipA_1}e^{-ipA_2}\Big). \quad (12)$$

Of course the choice of the components of A_μ is irrelevant, and the parameter $p \in \mathbb{R}$ can be considered as a "momentum".

Now $g(N)$ has to be tuned so that $\langle P\rangle$, $\langle W\rangle$ remain finite as $N \to \infty$. This is achieved by

$$g \propto 1/\sqrt{N}\ , \quad (13)$$

which leads to a beautiful large N scaling; Fig. 3 shows the invariance of $\langle P\rangle$ for $N = 16 \ldots 48$ in SUSY. Also the bosonic case scales accurately [7].

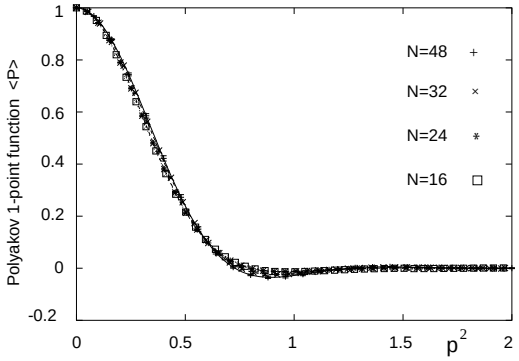

Figure 3. *The Polyakov function in the SUSY case for various values of N and $g^2N = const.$*

The historic 2d Eguchi-Kawai model [6] had a $(\mathbb{Z}_N)^2$ symmetry, which implied $\langle P(p \neq 0)\rangle = 0$, a property which was crucial for the proof of the Eguchi-Kawai equivalence to gauge theory. As we see, this property is not fulfilled here, but $\langle P(p)\rangle$ falls off rapidly, towards a regime where the assumption of this proof holds approximately.

We proceed to a more explicit test of Eguchi-Kawai equivalence by checking the area law for $\langle W(p)\rangle$. Fig. 4 shows that the area law seems to hold in a finite range of scale for the model with supersymmetry. Remarkably, the behavior is very similar [7] in the bosonic case[2]. There we further investigated the behavior at much larger N [14], and we observed that the power law regime does neither shrink to zero — as it was generally expected — nor extend to infinity — a scenario which seems possible from Fig. 4. At least in the bosonic case its range remains finite at large N as can be seen in Fig. 5.

[2]Recently the area law behavior was also observed in the 10d bosonic case [15].

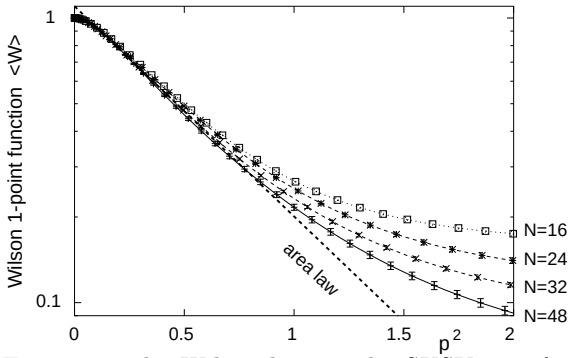

Figure 4. *The Wilson loop in the SUSY case for various values of N and $g^2N = const.$*

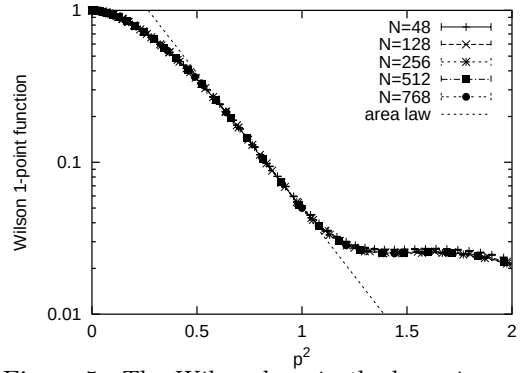

Figure 5. *The Wilson loop in the bosonic case for various values of N and $g^2N = const.$*

6. Multipoint functions

We now consider connected multipoint functions $\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_{con}$, $\mathcal{O}_i$ being a Polyakov or a Wilson loop. We wonder if it is possible to renormalize all of those multipoint functions simply by inserting $\mathcal{O}_i^{(ren)} = Z\mathcal{O}_i$, so that a single factor Z renders all functions $\langle \mathcal{O}_1^{(ren)} \mathcal{O}_2^{(ren)} \dots \mathcal{O}_n^{(ren)} \rangle_{con}$ (simultaneously) finite at large N.

It turns out that such a universal renormalization factor seems to exist in SUSY. We have to set again $g \propto 1/\sqrt{N}$, and then $Z \propto N$ provides large N scaling, as we observed for a set of 2, 3 and 4-point functions. Two examples are shown in Fig. 6. Our observation can be summarized by the SUSY rule

$$\langle \mathcal{O} \rangle = O(1) \ , \quad \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = O(N^{-n}) \quad (n \geq 2).$$

This implies that large N factorization holds, $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \langle \mathcal{O}_1 \rangle \dots \langle \mathcal{O}_n \rangle + O(N^{-2})$, as in gauge theory, although coupling expansions are not applicable here.

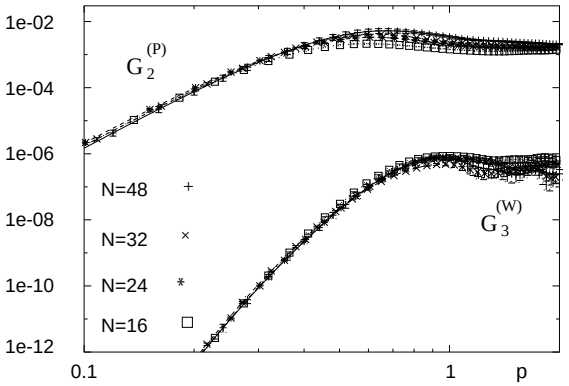

Figure 6. *A Polyakov 2-point function ($G_2^{(P)} = \langle [\mathrm{Im}P(k)]^2 \rangle$) and a Wilson 3-point function ($G_3^{(W)} = \langle [\mathrm{Im}W(k)]^2 \mathrm{Re}W(k) \rangle - \langle [\mathrm{Im}W(k)]^2 \rangle \langle \mathrm{Re}W(k) \rangle$) (both connected), with $g^2N = const.$ and renormalization factor $Z \propto N$, which leads to large N scaling in the SUSY case.*

For the bosonic case, a $1/d$ expansion [9] suggests large N factorization to hold as well, but it also predicts $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = O(N^{-2(n-1)})$ $(n \geq 2)$. This is confirmed numerically [7]: in particular the 3-point functions now require $Z^3 \propto N^4$. Therefore no universal renormalization factor Z exists in the bosonic case, which is an important qualitative difference from the SUSY case.

7. Conclusions

We reported results from numerical simulations of the 4d IIB matrix model, both, SUSY and bosonic. In the SUSY case we varied N up to 48, which turned out to be sufficient to study the large N dynamics.

We confirmed that the model is well-defined as it stands, hence g is a pure scale parameter. The space–time coordinates arise from eigenvalues of the bosonic matrices A_μ. The extent of space–time follows a power law in N with power of $1/4$. In SUSY this agrees with the branched polymer picture. Fermions leave the power unchanged but reduce the space–time uncertainty — though it

remains finite at large N. Space–time is quantum with the uncertainty in determining space–time points to scale together with the extent of space–time, surviving thus the large N limit.

The large N scaling of Polyakov and Wilson loops and their correlators requires $g \propto 1/\sqrt{N}$ in SUSY and in the bosonic case, but the wave function renormalization is qualitatively different: only in SUSY a universal renormalization exists. Using a rough argument presented in Ref. [7] this suggests that supersymmetry renders the world sheet smoother than in the bosonic case. Indeed such a phenomenon has been observed in the dynamical triangulation approach [16].

The area law for Wilson loops holds in a finite range of scale for the SUSY and the bosonic case. The latter comes as a surprise, and we checked up to $N = 768$ that this range remains indeed finite. Hence Eguchi-Kawai equivalence to ordinary gauge theory [6], even *without* quenching or twist, may hold in some regime.

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