Unitary Matrix Models:

A Study of

the String Equation

by

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ABSTRACT

In this thesis I review the Symmetric Unitary One Matrix Models (UMM). In the beginning, I discuss matrix models in general, with particular emphasis on their relation to string theory and two dimensional quantum gravity. The crux of matrix models lies in a single ordinary non-linear differential equation which, in a certain limit known as the double scaling limit, embodies the entire dynamical content of the continuum theory. This differential equation, called the string equation, may be solved and analyzed, yielding much insight into string theory and related physical models. Integrable hierarchies arise naturally from the local operators of the theory and describe the flows between multicritical points. The relevant hierarchy for UMM is the modified-KdV hierarchy. The Sato Grassmannian description of the flows is most appropriate for the computation of the space of solutions to the string equation and I discuss its connection to the τ-function formalism of the Japanese school and more conventional representations. The main results of this thesis are the discovery of the operator formalism for UMM, the computation of the space of solutions to the string equation and the derivation of the mKdV flows from the continuum limit of the local scaling operators.

CONTENTS

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CHAPTER 1

Introduction

The introduction of my thesis is a very brief review of the recent progress and development of zero dimensional field theories of unitary and hermitian matrices, called Matrix Models, and their contribution to understanding better string theories and two dimensional Quantum Gravity.

One-matrix models are quantum mechanical systems whose partition function is given by

$$Z_M = \int dM \, \exp\,(-\frac{N}{\lambda}\mathrm{tr}V(M)) \tag{1.1}$$

where M is a $N \times N$ matrix, usually chosen to be hermitian, unitary or orthogonal, the potential $V(M)$ is a polynomial in the matrix M and λ is called the cosmological constant. The latter nomenclature derives from the relation of Hermitian Matrix Models to two dimensional quantum gravity. The hamiltonian and the measure are invariant under the action of the unitary group $U(N)$ defined by $M \to V^{\dagger}MV$ with $V \in U(N)$. My thesis is on Unitary One-Matrix Models (UMM) where the matrix M belongs to the unitary group.

The interest in matrix models was revived a few years ago from the point of view of string theory and two dimensional quantum gravity and stimulated great activity among string theorists. It was found that HMM are related to a certain class of conformal field theories coupled to two dimensional quantum gravity. In the case of Hermitian One-Matrix Models (HMM), when one takes the limit $N \to \infty$ and λ is tuned to a critical value λ_c the models have a series of multicritical points labeled by an integer k. The authors of [1–4] discovered that when one takes those limits by keeping certain scaling variables fixed, the continuum theory obtained is exactly solvable! The dynamical content of the theory is all in a well known non-linear differential equation, called the string equation. These equations possess asymptotic solutions in the weak coupling limit, which are series in the "string" coupling κ corresponding to the genus expansion of string theory. The great excitement arose because of the discovery that every term in this expansion agrees exactly with the low genus calculation ($h \leq 2$) of the continuum theory for models of (p, q) minimal conformal matter coupled to two dimensional gravity. For one matrix models $q = 2$ and $p = 2k - 1$, where k is the order of the multicritical point considered. Later [5] it was found that the general (p, q) model can be obtained in a similar way from models of $q - 1$ hermitian matrices coupled linearly to each other (MHMM). It should be stressed that the connection of HMM to string theory is established only via the weak coupling expansion. Therefore all the non-perturbative information that is coming from the matrix model has to be considered as a consistent *definition* of string theory. Matrix models results, however, confirm the computations of low genus string theory and two dimensional quantum gravity since the latter make use of assumptions about the form of the measure of the path integral.

The connection of HMM to string theory and $2 - d$ gravity is most easily understood at the discrete level of the theory. The Feynman diagrams of the matrix model perturbation theory can be viewed as being dual to a discrete dynamical polygonation of a two dimensional Riemann surface. Then the perturbation series can be summed in the form

$$Z_M = \sum_{h=0}^{\infty} N^{\chi} Z_{M\,h} \tag{1.2}$$

where $\chi = V - E + L$ is the Euler characteristic of the corresponding surface and h is its genus given by $\chi = 2 - 2h$. Since the number of vertices V, edges E and loops L of the Feynman graph correspond, respectively, to the number of faces F, edges E and vertices V of the dual graph, the above series can be shown to correspond to the discretized version the partition function of string theory embedded in a zero dimensional target space

$$Z_{str} = \sum_{h} \sum_{T} \frac{1}{C(T)} \exp\Big(-\mu_B A + \frac{1}{4\pi G_B} \chi \Big) \,. \tag{1.3}$$

Every fixed h term in (1.3) is the discretized partition function of pure two dimensional quantum gravity on a background manifold of genus h. The precise relation between (1.2) and (1.3) is that Z_{str} is the free energy $\ln Z_M$ of the matrix model. This is because we need to consider only connected vacuum bubbles from the matrix model perturbation theory in order to obtain quantum gravity. In (1.3) A is the area of the surface, μ_B and G_B the bare cosmological and Newton's constant and $C(T)$ is the symmetry factor of the polygonation corresponding to dividing by the volume of the isometry group of the surface. The relation between (1.2) and (1.3) is established by identifying $\lambda = \mathrm{e}^{-\mu_B}$ and $N = \mathrm{e}^{\frac{1}{4\pi G_B}}$. The action in (1.3) is the discretized version of the action of a string theory embedded in zero dimensional spacetime

$$S_{str} = \ln \kappa_B \int d^2\xi \sqrt{g} R + \mu_B \int d^2\xi \sqrt{g}\,. \tag{1.4}$$

Then (1.2) gives the genus perturbation expansion with $\kappa_B = \frac{1}{N}$, the bare string coupling. The naive continuum theory is taken by letting $N \to \infty$. For a critical value μ_c of μ_B the increasing entropy of large surfaces compensates the Boltzmann factor and the system undergoes a (third order) phase transition. In this case the area A diverges and the polygonated surface is thought to approach a smooth Riemann surface. If the critical point is approached after the large N limit is taken, only the sphere $Z_{M\,0}$ contributes to (1.2). The remarkable observation [1–4] was that since the singular part of $Z_{M\,h} \sim (\mu_B - \mu_c)^{\chi(1+\frac{1}{2k})}$ with k a positive integer, one can obtain contributions from all genera by simultaneously taking the large N-limit and letting μ_B approach its critical value μ_c in a coordinated way. The integer k labels a series of multicritical points reached by tuning k parameters in the potential $V(M)$. Introducing a cutoff a in the theory, we define the string coupling κ_0 and renormalized cosmological constant μ_R to be

$$\kappa_0 = \frac{a^{-(2+\frac{1}{k})}}{N}\,, \qquad \mu_R = \frac{\mu_B - \mu_c}{a^2}\,. \tag{1.5}$$

The double scaling limit is defined by taking $N \to \infty$ and $\mu_B \to \mu_c$ while keeping κ_0 and μ_R fixed. Then the continuum limit of (1.2) becomes

$$Z_{str} = \sum_{h=0}^{\infty} \kappa^{\chi} Z_h\,, \tag{1.6}$$

with $\kappa = \frac{\kappa_0}{\mu_R}$. The series (1.6) is horribly divergent. It is non-Borel summable since every term increases as $(2h)!$. This reflects our ignorance in summing the perturbation series of string theory although the fixed genus partition function Z_h can be calculated and is well defined. Happily, the theory is exactly solvable at the multicritical points and its dynamical content is given by a single differential equation, the string equation. The string equation is a differential equation, in the variable x, satisfied by the matrix model specific heat $-\partial^2 \ln Z_M$, with $\kappa^2 = x^{-(2+\frac{1}{k})}$ and $\partial = \partial/\partial x$. It possesses solutions that in the weak coupling limit $\kappa \to 0$ are asymptotic to (1.6) and we say that the double scaling limit provides a non-perturbative definition of Z_{str}. Indeed comparison with calculations directly from the continuum theory indicates that Z_{str} corresponds to two dimensional gravity coupled to $(2k-1, 2)$ minimal conformal matter. Even more interesting is the discovery that the double scaling limit of $(q-1)$ or 2 MHMM gives two dimensional gravity coupled to (p, q) minimal conformal matter [5,6,7].

Unitary One Matrix Models (UMM) form another interesting class of matrix models. These are defined by (1.1) with M being a unitary matrix U. The interest in those models arose a long time ago when Gross and Witten [8] showed that the partition function of two dimensional $U(N)$ QCD on a lattice is given by $Z_{QCD} = (Z_U)^{\frac{V}{a^2}}$ and that the theory undergoes a third order phase transition in the large N limit (V is the volume of the two dimensional world and a is the lattice cutoff). The theory was also shown to possess a double scaling limit $N \to \infty$ and $\lambda \to \lambda_c$ with $t = (1 - \frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y = (1 - \frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed [9,10]. The string equation is a $2k^{\text{th}}$ order differential equation of the function v in the variable $x = t + y$, with $v^2 = -\partial^2 \ln Z$. It has solutions that are asymptotic to (1.6) in the limit $x \to \infty$ with $\kappa^2 = x^{-(2+\frac{1}{k})}$. The identifications of those solutions with conformal field theories coupled to two dimensional gravity or other interesting systems is still, however, an interesting open problem. Some interesting suggestions have been made in [11] . These are discussed in chapter 6. Moreover, the surface interpretation of UMM is not as clear as in the case of HMM. In [12] Neuberger views the unitary matrix as $U = \mathrm{e}^{iM}$ where M is hermitian and introduces $N \times N$ hermitian fermionic matrices ψ and $\overline{\psi}$ to exponentiate the Haar measure $dU \to dM det(\frac{\delta U}{\delta M})$. The resulting surfaces contain an infinite number of types of bosonic vertices forming bosonic "webs" and fermionic loops forming their boundaries that might allow a stringy interpretation of the UMM. For another interesting suggestion see [13] . It is also interesting to note that UMM belong to the same universality class as the HMM in a different class of multicritical points, the double-cut HMM [11,14]. This is expected since the critical behaviour is governed by the scaling of the density of the eigenvalues at the edge of its support [15] and the eigenvalues of the two models models scale identically there. The surface interpretation for HMM described above does not hold for the double-cut HMM, because the multicritical potentials lead to complex values of the cosmological constant μ_B.

The continuum theory obtained in the double scaling limit has a very rich mathematical structure. Well known integrable hierarchies are found to describe flows between multicritical points [5,16]. For example, the Korteweg-de Vries (KdV) hierachy

$$\frac{\partial u}{\partial t_{2l+1}} = \partial \mathrm{R}_l^{KdV}[u] \tag{1.7}$$

describes the flow of the scaling function u giving the specific heat of HMM and the modified-KdV (mKdV) hierachy

$$\frac{\partial v}{\partial t_{2l+1}} = -\partial \mathrm{R}_{l+1}^{mKdV}[v] \tag{1.8}$$

describes the flow of the scaling function v giving the square root of the specific heat of UMM. The right hand sides of (1.7) and (1.8) are polynomials of the scaling functions u and v and their derivatives. Their precise definition can be found in chapters 3 and 6 respectively. From the point of view of string theory, the flows of HMM describe flows between different string backgrounds. These are well defined only for flows between odd multicritical points, since only these flows evolve physically acceptable solutions of a multicritical point to physically acceptable solutions of another multicritical point. The flows arise when one considers the operators σ_l obtained by coupling the $\mathrm{l^{th}}$ multicritical potential $V_l(M)$ to the critical potential $V_k(M)$ of the k^{th} multicritical point. In this way one defines a set of non-critical (massive) models which interpolate between the multicritical models. The integrable hierarchies (1.7) and (1.8) are obtained by considering the dependence of the scaling functions u and v on the sources ("times") t_{2l+1} that couple to the operators σ_l.

An alternative description of the hierarchies (1.7) and (1.8) is given by the corresponding τ-functions. These are related to u and v by $u = -2\partial^2 \ln \tau$ and $v = \partial \ln \frac{\tau_2}{\tau_1}$. The KdV and mKdV τ-functions are solutions to the Hirota bilinear equations, which are equivalent to (1.7) and (1.8). The τ-functions can be associated to points of the Universal Sato Grassmannian, which is an appropriate infinite dimensional generalization of finite dimensional grassmannians. This is accomplished by mapping the points of the Universal Grassmannian, which are infinite dimensional vector spaces V, to states of a two dimensional fermionic free field theory. The τ-functions, considered as formal power series in the times t_{2l+1}, represent states of a two dimensional bosonic free field theory. The connection is established by the well known equivalence of the two dimensional fermionic and bosonic theories. The partition functions of HMM and UMM are found to be given by the corresponding τ-functions [17–19]. The τ-functions that solve the string equation must be annihilated by constraints which for the one-matrix model obey the centerless Virasoro algebra and are called the Virasoro constraints [17–20]. All of those results have counterparts in the discrete theory. The integrable flows are now with respect to the couplings in the potential $V(M)$. For the UMM these are given by Toda flows on the half line [21] and the partition function is given by the product of two Toda-chain τ-functions. The Virasoro constraints L_n have the simple interpretation of corresponding to invariance of the partition function under specific transformations, which for the UMM are given by $\delta U = \epsilon_n(U^{n+1} - U^{1-n})$.

An interesting observation is that the string equation can be written in the form $[P, Q] = 1$ where P and Q are differential operators for the HMM [5] and 2×2 matrices of differential operators for the UMM [22] . They correspond to the continuum limits of operators acting on the space of orthonormal functions used to solve the model. One can use this form of the string equation to determine easily the points in the Universal Grassmannian that solve the string equation [23]. For the UMM [24] these are found to correspond to a pair of points V_1 and V_2 in the (big cell of the) Sato Grassmannian satisfying certain invariance conditions. It is very important that the mKdV evolution of V_1 and V_2 gives new solutions to the string equation. The τ-functions that correspond to V_1 and V_2 are shown to satisfy the Virasoro constraints in this formalism [24] since the constraints are derived from the same invariance conditions that solutions to the string equation satisfy [25–28].

The organization of this thesis is as follows.

In chapter 2 I review some results of bosonic string theory. For a fixed genus surface this is two dimensional quantum gravity coupled to conformal matter, where the two dimensional world-sheet of string theory is viewed as a two dimensional (Euclidean) space-time and the embedding coordinates as the matter fields. This is done in the path integral approach which reduces the theory to the study of quantum Liouville theory. The Liouville field is the conformal factor ϕ of the metric $g_{ab} = \mathrm{e}^{\phi}\,\widehat{g}_{ab}$. The quantization of the Liouville field results in the "dressing" of the operators of the conformal field theory and a change in their scaling properties. The scaling of these operators can be computed and compared to the matrix model results. For MHMM they agree for (p, q) minimal conformal matter fields coupled to gravity.

In chapter 3 I review the results obtained from HMM. MHMM are also briefly discussed. I discuss the double scaling limit and how it can be solved by using the orthogonal polynomial method. The solutions to the string equation give local or "microscopic" scaling operators that in the weak coupling limit have scaling properties that can be compared to the string theory calculations. Another class of operators, the loop or "macroscopic" operators can be computed in the spherical limit and found to agree with the corresponding operators of Liouville theory in the minisuperspace approximation. The flows between multicritical points are described by the KdV integrable hierarchy. I discuss the implications of this result as well as the operator formalism representation of the string equation and the flows. I discuss very briefly the Sato Grassmannian description of the string equation and the flows and how it leads to the computation of the space of solutions to the string equation.

In Chapter 4 I discuss the large-N or spherical limit of UMM and the motivation that leads to the study of these models. The connection of the simplest of these models to 2-d pure QCD is discussed, as well as the possibilities for a world-sheet interpretation of the model. The existence of multicritical potentials is already revealed in this limit and they are not affected by taking the double scaling limit. The critical behaviour is understood by studying the scaling of the density of eigenvalues at the edge of its support. This governs the scaling of the operators in the double scaling limit as well.

In chapter 5 I discuss the method of orthogonal polynomials for UMM. This method makes possible the solution of the model even before taking the double scaling limit. The integrability of the model is revealed at this level as well, where the solutions are given by the modified Voltera hierarchy or the Toda chain on the half line. There are two convenient choices for orthogonal polynomials which I present, the Periwal-Shevitz (PS) basis and the trigonometric basis.

In chapter 6 I discuss the double scaling limit of UMM. I discuss how to obtain the string equation and study its "physical" solutions, which have a genus expansion in the weak coupling limit. The scaling operators can be computed exactly and the mKdV flows revealed. There is an operator formalism for the string equation and the mKdV flows similar to the one for HMM. I also give a very brief review of the double cut HMM and its relation to UMM.

In chapter 7 the main topic is the computation of the space of solutions to the string equation. This is most conveniently done by studying the solutions in the Sato Grassmannian. For this reason I review the τ-function representation of the mKdV (and the more general mKP) hierarchies and discuss its connection to the Sato Grassmannian formalism and the formalism presented in chapter 6. This is done by mapping the points of the Grassmannian to fermionic states of a two dimensional free field theory of fermions. This is called the Plücker embedding. The τ-functions are states that belong to the GL(∞) orbit of filled fermionic states and can be mapped to polynomials of the times t_k of the KP or mKP hierarchies by using the bosonization of the fermionic theory. In this way the hierarchy has a differential operator representation that derives from pseudo-differential (ΨDO

) operators. This way the solutions to the string equation can be easily obtained by solving the conditions that the spaces V_1 and V_2 must satisfy. The Virasoro constraints are a simple corrolary of these conditions and the compatibility of the string equation with the mKdV flows is easily understood.

My research, in collaboration with my advisor Mark Bowick, Nobiyuki Ishibashi and Albert Schwarz, contributed to the discovery of the operator formalism [24] for UMM, the formulation of the string equation as conditions on points of the Sato Grassmannian and the computation of solutions to the string equation. The general proof that UMM lead to the mKdV hierarchies has never been presented in the literature before - it was obtained by studying the double cut HMM - although the result was known before.

CHAPTER 2

A Glance at 2-d Quantum Gravity and Liouville Theory

In this chapter I make a quick tour of string theory, 2-d gravity and Liouville theory. I describe how one obtains the conformal anomaly in the path integral approach of string theory, which makes the effective quantum theory equivalent to the study of quantum Liouville theory. I review how one obtains the string susceptibility, or the anomalous dimension of the scaling of the fixed area partition function with the area of the world sheet, and the anomalous dimensions of the primary field (local) operators. In the canonical approach, one can solve the minisuperspace Wheeler-deWitt equation and obtain loop operators $\psi_{\mathcal{O}}(l)$ which is important for comparing the theory with matrix models results. The purpose of this chapter is to establish a connection of the Liouville theory results and matrix models computations. No attempt for rigor is made and the interested reader is referred to an excellent review by Ginsparg and Moore [29,30] and the references therein. Most of the ideas for this exposition were taken from there.

2.1. 2-d Quantum Gravity Coupled to Conformal Matter with $c < 1$.

In (bosonic, closed) string theory one wishes to study the path integral

$$\begin{aligned} Z &= \sum_{h=0}^{\infty} \int_{\Sigma_h} \frac{\mathcal{D}g\,\mathcal{D}X}{Vol(Diff(\Sigma_h))}\; \mathrm{e}^{-S_{str}} \\ &= \sum_{h=0}^{\infty} \kappa^{\chi} Z_h \,, \end{aligned} \tag{2.1.1}$$

with

$$S_{str} = \ln\kappa \int d^2\xi\,\sqrt{g}R + \mu_B \int d^2\xi\,\sqrt{g} + S_{Matter}\,. \tag{2.1.2}$$

κ is the string coupling constant and μ_B the bare cosmological constant. S_{Matter} is chosen to be the conformally invariant action

$$S_{Matter} = \int d^2\xi\,\sqrt{g}g^{ab}\partial_a X^{\mu}\partial_b X^{\nu} G_{\mu\nu}(X)\,. \tag{2.1.3}$$

Here $\mu, \nu = 1, 2, \ldots, D$ where D is the dimension of spacetime with Euclidean signature metric $G_{\mu\nu}(X)$, $a, b = 1, 2$ are the world sheet indices whose metric is $g_{ab}(\xi)$ and $X^\mu(\xi)$ are the embedding coordinates of the world sheet Σ_h into the spacetime. The path integral is performed over all admissible Riemannian metrics on the world sheet Σ_h whose genus is h. The summation is over all two dimensional topologies and the division by the volume of the diffeomorphism group $Diff(\Sigma_h)$ is made because the measure and the action are diffeomorphism invariant.

Each term in (2.1.1) is a theory of Euclidean two dimensional quantum gravity on the background manifold Σ_h coupled to D conformal scalar fields. In string perturbation theory we are interested in analyzing such a theory separately and in the end to study the sum (2.1.1). It has been proven a formidable and unsolved problem to analyze the theory in detail for large h. Even if someone could, the series is known to be very badly behaved (it is non Borel-summable) and no one would know how to define the sum (2.1.1). The problem of a non-perturbative definition of string theory -maybe through a second quantized string field theory- in physical dimensions is still outstanding. Lately, a consistent closed string field theory has been defined (for a review see [31]). We will see that matrix models shed some light in these problems in some simple, unphysical cases by providing a sensible non-perturbative definition of (2.1.1).

Following, I give an outline of how to calculate Z_h. Most of the technical details can be found in many string textbooks and papers, such as [32–34]. The measures $\mathcal{D}X$ and $\mathcal{D}g$ are defined from the following metrics in the space of metrics and embeddings:

$$\|\delta g\|_g^2 = \int d^2\xi \sqrt{g}\,(g^{ac}g^{bd} + cg^{ab}g^{cd})\delta g_{ab}\delta gcd$$

and

$$\|\delta X\|_g^2 = \int d^2\xi \sqrt{g}\,\delta X^\mu \delta X_\mu$$

where δg and δX are tangent vectors in the space of metrics and embeddings respectively. Both measures are invariant under the group of diffeomorphisms $Diff(\Sigma_h)$, but although the classical action S_{Matter} is invariant under the action of Weyl transformations $g \to e^{\varphi} g$, the measures $\mathcal{D}g$ and $\mathcal{D}X$ are not. This is the origin of the so called conformal anomaly.

The integral can be analyzed conveniently in the conformal gauge [35]. The space of metrics modulo diffeomorphisms and Weyl rescalings is called the moduli space $\mathcal{M}_h$ which is finite dimensional and compact for a two dimensional manifold Σ_h. For genus 0 it is 0-dimensional, for genus 1 it is 2-dimensional and for higher genus h it has $6h-6$ real dimensions. For every point $\tau \in \mathcal{M}_h$ we choose a representative metric $\widehat{g}_{ab}(\tau)$ and then every metric can be written in the form

$$f^* g = e^{\varphi} \widehat{g}(\tau) \tag{2.1.4}$$

with $f : \Sigma_h \to \Sigma_h$ a diffeomorphism.

Integration over diffeomorphism equivalent metrics introduces an infinite factor which is absorbed by dividing by the volume of $Diff(\Sigma_h)$. Restricting the integration to a gauge slice introduces ghosts. Under the infinitesimal action of a diffeomorphism generated by a vector field v^a the metric changes by

$$\delta g_{ab} = \pounds_v g_{ab} = 2\, \nabla_{(a}\, v_{b)}\, .$$

Then the integration over the measure $\mathcal{D}g$ splits to an integration over the moduli $[d\tau]$, the conformal factor $\mathcal{D}\varphi$ and the diffeomorphisms $\mathcal{D}v$. In complex coordinates ($z = \xi_1 + i\xi_2$, $\overline{z} = \xi_1 - i\xi_2$) $\delta g_{zz} = \nabla_z\, v_z$ and $\delta g_{\overline{zz}} = \nabla_{\overline{z}} v_{\overline{z}}$ and integrating over v gives the determinants

$$det\, \nabla_z\; det\, \nabla_{\overline{z}} = \int \mathcal{D}b \mathcal{D}\bar{b} \mathcal{D}c \mathcal{D}\bar{c}\, e^{-S_{gh}}\, , \tag{2.1.5}$$

where

$$S_{gh} = \int d^2\xi \sqrt{g}\; (b_{zz}\, \nabla_{\overline{z}}\, c^z + b_{\overline{zz}}\, \nabla_z\, c^{\overline{z}})\, . \tag{2.1.6}$$

The conformal anomaly arises when we try to express the measures $\mathcal{D}_g(ghost) = \mathcal{D}_g b \mathcal{D}_g \bar{b} \mathcal{D}_g c \mathcal{D}_g \bar{c}$, $\mathcal{D}_g X$ and $\mathcal{D}_g g$ in terms of the reference metric $\widehat{g}$ of (2.1.4). The celebrated results of Polyakov [35] are

$$\mathcal{D}_{e^{\varphi}\widehat{g}} X = e^{\frac{D}{48\pi} S_L(\varphi, \widehat{g})} \mathcal{D}_{\widehat{g}} X\, , \tag{2.1.7}$$

and

$$\mathcal{D}_{e^\varphi \widehat{g}}(ghost) = e^{-\frac{26}{48\pi} S_L(\varphi,\widehat{g})} \mathcal{D}_{\widehat{g}}(ghost)\,, \tag{2.1.8}$$

where

$$S_L(\varphi,\widehat{g}) = \int d^2\xi \sqrt{\widehat{g}}\,(\frac{1}{2}\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi + \widehat{R}\varphi + \mu e^{\varphi}) \tag{2.1.9}$$

is the well known Liouville action. At $D = 26$ we are led to the decoupling of the Liouville mode and critical string theory. The measure $\mathcal{D}_g\varphi$, however, does not have an obvious simple dependence on $\mathcal{D}_{\widehat{g}}\varphi$. the reason is that the metric $\|\delta\varphi\|_g^2 = \int d^2\xi\,\sqrt{g}\,\delta\varphi\delta\varphi$ that defines the measure, depends on φ itself! The authors of [36,37] *assume* that

$$\begin{aligned} Z_h = \int [d\tau]\mathcal{D}_{\widehat{g}}(ghost)\mathcal{D}_{\widehat{g}}X\mathcal{D}_{\widehat{g}}\varphi\ e^{-S_{Matter}-S_{ghost}} \\ \times \exp\{\int d^2\xi\sqrt{\widehat{g}}\,(a\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi + b\widehat{R}\varphi + \mu e^{c\varphi})\}\,. \end{aligned} \tag{2.1.10}$$

The constants a, b and c can be determined by using the simple fact that Z_h is invariant under the simultaneous transformation:

$$\begin{aligned} \widehat{g} \to& \widehat{g}e^{\sigma} \\ \varphi \to& \varphi - \sigma \end{aligned} \tag{2.1.11}$$

that leaves $g = e^{\varphi}\widehat{g}$ invariant. Using the known conformal anomalies for X and the ghosts, observing that (2.1.11) is independent of φ $(i.e. \mathcal{D}_{e^\sigma\widehat{g}}\varphi = e^{\frac{1}{48\pi}S_L(\sigma,\widehat{g})}\mathcal{D}_{\widehat{g}}\varphi)$ and taking into account the renormalization of the $e^{c\varphi}$ term, we obtain

$$(\frac{D-26+1}{48\pi} - b)\widehat{R} - (2a-b)(\Delta_{\widehat{g}}\sigma - \Delta_{\widehat{g}}\varphi) - \mu(1 - c - \frac{c^2}{16\pi a})e^{c\varphi}e^{\sigma(1-c-\frac{c^2}{16\pi a})} = 0\,. \tag{2.1.12}$$

Then

$$\begin{aligned} a =& \frac{1}{2}b \\ b =& \frac{D-25}{48\pi} \\ c =& \frac{1}{12}(25 - D - \sqrt{(D-25)(D-1)}) \end{aligned} \tag{2.1.13}$$

where the choice of cut of the square root is such that we obtain the classical limit as $D \to -\infty$ and the effective coupling $(25-d)^{-1}$ goes to zero. We renormalize $\varphi \to \sqrt{\frac{12}{25-D}}\varphi$ and we obtain the Liouville action as

$$S_L(\varphi,\widehat{g}) = \frac{1}{8\pi}\int d^2\xi\sqrt{\widehat{g}}\,(\widehat{g}^{ab}\partial_a\varphi\partial_b\varphi + Q\widehat{R}\varphi + \mu e^{\gamma\varphi}) \tag{2.1.14}$$

where

$$\begin{aligned} \gamma =& \frac{1}{\sqrt{12}}(\sqrt{25-D} - \sqrt{1-D}) \\ Q =& \sqrt{\frac{25-D}{3}} = \frac{2}{\gamma} + \gamma\,. \end{aligned} \tag{2.1.15}$$

Then the energy momentum tensor $T(z) = -\frac{1}{2}\partial\varphi\partial\varphi + \frac{Q}{2}\partial^2\varphi$ $(\partial = \partial_z)$ gives an operator product expansion (OPE) $T(z)T(w) \sim \frac{1}{2}c_L/(z-w)^4 + \ldots$ with $c_L = 1 + 3Q^2$. Happily, the total central charge

$$c_{Matter} + c_{ghost} + c_L = D - 26 + (26 - D) = 0$$

and the conformal anomaly vanishes! Moreover, the conformal weights Δ and $\overline{\Delta}$ of the operator $e^{\gamma\varphi}$ defined by $T(z)e^{\gamma\varphi(w)} \sim \frac{\Delta e^{\gamma\varphi(w)}}{(z-w)^2} + \ldots$ and similarly for $\overline{\Delta}$ are equal to $\Delta = \overline{\Delta} = -\frac{1}{2}\gamma(\gamma - Q) = 1$. This means that the operator $e^{\gamma\varphi}$ behaves like a $(1,1)$ conformal field which is consistent with the requirement that $\int d^2\xi\sqrt{\widehat{g}}\,e^{\gamma\varphi}$ gives the total area of the surface.

Note that for $D \leq 1$, both γ and Q are real and the Liouville theory is well defined. At $D = 25$ $Q = 0$ and one might interpret φ as a Euclidean time coordinate. This is most easily recognized by Wick rotating $\varphi \to -i\varphi$ and obtaining an extra timelike coordinate from the Liouville field. Therefore for strings naively embedded in 25 Euclidean dimensions, the Liouville mode provides an extra 26$^{\text{th}}$ timelike dimension, dynamically realizing a 26 dimensional Minkowski spacetime. For $D \geq 25$ the kinetic term of the Liouville field changes sign and we have a ghost field of negative norm. For $1 < D < 25$, γ is complex and Q imaginary. It is not known of how to make sense of Liouville theory in this (most interesting) regime and it is still an outstanding problem in string theory and two dimensional quantum gravity.

A very important critical exponent is the string susceptibility Γ_{str}. It was the agreement of this exponent with the one obtained from matrix model calculations that originally confirmed the assumptions made by both methods. Γ_{str} is defined from the scaling of the fixed area partition function $Z_h(A)$ with the area A as $A \to \infty$. $Z_h(A)$ is given by

$$\begin{aligned} Z_h(\mu) &= \int dA\, e^{-\mu A}\, Z_h(A) \\ Z_h(A) &= \int \mathcal{D}g\mathcal{D}X\,\delta(\int d^2\xi\sqrt{g} - A)\, e^{-S} \end{aligned} \tag{2.1.16}$$

and for large A it behaves like

$$Z_h(A) \sim A^{(\Gamma_{str}-2)\frac{\chi}{2}-1}\,. \tag{2.1.17}$$

Γ_{str} is determined by a simple scaling argument. First write

$$Z_h(A) = \int [d\tau]\mathcal{D}_{\widehat{g}}(ghost)\mathcal{D}_{\widehat{g}}\varphi\mathcal{D}_{\widehat{g}}X e^{-S_{Matter}-S_{gh}}\mathcal{D}g\mathcal{D}X\,\delta(\int d^2\xi\sqrt{\widehat{g}}e^{\gamma\varphi} - A)\, e^{-S_L}$$

and consider the change $\varphi \to \varphi + \frac{\rho}{\gamma}$ where ρ is a *constant*. The measure terms are invariant and the only change in $Z_h(A)$ comes from S_L and the δ-function:

$$\begin{aligned} \frac{Q}{8\pi}\int d^2\xi\sqrt{\widehat{g}}\,R\varphi \to &\frac{Q}{8\pi}\int d^2\xi\sqrt{\widehat{g}}\,R\varphi + \frac{Q}{8\pi\gamma}\int d^2\xi\sqrt{\widehat{g}}\,R \\ =&\frac{Q}{8\pi}\int d^2\xi\sqrt{\widehat{g}}\,R\varphi + \frac{Q}{2\gamma}\rho\chi \end{aligned} \tag{2.1.18}$$

$$\delta(\int d^2\xi\sqrt{\widehat{g}}e^{\gamma(\varphi+\frac{\rho}{\gamma})} - A) = e^{-\rho}\,\delta(\int d^2\xi\sqrt{\widehat{g}}e^{\gamma\varphi} - A)$$

so that

$$Z_h(A) = e^{-\frac{Q\rho}{2\gamma}\chi-1}\,Z_h(e^{-\rho}A)\,.$$

Setting $e^{\rho} = A$ we obtain

$$Z_h(A) = A^{-\frac{Q\chi}{2\gamma}-1}\,Z(1) = A^{(\Gamma_{str}-2)\frac{\chi}{2}-1}Z(1)\,,$$

so that

$$\Gamma_{str} = 2 - \frac{Q}{\gamma} = \frac{1}{12}(D - 1 - \sqrt{(D-25)(D-1)})\,. \tag{2.1.19}$$

Notice that for $D > 1$ Γ_{str} is complex, indicating that the above considerations break down for $D > 1$.

There is a certain class of conformal field theories (CFT) with a finite number of primary fields (for a review see *e.g.* [38] and references therein) called "minimal CFT". They are characterized by a pair of relatively prime integers (p, q) and have central charge

$$c_{p,q} \equiv D = 1 - \frac{6(p-q)^2}{pq}\,.$$

The unitary subseries are given by $(p, q) = (k+1, k)$. The string susceptibility of these models coupled to gravity is

$$\Gamma_{str} = -\frac{2}{(p+q-1)}\,. \tag{2.1.20}$$

In the double scaling limit (which is discussed in the next chapter), HMM admit exact solutions for a series of multicritical points labeled by an integer k. For these solutions $\Gamma_{str} = -1/k$ which is the string susceptibility obtained from the $(2k-1, 2)$ minimal CFT.

In order to determine the scaling of the operators of a CFT coupled to gravity, we apply the previous simple arguments. For example, the one point function of the operator Φ_0 for fixed area is calculated by the integral

$$\begin{aligned}\langle \int d^2\xi\sqrt{\hat{g}}\,\mathrm{e}^{\alpha\varphi}\Phi_0\rangle_A =& \frac{1}{Z_h(A)}\int \mathcal{D}\varphi\mathcal{D}X\mathrm{e}^{-S}\delta(\int d^2\xi\sqrt{\hat{g}}\,\mathrm{e}^{\gamma\varphi} - A)\int d^2\xi\sqrt{\hat{g}}\,\mathrm{e}^{\alpha\varphi}\Phi_0 \\ \sim& A^{1-\Delta}\,,\end{aligned} \tag{2.1.21}$$

where the last line is the definition of the exponent Δ. Therefore we see that the coupling of Φ_0 to gravity is effectively described by the dressing of the operator Φ_0 such that $\Phi = \mathrm{e}^{\alpha\varphi}\Phi_0$. The constant α is fixed by the requirement that Φ has dimension $(1, 1)$ so that the integral $\int d^2\xi\sqrt{\hat{g}}\,\mathrm{e}^{\alpha\varphi}\Phi_0$ does not break conformal invariance. Since the weight of $\mathrm{e}^{\alpha\varphi}$ is $-\frac{1}{2}\alpha(\alpha - Q)$ and that of Φ_0 is Δ_0, we have that

$$\Delta_0 - \frac{1}{2}\alpha(\alpha - Q) = 1$$

or

$$\begin{aligned}\alpha =&\frac{1}{2}Q - \sqrt{\frac{1}{4}Q^2 - 2 + 2\Delta_0} \\ =&\frac{1}{\sqrt{12}}(\sqrt{25-D} - \sqrt{1-D+24\Delta_0})\,.\end{aligned} \tag{2.1.22}$$

Rescaling $\varphi \to \varphi + \frac{\rho}{\gamma}$ with $\mathrm{e}^{\rho} = A$ we obtain

$$\begin{aligned}\langle \int d^2\xi \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi}\Phi_0\rangle_A =&\frac{A^{-\frac{q\chi}{2\gamma}-1+\frac{\alpha}{\gamma}}}{A^{-\frac{q\chi}{2\gamma}-1}}\langle \int d^2\xi \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi}\Phi_0\rangle_1 \\ =&A^{\frac{\alpha}{\gamma}}\langle \int d^2\xi \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi}\Phi_0\rangle_1\,,\end{aligned} \tag{2.1.23}$$

or $\Delta = 1 - \frac{\alpha}{\gamma}$. The additional $\mathrm{e}^{\rho\alpha/\gamma}$ comes from the integral $\int d^2\xi \sqrt{\hat{g}}\, \mathrm{e}^{\alpha\varphi}\Phi_0$. Substituting α and γ we obtain

$$\Delta = \frac{\sqrt{1-D+24\Delta_0} - \sqrt{1-D}}{\sqrt{25-D} - \sqrt{1-D}}\,. \tag{2.1.24}$$

For the (p,q) series, the operators $\Phi_0^{(r,s)}$ are labelled by the integers r, s with $0 < r < q$, $0 < s < p$, $p > q$ and the conformal weights $\Delta_0 = \frac{(pr-qs)^2-(p-q)^2}{4pq}$. For these operators ($D = c = 1 - \frac{6(p-q)^2}{pq}$)

$$\frac{\alpha}{\gamma} = \frac{p+q-|pr-qs|}{2q}\,. \tag{2.1.25}$$

(2.1.25) is in agreement with the (p,q) MHMM calculations as we will see in chapter 3.

Comparison with the matrix models results can also be done in the canonical quantization of Euclidean two dimensional gravity. The wave functions Ψ are functions of the only diffeomorphism invariant quantity that can be constructed from $\varphi(\sigma)$ on a spatial slice, namely the length of the universe l. The wavefunctions corresponding to physical states must be annihilated by the hamiltonian constraint *i.e.* satisfy the Wheeler-deWitt (WdW) equation. In the minisuperspace approximation, where $\varphi(\sigma,t) = \varphi(t)$, the WdW equation $(\mathrm{L}_0 + \overline{\mathrm{L}}_0 - 2)\Psi(l) = 0$ takes the form [39]

$$(-l(\frac{\partial}{\partial l})^2 + 4\mu l^2 + \nu^2)\psi(l) = 0\,, \qquad \nu = \pm\frac{2}{\gamma}(a - \frac{1}{2}Q)\,, \tag{2.1.26}$$

with (non-normalizable) solutions decaying at large lengths

$$\Psi_{\mathcal{O}}(l) \propto K_\nu(2\sqrt{\mu}l)\,. \tag{2.1.27}$$

Semiclassically these correspond to metrics

$$e^{\gamma\varphi}(dt^2 + d\sigma^2) = \frac{4}{\mu}\frac{\nu^2}{\sinh^2(\nu t)}(dt^2 + d\sigma^2) \quad t < 0\,,$$

which correspond to "funnel" like Universes. We may interpret $\Psi_{\mathcal{O}}(l)$ as a local operator inserted at the tip of the funnel at $t \to -\infty$ [39]. The result -with these boundary conditions-, although obtained in the minisuperspace approximation agrees *exactly* with the matrix models results! $\Psi_{\mathcal{O}}(l)$ will be found to correspond to correlators $\langle \widehat{\sigma_j} W(l)\rangle$. In the language of two dimensional gravity, $W(l)$ is the operator that creates a universe of size l and $\widehat{\sigma_j}$ the one that inserts a local operator at $t \to -\infty$.

I close this section with a last remark. In a string theory given by a product of D gaussian models ($G_{\mu\nu} = \delta_{\mu\nu}$ in (2.1.3)) for which $c_{Matter} = D$, we identify each boson with a space-time dimension. Therefore we identify an arbitrary CFT with central charge c with a *non-critical* string theory embedded in an (abstract) target space of dimension $D = c_{Matter}$. In that case the Liouville mode is responsible for the cancellation of the conformal anomaly. A different point of view is to consider the Liouville mode as a dynamically realized space-time direction. In that case we are talking about a *critical* string theory embedded in $d = D + 1$ dimensions. For the cases of interest of this thesis, *i.e.* for zero dimensional HMM (or MHMM), the central charge $c < 1$ and corresponds to minimal CFT. Therefore we are describing non-critical strings living in $D < 1$ dimensions, or critical strings living in $d < 2$ dimensions. Another class of interesting solvable matrix models, a model of large N matrix quantum mechanics (one dimensional matrix models), deals with the case $c = 1$ or $D = 1$ and $d = 2$. For the moment there is no known matrix model surpassing the $c = 1$ barrier and it remains a very interesting open problem.

CHAPTER 3

On Hermitian Matrix Models and their Relation to $c < 1$ strings.

As I mentioned in the introduction, HMM provide a naturally regularized non-perturbative definition of non-critical string theories with target spaces of dimension $D < 1$. In this chapter I will review the progress made in the field during the last four years. I will describe the results that show the relation of the models to string theory. I will also try to establish a connection to UMM by discussing the rich mathematical structure that arises in the continuum limit that is so similar for the two models. This is the appearance of KdV flows, which physically describe the flows between multicritical points. I will briefly mention matrix chains (MHMM) that are related to the general (p, q) minimal CFT coupled to gravity. Because of lack of space, I will skip most of the technical details since these are similar to the UMM and will be discussed in later chapters. I will emphasize results that relate the models to two dimensional quantum gravity and the interested reader should consult the excellent reviews by Ginsparg and Moore [29,30,40] and the references therein.

3.1. One Hermitian Matrix Models (HMM) and string theory

The models I will describe are zero dimensional field theories of $N \times N$ hermitian matrices Φ with a partition function

$$Z_\Phi = \int d\Phi \, \exp\big(-\frac{N}{\lambda}\mathrm{tr}V(\Phi)\big)\,, \tag{3.1.1}$$

where

$$\frac{N}{\lambda}V(\Phi) = \sum_{k\geq 2} g_k \lambda^{\frac{k}{2}-1} N \Phi^k$$

and $g_2 = \frac{1}{2}$. In perturbation theory the free energy $\ln Z_\Phi$ of this model is given by the sum of all one point irreducible Feynman diagrams. The main motivation in [41,42] was to use (3.1.1) to enumerate such graphs. Since this is a theory of hermitian matrices these graphs are double lined diagrams, each line connecting two indices and having a preferred direction. The ribbon like structure of these diagrams permits to assign to them a two dimensional orientable compact surface of given genus. This is the minimum genus, orientable surface one can draw without self crossings of the lines. The dual graphs of these diagrams provide a triangulation (more accurately "polygonation") of such a surface. The direction of lines of the Feynman diagrams are compatible with the choice of orientation on such a surface. By organizing the terms in (3.1.1) in powers of N, we observe that each term gets contributions from graphs of the same genus. This is because of the normalization chosen in (3.1.1). The propagator is of order N^{-1}, a k-vertex of order $\lambda^{\frac{k}{2}-1}N$, and every loop of order N since we have to sum over all matrix indices. Therefore the contribution of each graph G with $V=\sum_{k\geq 3} V_k$ vertices, E propagators and L loops is proportional to

$$\begin{aligned} G \sim & \prod_{k\geq 3} (\lambda^{\frac{k}{2}-1}N)^{V_k} \frac{1}{N^E} N^L \\ = & N^{V-E+L} \lambda^{\sum_k (\frac{k}{2}-1)V_k} \\ = & N^{\chi} \lambda^A \end{aligned} \tag{3.1.2}$$

where $\chi = V - E + L = 2 - 2h$ is the Euler character of the surface and $A = \frac{1}{2}\sum_k (k-2)V_k$. The Euler character is obtained because the number of loops, propagators and vertices of the Feynman diagram is equal to the number of nodes, edges and faces of the discretized surface. Therefore the expression for $\ln Z_\Phi$ can be organized perturbatively in a large N topological expansion of the form

$$\ln Z_\Phi = \sum_h \sum_G N^{\chi} \lambda^A Z_\Phi^{(G)} \, , \tag{3.1.3}$$

where $Z_\Phi^{(G)}$ is given by the products of vertex weights and is divided by the order of the symmetry group $C(G)$ of the graph G.

In order to relate (3.1.1) to quantum gravity we consider the triangulated by the dual graph G^* 2-d surface and we define a Riemannian metric on it by fixing the lengths of each equilateral triangle to be $\frac{1}{2}$. Then $A = \frac{1}{2}\sum_k (k-2)V_k$ is equal to the area of the surface since the number of k-vertices of the dual graph corresponds to a k-polygon of the triangulation that splits into $k-2$ equilateral triangles of area $\frac{1}{2}$. By defining $N = \mathrm{e}^{\frac{1}{4\pi G_B}}$ and $\lambda = e^{-\mu_B}$ we obtain the disretized version of the partition function of string theory embedded in a zero dimensional target space:

$$Z_{str} = \sum_h \sum_G \frac{1}{C(G)} \exp\Big(-\mu_B A + \frac{1}{4\pi G_B}\chi\Big) . \tag{3.1.4}$$

The sum over all triangulations simulates the sum over all metrics, where the distance between metrics defined by triangulations G and G' is given by $d^2(G, G') = \frac{1}{2}\sum(C_{ij} - C'_{ij})^2$ where C_{ij} and C'_{ij} are the adjacency matrices of the corresponding triangulations. $C(G)$ is the symmetry factor corresponding to dividing by the volume of the isometry group of each surface. Note that in this way, the terms in (3.1.4) sum over only diffeomorphically inequivalent metrics and the division by the volume of the diffeomorphism group is already taken into account. As I mentioned in the introduction each fixed genus term behaves as $(\lambda - \lambda_c)^{\chi(1+1/k)}$ when $\lambda \to \lambda_c$. By taking the limit $N \to \infty$ and $\lambda \to \lambda_c$ such that $t = (1 - \frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y = (1 - \frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ are held fixed, (so that the string coupling κ of (1.5) and (1.6) is held fixed), every term in (3.1.4) diverges and (3.1.4) gets contributions from triangulations of arbitrary genus. In this double scaling limit the mean area

$$\langle A \rangle \sim \frac{\partial}{\partial \mu_B} \ln Z_{str} \sim \frac{1}{\lambda - \lambda_c}$$

diverges, and the sum is dominated by configurations with an infinite number of simplices. For a fixed area partition function one must take the cuttoff to zero and this way a continuum limit of the theory can be defined.

3.2. The Method of Orthogonal Polynomials and the Double Scaling Limit

In order to solve (3.1.1) we first integrate the angular degrees of freedom *i.e.* the unitary group $U(N)$ [41,42]. By writing $\Phi = U^\dagger \Lambda U$ where $\Lambda = diag(\phi_1, \ldots, \phi_N)$ the measure splits in

$$\begin{aligned} d\Phi =& \prod_i d\Phi_{ii} \prod_{l<m} d(\mathrm{Re}\Phi_{lm}) d(\mathrm{Im}\Phi_{lm}) \\ =& \prod_i d\phi_i \prod_{l<m} dK_{lm} J(\Phi, (\phi, K)) \end{aligned}$$

where $U = e^{iK}$ and $\Phi_{ij} \approx \phi_i \delta_{ij} + i(\phi_i - \phi_j)K_{ij}$. The jacobian

$$J(\Phi, (\phi, K)) = \frac{\partial(\Phi_{ii}, \Phi_{ij})}{\partial(\phi_i, K_{ij})} = \prod_{i<j} (\phi_i - \phi_j) = \Delta^2(\phi)$$

is related to the Vandermonde determinant $\Delta(\phi)$. Integrating out the unitary group we obtain

$$Z_\Phi = \int [\prod_{i=1}^{N} d\phi_i] \, \Delta^2(\phi) \, e^{-\frac{N}{\lambda} \sum_i V(\phi_i)} \, . \tag{3.2.5}$$

We define real orthogonal polynomials [42] $P_n(\phi) = \phi^n + \alpha_{n,n-1}\phi^{n-1} + \ldots$ with respect to the measure $d\mu = d\phi \, e^{-\frac{N}{\lambda} V(\phi)}$ from the relation

$$\int d\mu(\phi) \, P_n(\phi) P_m(\phi) = h_n \delta_{nm} \, . \tag{3.2.6}$$

By adding linear combinations of columns in $\Delta(\phi)$ we can prove that

$$\Delta(\phi) = det \, [\phi_m^{n-1}] = det \, [P_{n-1}(\phi_m)]$$

and that

$$Z_\Phi = N! \prod_{i=0}^{N-1} h_i = N! \, h_0^N \prod_{i=1}^{N-1} R_n^{N-i} \, , \tag{3.2.7}$$

where $R_n = h_n / h_{n-1}$.

For even potentials, the orthogonal polynomials satisfy the recursion relation

$$\phi P_n(\phi) = P_{n+1}(\phi) + R_n P_{n-1}(\phi) \, . \tag{3.2.8}$$

The dynamical information of the theory is contained in the "string equation"

$$\int d\mu P_{n-1}(\phi) V'(\phi) P_n(\phi) = \frac{\lambda n}{N} h_n \tag{3.2.9}$$

which is obtained by integrating by parts the trivial relation $\int d\mu P'_{n-1} P_n = 0$. The left hand side of this equation can be calculated using (3.2.8) and it is a function of the R_n's and the couplings N, λ and g_k. As we take the limit $N \to \infty$ the model becomes critical for $\lambda \to \lambda_c$. The potentials in the neighbourhood of the critical potentials constrain the eigenvalues of Φ to be in the closed interval $[-\phi_c, \phi_c]$. In the large N limit the density of eigenvalues $\rho(\phi) = \frac{d(n/N)}{d\phi}$ scale for $\phi \to \phi_c$ as

$$\rho(\phi) \sim P(\phi)\sqrt{\phi_c^2 - \phi^2}\,, \tag{3.2.10}$$

where $P(\phi)$ is a polynomial in ϕ. We can tune k coefficients in the potential so that as

$$\rho_k(\phi) \sim (\phi_c - \phi)^{k-\frac{1}{2}}\,. \tag{3.2.11}$$

Then a non trivial multicritical scaling behaviour is observed as $\lambda \to \lambda_c$ near the edge of the support of $\rho(\phi)$. It is this region that contributes to the scaling part of the operators of the theory [15].

If we take the large N double scaling limit, $\lambda \to \lambda_c$ by keeping $t = (1 - \frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y = (1 - \frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ fixed, we can prove that R_n's scale as

$$R_n = R_c + N^{-\frac{2}{2k+1}}u(x) \tag{3.2.12}$$

with $x = t + y$. The constant $R_c \neq 0$ comes from the naive large N (spherical) limit. Then the limit of (3.2.9) gives a differential equation of order $2k - 2$ [1–4]

$$x = \frac{2k!}{(2k-1)!!}\mathrm{R}_k[u]\,. \tag{3.2.13}$$

The $\mathrm{R}_k[u]$, called the Gelfand-Dikii potentials [43], are polynomials of u and its x-derivatives up to $u^{(2k-2)}$. The general term is of the form $(u)^{a_0}(u')^{a_1}\ldots(u^{(2k-2)})^{a_{2k-2}}$ such that $a_0 + \frac{3}{2}a_1 + \ldots + ka_{2k-2} = k$. They are defined from the asymptotic expansion of the diagonal of the resolvent $\mathrm{R}(x, y; \xi)$ of the operator $\xi - Q$ where $Q = \partial^2 - u$ and $\partial \equiv \partial_x$

$$\mathrm{R}(x, x; \xi) = \sum_{l=0}^{\infty} \frac{\mathrm{R}_l[u]}{\xi^{l+\frac{1}{2}}}\,, \tag{3.2.14}$$

and they are given by the recursion relations

$$\begin{aligned} \partial\mathrm{R}_{k+1}[u] =& (\frac{1}{4}\partial^3 - \tfrac{1}{2}(\partial u + u\partial))\mathrm{R}_k[u]\,, \quad \mathrm{R}_0[u] = \tfrac{1}{2} \\ =& M^{KdV}\,\mathrm{R}_k[u]\,. \end{aligned} \tag{3.2.15}$$

We may obtain the weak coupling expansion $\kappa^2 = x^{-(2+1/k)} \to 0$ by observing that

$$u \sim x^{1/k}(1 - \sum_m Z_m \kappa^{2m}) \tag{3.2.16}$$

is an asymptotic solution to (3.2.13). Then the string susceptibility defined by $Z \sim (\lambda_c - l)^{(2-\Gamma_{str})\frac{\chi}{2}}$ is

$$\Gamma_{str} = -\frac{1}{k}\,. \tag{3.2.17}$$

According to (2.1.20), this is consistent with a minimal (non-unitary for $k > 2$) CFT with $(p, q) = (2k - 1, 2)$.

The coefficients Z_m in (3.2.16) grow as $(2m)!$ making the series non-Borel summable [44]. This means that there is an ambiguity in defining u non-perturbatively from (3.2.16). Every function that has an asymptotic expansion (3.2.16) will differ from another by exponentially small terms. Therefore the double scaling limit is a non-perturbative *definition* of the sum over genera of $(2k - 1, 2)$ minimal CFT. The $(2m)!$ growth is typical to string theory [45] as opposed to the $m!$ growth in field theory.

If the k$^{\text{th}}$ multicritical point is obtained form the multicritical potential $V_k(\lambda, g)$, the scaling operators σ_k of the theory are obtained by introducing sources t_{2k+1} and perturbing V_k to $V_k - \sum_l t_{2l+1} V_l N^{\frac{2(k-l)}{2k+1}}$. This defines the massive model and the solutions are given by the massive string equation

$$\sum_l t_{2l+1}(2l + 1)\mathrm{R}_l[u] = x\,, \tag{3.2.18}$$

which give the k$^{\text{th}}$ multicritical point when $t_{2k+1} = \frac{2k!}{(2k+1)!!}$ and all other t_{2l+1} are zero. This gives the partition function $Z_\Phi(t_1, t_3, \ldots)$ as a function of the "masses" or "times" t_{2l+1} and we define

$$\langle\sigma_k\rangle = \frac{\partial}{\partial t_{2k+1}} \ln Z_\Phi\,. \tag{3.2.19}$$

In the spherical limit $x \to \infty$, (3.2.18) reduces to

$$x = u^k - \sum_l t_{2l+1} u^l \tag{3.2.20}$$

which gives (see [4] for an elegant derivation)

$$\begin{aligned}\langle\sigma_{l_1}\dots\sigma_{l_p}\rangle &= \frac{\partial}{\partial t_{2l_1+1}}\dots\frac{\partial}{\partial t_{2l_p+1}}\ln Z_\Phi[t] \\ &\propto x^{(\Sigma+1-(p-2)k)/k}\end{aligned} \tag{3.2.21}$$

where $\Sigma=\sum_i l_i$. For the operator σ_l we obtain

$$\langle\sigma_l\rangle \sim x^{\frac{l+k+1}{k}} . \tag{3.2.22}$$

From the point of view of $(2k-1,2)$ minimal CFT coupled to gravity via Liouville theory, x is coupled to the lowest dimensional operator of the theory, which in the notation of (2.1.23) and (2.1.25) scales with the area as $\alpha_0/\gamma=k/2$. (3.2.20) gives the scaling of the *gravitational* operators with respect to the lowest dimensional operator to be $\alpha/\alpha_0=\frac{l-k}{k}$. The scaling with respect to the area is given by $\frac{\alpha}{\gamma}=\frac{\alpha}{\alpha_0}\frac{\alpha_0}{\gamma}=\frac{l-k}{2}$ with $l=0,1,\dots,k-1$ which coincides with (2.1.25) for $(p,q)=(2k-1,2)$. The same scaling can be obtained from (3.2.22) by dividing by the scaling of the *quantum gravity* partition function which scales as $\sim x^{2+\frac{1}{k}}$. This normalization has to be taken into account since (3.2.22) gives *disconnected* quantum gravity correlation functions (but connected *matrix model* correlation functions since $\ln Z_\Phi=Z_{str}$. Note that $l=0,1,\dots,k-1$ are the relevant operators of the theory.

The agreement of the matrix model results with Liouville theory can be further established by calculating loop operators. An operator of the form $\mathrm{tr}\Phi^p$ inserts a loop of p lattice lengths on the surface. We may obtain a loop of finite length L in the continuum limit -a *macroscopic* loop- by carefully taking the limit $p\to\infty$ while keeping $L=\frac{1}{2R_c}pa^{2/k}$ fixed. R_c is given in (3.2.12) and $a=N^{-k/(2k+1)}$ is the lattice cutoff. In this limit we can prove that $\mathrm{tr}\Phi^p$ tends to a macroscopic loop operator [16] $w(L)$ whose correlator is given by

$$\langle w(L)\rangle = \int_{\sqrt{\mu}}^{\infty} dx\ < x|\mathrm{e}^{LQ}|x> . \tag{3.2.23}$$

The states $|x>$ are the limits in the continuum of the discrete orthonormal states $|n>= \frac{1}{\sqrt{h_n}} e^{-\frac{N}{2\lambda}V} P_n$. The operator Q is (minus) the Schrödinger operator

$$Q = \partial^2 - u\,. \tag{3.2.24}$$

The two loop corelation function is given by

$$\langle w(L) w(L') \rangle = \int_{\sqrt{\mu}}^{\infty} dx\, dx' \ < x|e^{LQ}|x' >< x'|e^{L'Q}|x > \ . \tag{3.2.25}$$

The operator $w(L)$ can be expanded in terms of σ_l as

$$\begin{aligned} w(L) =& \sum_{l\geq 0} L^{l+\frac{1}{2}} \sigma_l \\ =& 2\sum_{l\geq 0} (2l+1) \frac{I_{l+\frac{1}{2}}(L\sqrt{\mu})}{\sqrt{(\mu)}^{l+\frac{1}{2}}} \widehat{\sigma_l} \end{aligned} \tag{3.2.26}$$

The operators $\widehat{\sigma_l}$ and σ_l are related by an upper triangular operator whose coefficients are analytic functions of μ. Since we are interested only in the non-analytical behaviour of the operators, there is really no distinction between them in the physics of the continuum. In the spherical limit it is easy to calculate (3.2.25) and obtain the correlator $\langle \widehat{\sigma_l} w(L) \rangle$ in the limit $L' \to 0$. The result is [16]

$$\langle \widehat{\sigma_l} w(L) \rangle = \sqrt{\mu}^{l+\frac{1}{2}} K_{l+\frac{1}{2}}(\sqrt{\mu} L) \tag{3.2.27}$$

and we see that it agrees with (2.1.27)! This is remarkable because (2.1.27) results from a minisuperspace approximation. It is not clear why this should be so.

I close this section hoping that I gave to the reader a flavor of some of the successes of matrix models in confirming perturbative results of Liouville theory and providing a sensible definition of non-perturbative non-critical string theory in dimensions less than one. Despite this success, however, we still cannot get any lessons on important problems of Quantum Gravity, such as the quantum nature of singularities, loss of information in black holes etc. We got some important lessons, like for example the $(2h)!$ growth of the perturbative terms of string theory, the importance of singular geometries with non-normalizable wave functions in quantum gravity and string theory and the beautiful mathematical structure that arises in the continuum that will be the subject of the next section. There are still many interesting open problems in the field, such as the solution of quantum Liouville theory, the better understanding of the backgrounds that arise in matrix models and, most important for string theory, the passage of the $c = 1$ barrier that will lead to the understanding of physics in realistic dimensions.

The results that we discussed in this section were confined to one hermitian matrix models (HMM). The more general model of a chain of matrices (MHMM) will be discussed in the last section of this chapter. These will lead to the solution of general (p, q) CFT.

3.3. Hermitian Matrix Models and KdV Flows

The integrable nature of HMM in the double scaling limit arises most naturally in the operator formalism [5]. The KdV hierarchies describe the flows between multicritical points [5,16] which are induced by perturbing away from the multicritical points using the operators σ_l. The "times" t_{2l+1} of the flows are the sources of these operators. The flows are compatible with the string equation, *i.e.* solutions to the string equation flow to other solutions of the string equation. The restriction put by the string equation into the space of solutions to the KdV hierarchy is described by a set of constraints that these solutions must satisfy. Those constraints act on the square root of the partition function and satisfy the centerless Virasoro algebra.

At the discrete level we choose an orthonormal basis

$$\begin{aligned} |n> &= \pi_n(\phi) \\ &= \frac{1}{\sqrt{h_n}} \mathrm{e}^{-\frac{N}{2\lambda}V} P_n(\phi) \end{aligned} \tag{3.3.28}$$

which satisfy

$$< n|m > = \int d\phi \, \pi_n(\phi)\pi_m(\phi) = \delta_{nm} \, .$$

The operator of multiplication by ϕ and the derivative operator act on this space of states by

$$\begin{aligned}\phi\,\pi_n =& Q_{nm}\pi_m \\ =& \sqrt{R_{n+1}}\pi_{n+1} + \sqrt{R_n}\pi_{n-1}\end{aligned} \tag{3.3.29}$$

and

$$\frac{d}{d\phi}\pi_n = P_{nm}\pi_m\,. \tag{3.3.30}$$

For the k^{th} multicritical point we take the minimal degree multicritical potential, which is of order $2k$. Then, because of (3.3.29), P_{nm} has only $2k-1$ non-vanishing lines off the diagonal. Therefore we can tune Q_{nm} and P_{nm} to a second and $(2k-1)^{\text{th}}$ order differential operator, respectively, that take the form

$$\begin{aligned}Q_{nm} \to& Q = \partial^2 - u \\ P_{nm} \to& P = \sum_{l=0}^{2k-1} \alpha_l(x)\partial^l\,.\end{aligned} \tag{3.3.31}$$

The string equation is obtained from the obvious relation $[\frac{d}{d\phi}, \phi] = 1$ that in the continuum limit becomes

$$[P, Q] = 1\,. \tag{3.3.32}$$

This equation is highly nontrivial. The left hand side is a differential operator of order $2k-1$ and gives $2k$ equations that the $2k-1$ functions $\alpha_l(x)$ and $u(x)$ must satisfy (we chose a basis so that $\alpha_{2k-1} = 1$). These equations determine the functions $\alpha_l(x)$ in terms of $u(x)$ and its derivatives and give the string equation for $u(x)$. The result is

$$\begin{aligned}P_k =& Q_+^{k-\frac{1}{2}} \\ x =& 2\mathrm{R}_k[u]\,.\end{aligned} \tag{3.3.33}$$

The unique operator $Q^{\frac{1}{2}} = \partial + \sum_{i=1}^{\infty} q_i(x)\partial^{-i}$ is a pseudo-differential operator (ΨDO). A ΨDO is a formal laurent series of the ∂ symbol with coefficients functions of x (which will be considered formal series of x). The negative powers of ∂ are defined by analytically continuing the Leibnitz rule

$$\partial^{-1} f(x) = \sum_{k=0}^{\infty} (-1)^k f^{(k)}(x)\partial^{-1-k}\,.$$

We define by Ψ_+ and Ψ_- the part of the ΨDO $\Psi = \sum_{i<+\infty} \psi_i(x)\partial^i$ with positive and negative powers of ∂ respectively. We also define by $Res\,\Psi$ to be the function ψ_{-1}. Then (3.3.33) derives from the simple relation $[Q_+^{k-\frac{1}{2}}, Q] = -[Q_-^{k-\frac{1}{2}}, Q] = Res\, Q^{k-\frac{1}{2}}$ and from the result of [43] that $Res\, Q^{k-\frac{1}{2}} = 2\mathrm{R}_k[u]$.

We may perturb away from the multicritical point using the operators σ_l. Then $V_k \to V_k + \sum_l t_{2l+1} V_l$ and $P_k \to P = P_k + \sum_l t_{2l+1} P_l$. By normalizing appropriately we obtain the string equation of the massive model

$$
\begin{aligned}
P &= \sum_l t_{2l+1} Q_+^{l-\frac{1}{2}} \\
\sum_{l>1} &t_{2l+1} \mathrm{R}_l[u] = x\,.
\end{aligned}
\tag{3.3.34}
$$

(3.3.34) gives $u = u(t)$ and consequently $Z = Z(t)$. The t evolution of u is given by the KdV integrable flows [16]

$$
\frac{\partial u}{\partial t_{2k+1}} = \partial \mathrm{R}_k[u]\,. \tag{3.3.35}
$$

This is proven by taking the continuum limits of the operators σ_l as I will explain in chapter 6 for the UMM.

A very important question that someone can ask at this point is whether we can start from a given background described by the k^{th}multicritical point and use the flow equation to reach a different multicritical point. This is in principle true, since (3.3.34) and (3.3.35) are compatible with each other. A physical solution, however, must obey certain boundary conditions. In the case of string theory the solutions should be asymptotic to the weak coupling asymptotic expansion (3.2.16). They should also be real and free or real poles. Reality is imposed by the requirement of reality of the partition function. A generic solution of (3.3.34) will be functions with second order poles on the real axis [1–4]. These correspond to zeroes of the partition function. The position of the pole is a free parameter of the theory and the solution cannot be specified by the genus expansion requirement. Moreover, [46] found that solutions with poles on the real axis are incompatible with the loop equations - the Schwinger-Dyson equations of the matrix model.

For k odd there exist well defined, unique physical solutions specified by the weak coupling expansion [47]. For k even there exist no real, pole free solution satisfying (3.2.16). This reflects the fact that a minimal even order multicritical potential is unbounded from below - the coefficient of the highest power term is negative- , making the path integral ill defined. Then one must use the path integral as the definition of perturbation theory and the differential equation as the *definition* of the non-perturbative, continuum theory. By using analytic continuation, we obtain *complex* boundary conditions ($\sim x^{1/k}$) for $x \to -\infty$. In that case the solutions are pole free on the real axis but they are complex. The imaginary part indicates a non-perturbative instability that is exponentially small for $x \to +\infty$. If one *defines* the matrix model at an even order multicritical point by using a higher order potential bounded from below, the real solutions will necessarily have poles on the real axis and they will be incompatible with the loop equations.

From the discussion above, we expect that the flows from odd order multicritical point physical solutions to even order ones will be impossible. If one tries to flow from the $k = 3$ points - the Yang-Lee edge singularity [48] - to the $k = 2$ - pure gravity - using (3.3.35), an instability forms and the $k = 3$ background never decouples from the $k = 2$ operator [49]. This was proven numerically in [49] and analytically in [50].

In the following I will mostly ignore physical questions of this nature and focus on the mathematical structure of the solutions to the string equation (3.3.32). I will discuss formal solutions to (3.3.32) without referring to any specific boundary condition. We can get a geometrical picture and compute the space of solutions to the string equation by introducing the Sato Grassmannian.

Consider the space of formal Laurent series

$$H = \{\sum_n a_n z^n \,, \quad a_n = 0 \quad \text{for} \quad n \gg 0\,\}$$

and its decomposition

$$H = H_+ \oplus H_- \,,$$

where $H_+ = \{ \sum_{n\geq 0} a_n z^n \,, \quad a_n = 0 \quad \text{for} \quad n \gg 0 \}$. Then the big cell of the Sato Grassmannian $Gr^{(0)}$ consists of all subspaces $V \subset H$ comparable to H_+, in the sense that the natural projection $\pi_+ : V \to H_+$ is an isomorphism.

Consider the space Ψ of pseudodifferential operators $W = \sum_{i\leq k} w_i(x)\partial^i$ where the functions $w_i(x)$ are taken to be formal power series (i.e. $w_i(x) = \sum_{k\geq 0} w_{ik}x^k \,, \; w_{ik} = 0 \,, \; k \gg 0$). W is then a pseudodifferential operator of order k. It is called monic if $w_k(x) = 1$ and normalized if $w_{k-1}(x) = 0$. The space Ψ forms an algebra. The space of monic, zeroth-order pseudodifferential operators forms a group $\mathcal{G}$.

There is a natural action of Ψ on H defined by

$$\begin{aligned} x^m\partial^n &: H \to H \\ &\phi \to (-\frac{d}{dz})^m (z)^n \,\phi \,. \end{aligned}$$

Then it is well known [51] that every point $V \in Gr^{(0)}$ can be uniquely represented in the form $V = SH_+$ with $S \in \mathcal{G}$.

Then, given the operator $Q = \partial^2 - u$, we may associate to it uniquely a point $V \in Gr^{(0)}$ by demanding that

$$Q = S^{-1}\partial^2 S \,. \tag{3.3.36}$$

Then the string equation (3.3.32) gives a unique operator A_k such that [23]

$$P = S^{-1}A_k S \,, \tag{3.3.37}$$

where

$$A_k = \frac{d}{dz^2} + \sum_{\substack{l=-2 \\ l\,\text{odd}}}^{2k-1} \alpha_l z^l \,, \qquad \alpha_{-2} = -\frac{1}{2} \,. \tag{3.3.38}$$

Since a differential operator leaves H_+ invariant, the conditions $Q\,H_+ \subset H_+$ and $P\,H_+ \subset H_+$ give

$$\begin{aligned} Q\,H_+ \subset H_+ &\Rightarrow QS^{-1}\,V \subset S^{-1}V \\ &\Rightarrow \partial^2\,V \subset V \\ &\Rightarrow z^2\,V \subset V \end{aligned} \tag{3.3.39}$$

and similarly [23,25,26,27]

$$A_k V \subset V\,. \tag{3.3.40}$$

It follows that the solutions to the string equation (3.3.32) are given by points $V \in Gr^{(0)}$ that satisfy (3.3.39) and (3.3.40). The KdV evolution of those solutions are given by spaces $V(t) = V(t_1, t_3, \ldots)$ that evolve according to the equations

$$\frac{\partial}{\partial t_{2l+1}} V(t) = z^{2l-1} V(t) \tag{3.3.41}$$

which imply $V(t) = \exp(\sum_l t_{2l+1} z^{2l+1})\, V \equiv \gamma(t,z)\, V$.

Then solutions to the string equation flow to solutions of the string equation since

$$\begin{aligned} z^2 \gamma V \subset \gamma V &\Rightarrow z^2 V(t) \subset V(t) \\ \gamma A_k \gamma^{-1} \gamma V \subset \gamma V &\Rightarrow A_k(t)\, V(t) \subset V(t)\,, \end{aligned} \tag{3.3.42}$$

where

$$A_k(t) \equiv \gamma A_k \gamma^{-1} = A_k - \frac{1}{2} \sum_{\substack{l=-2 \\ l\,\mathrm{odd}}}^{2k-1} (l+2) t_{l+2} z^l\,. \tag{3.3.43}$$

By formulating the string equation in terms of conditions on points in $Gr^{(0)}$, A. Schwarz [23] computed the space of solutions to the string equation (3.3.32). If $\mathcal{M}_{2k-1,2}$ is the space of 2×2 matrices $P = (P_{ij})$ with polynomial entries satisfying $2k - 1 = \max_{j=1,2}(j - i + 2\, deg\, P_{ij})$ for every i, and $\mathcal{T}_2$ is the group of invertible 2×2 upper triangular matrices $T = (T_{ij})$ acting on $\mathcal{M}_{2k-1,2}$ by $P \to TPT^{-1}$, then the space of solutions is given by $\mathcal{M}_{2k-1,2}/\mathcal{T}_2$.

The solutions to the string equation can be given in terms of the τ-function

$$u = -2\partial^2 \ln \tau\,.$$

The KdV τ-function obey the bilinear Hirota equations and will be discussed in detail in chapter 7. The matrix model partition function for models with even potentials is given by

$$Z = \tau^2\,. \tag{3.3.44}$$

There is a $1-1$ relation between the points $V \in Gr^{(0)}$ and the τ-functions via the Plücker embedding and the two dimensional equivalence between bosons and fermions. Then the relation

$$z^{2(n+1)} A V \subset V \qquad n \geq -1 \tag{3.3.45}$$

implies a set of constraints for the τ-function associated with V:

$$L_n \tau = 0 \qquad n \geq -1 \, . \tag{3.3.46}$$

The L_n satisfy the centerless Virasoro algebra $[L_n, L_m] = (n-m)L_{n+m}$. The action of L_n's on τ is given by

$$\mathrm{L}_n = \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2(k+n)+1}} + \frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2}{\partial t_{2k-1}\partial t_{2(n-k)+1}} + \frac{1}{16}\delta_{n,0} + \frac{1}{8}t_1^2\delta_{n,-1} \tag{3.3.47}$$

Note that the Virasoro constraints on the τ-function are the necessary conditions for the compatibility of the string equation and the KdV flows. The authors of [17,18] derive the Virasoro constraints as a consequence of the loop equations that the macroscopic loop operators should satisfy. I will discuss the connection of the Virasoro constraints in the context of the fermion-boson formalism in chapter 7. In this formalism the τ-function corresponds to a state $|\Omega>_t$ in the bosonic Fock space. The Virasoro operators (3.3.47) can be considered as the negative order Laurent expansion coefficients of the stress energy tensor [18]

$$T(z) = -\frac{1}{2} : \partial\varphi(z)^2 : + \frac{1}{16z^2}$$

of a free twisted bosonic field $\varphi(z) = -\varphi(2\pi i z)$. Acting on the states $|\Omega>_t$ the positive modes of the field $\partial\varphi(z)$ creates a loop operator $w(z)$ where $w(z) = \int dL \mathrm{e}^{-Lz} w(L)$ is the Laplace transform of the loop operator $w(L)$

$$\frac{<0|\partial\varphi(z)|\Omega>_t}{<0|\Omega>_t} = <w(z)>_t \, . \tag{3.3.48}$$

The negative modes of the field $\partial\varphi(z)$ are similarly annihilating a loop $w(z)$. The authors of [18] speculate that $\varphi(z)$ plays the role of a second quantized string field in a possible formulation of string field theory.

3.4. Matrix Models of Hermitian Chains of Matrices.

A matrix model of a chain of $q-1$ $N \times N$ hermitian matrices (MHMM) $\Phi(t)$ linearly coupled to each other, is defined by the partition function

$$Z_q = \int [\prod_{t=1}^{q-1} d\Phi(t)] \exp\{-\frac{N}{\lambda}(\sum_{t=1}^{q-1} \mathrm{tr} V_t(\Phi(t)) + \sum_{t=1}^{q-2} c_t \mathrm{tr}\Phi(t)\Phi(t+1))\}\,. \tag{3.4.49}$$

The interpretation in terms of discrete random surfaces is that (3.4.49) describes the perturbation theory of a statistical system of $(q-1)$-valued spins lying on the nodes of a discretized Riemann surface. Mehta [52,53] explained how to integrate the angular degrees of freedom and reduce the integral (3.4.49) to an integral over the eigenvalues $\phi_i(t)$ of the matrices $\Phi(t)$. The result is

$$Z_q = \int [\prod_{i,t} d\phi_i(t)]\, \Delta(\phi(1))\Delta(\phi(q-1)) \exp\{-\frac{N}{\lambda} \sum_{i,t} (V_t(\phi_i(t)) + c_t\phi_i(t)\phi_i(t+1))\} \tag{3.4.50}$$

with $\Delta(\phi(t)) = \prod_{i<j}(\phi_i(t) - \phi_j(t))$.

In this form the matrix model can be solved with the aid of orthogonal polynomials as described in the previous section. The operators P and Q discussed for HMM are defined similarly

$$\begin{aligned} \frac{d}{d\phi}\pi_n(\phi,t) &= P_{nm}(t)\pi_m(\phi,t) \\ \phi\pi_n(\phi,t) &= Q_{nm}(t)\pi_m(\phi,t)\,. \end{aligned} \tag{3.4.51}$$

There is a recursion relation [5] relating $P(t)$ and $Q(t)$ to $P(t-1)$ and $Q(t-1)$. Therefore all the operators $P(t)$ and $Q(t)$ are determined from $P \equiv P(1)$ and $Q \equiv Q(1)$. The double scaling limit at the p^{th} multicritical point is reached by taking the limit $N \to \infty$ and $\lambda \to \lambda_c$. In this limit

$$\begin{aligned} P &= \sum_{i=0}^{p} \alpha_i(x)\partial^i \\ Q &= \sum_{i=0}^{q} u_i(x)\partial^i \end{aligned} \tag{3.4.52}$$

where $\alpha_p = 1$, $u_q = 1$, $u_{q-1} = 0$ and $u_{q-2} \equiv u$ gives the scaling part of the specific heat. The string equation is given by (3.3.32) with solution

$$P = Q_+^{p/q} . \tag{3.4.53}$$

The operators P and Q act on $Gr^{(0)}$ via the operators [23]

$$\begin{aligned} P =& S^{-1} A S \\ Q =& S^{-1} \partial^q S , \end{aligned}$$

with $A_{p,q} = \frac{d}{dz^q} + \sum_{i=-q}^{p} \alpha_i z^i$, $\alpha_{-q} = -\frac{q-1}{2}$ and $\alpha_m = 0$ for $m = 0 \, mod \, q$. The string equation has solutions given by the points $V \in Gr^{(0)}$ satisfying [23,25,26,27]

$$\begin{aligned} A \, V \subset& V \\ z^q \, V \subset& V . \end{aligned} \tag{3.4.54}$$

The evolution of the solutions V and the operator A with the couplings t is given by

$$\begin{aligned} \frac{d}{dt_r} V(t) =& z^r \, V(t) \\ A(t) =& \gamma(t,z) A \gamma^{-1}(t,z) \end{aligned} \tag{3.4.55}$$

with $\gamma(t,z) = \exp(\sum_{k \neq 0 \, mod \, q} t_k z^k)$. The first of these equations are the q-reduced KP hierarchy, alternatively given by

$$\frac{dQ}{dt_r} = [Q_+^{r/q}, Q] , \qquad r \neq 0 \, mod \, q . \tag{3.4.56}$$

The τ-function associated with V is now annihilated by the Virasoro constraints

$$L_k \, \tau = 0 , \qquad k \geq -1$$

with

$$L_k = \frac{1}{q} \sum_{2a<kq} J_a J_{kq-a} + \frac{1}{2q} J_{kq/2}^2 + \frac{q^2-1}{24q} \delta_{k,0}$$

which is a consequence of the relation $z^{qk} A V \subset V$. Here the currents $J_a \equiv \partial/\partial t_a$ for $a > 0$ and $J_a \equiv -a t_{-a}$ for $a < 0$. There are additional constraints on the τ-function that obey the W-algebra. They are given by the relation

$$W_{kl} V = z^{kq} A^l V \subset V \,. \tag{3.4.57}$$

These are extra non-trivial constraints only when $q > 2$. For $q = 2$ they reduce to the Virasoro constraints [54].

The space of solutions to (3.3.32) is now given by the quotient $\mathcal{M}_{p,q}/\mathcal{T}_q$, where $\mathcal{M}_{p,q}$ is the space of polynomial $q \times q$ matrices with entries $P_{ij}(z)$ satisfying $p = \max_{1 \le j \le q}(j - i + q \, deg \, P_{ij})$ and $\mathcal{T}_q$ is the group of $q \times q$ upper triangular matrices.

MHMM are very important because they are related to the (p,q) minimal CFT coupled to gravity. The weak coupling expansion of the specific heat u_{q-2} gives the string susceptibility exponent $\Gamma_{str} = -\frac{2}{p+q-1}$. The scaling operators are given by perturbations $P \to P + t Q_+^{|pr-qs|/(q-1)}$ and scale with the area with exponent $\frac{\alpha}{\gamma} = \frac{2q}{p+q-1}$, exactly as in these theories. There is also a more economical way to obtain the (p,q) minimal CFT models by simply considering a two MHMM [6,7]. By choosing potentials $V_1(\Phi(1))$ and $V_2(\Phi(2))$ of order p and q respectively, one can reach a series of multicritical points labeled by (p,q) such that in the double scaling limit the operators P and Q are differential operators of order p and q respectively. One finds the same set of operators with the correct scaling behaviour, giving the same string equation and the generalized KdV hierarchy as the one obtained from the $(q-1)$ MHMM.

CHAPTER 4

The Large-N Limit of Unitary Matrix Models

From this chapter on I discuss Unitary One Matrix Models (UMM). In this chapter I discuss the definition of the model, the motivation that led to the study of the problem and the multicritical behaviour of UMM in the naive large N ("planar" or "spherical") limit. The latter is better understood in the Coulomb gas like behaviour of the eigenvalues of the unitary matrix on the unit circle. I restrict the discussion to a certain class of potentials which are real, polynomial and symmetric under reflection on the x axis of the complex z-plane where the eigenvalues lie. This is called the symmetric UMM or simply UMM. For a brief discussion of more general potentials leading to a more general multicritical behaviour see the last section of chapter 6.

4.1. Definitions and Motivation

The partition function of UMM is given by the integral

$$Z_U = \int dU \, \exp\{-\frac{N}{\lambda}\mathrm{tr}V(U+U^\dagger)\}\,, \tag{4.1.1}$$

where U is a $2N \times 2N$ (or $(2N+1)\times(2N+1)$) unitary matrix,

$$\begin{aligned} V(U+U^\dagger) &= \sum_{k\geq 1} g_k (U+U^\dagger)^k \\ &= \sum_{k\geq 1} \tilde{g}_k (U^k + U^{-k})\,, \end{aligned} \tag{4.1.2}$$

and dU is the Haar measure of the unitary group $U(2N)$.

These models were studied by Gross and Witten [8] and Wadia [55] in 1980 in the context of two dimensional pure QCD. Pure 2-d $U(2N)$ gauge theory on the lattice is defined by the Wilson action

$$S(U) = \sum_p \frac{1}{g^2}\mathrm{tr}(\prod_p U + h.c.)\,, \tag{4.1.3}$$

where $\prod_p U = U_{n,i_0} U_{n+i_0,i_1} U_{n+i_0+i_1,-i_0} U_{n+i_1,-i_1}$ is the product over each plaquette of the parallel transport matrices U_{n,i_k} from the lattice site n to the lattice site $n+i_k$ and i_0 and i_1 are the unit vectors on the square lattice in the time and space direction respectively. The partition function is

$$Z_{QCD} = \int \mathcal{D}U \exp\{-S(U)\}\,. \tag{4.1.4}$$

$\mathcal{D}U = \prod_{n,i} dU_{n,i}$, where $dU_{n,i}$ is the Haar measure of the unitary group $U(2N)$ on the link (n,i). dU is invariant under $U \to VU$ or $U \to UV$ with $V \in U(2N)$. The theory is gauge invariant under $U_{n,i} \to V_n U_{n,i} V^\dagger_{n+i}$. If we fix the gauge $A_0 = 0$ ($U_{n,\pm i_0} = 1$) and change variables $U_{n+i_0,i_1} = W_n U_{n,i_1}$ we obtain

$$\begin{aligned} Z_{QCD} =& \int [\prod_n dW_n]\, \exp\{-\frac{1}{g^2}\sum_n \mathrm{tr}(W_n + W_n^\dagger)\} \\ =& (Z_W)^{\frac{V}{a^2}}\,. \end{aligned} \tag{4.1.5}$$

V is the volume of the 2-d world and a is the cutoff. The evaluation of (4.1.5) in the large N limit using the techniques described in the following chapter, reveals a third order phase transition as $\lambda \equiv g^2 N \to 1$ from the weak to the strong coupling phase. Important lessons for real QCD can be learned for the validity or not of the interchangeability of the strong and weak coupling limits with the large N limit, the area law and stringy behaviour of QCD and the possibility of a large N third order phase transition for real QCD separating the weak from the strong coupling phase.

For more general potentials, UMM have a similar multicritical structure as HMM with multicritical points labelled by an integer k. After the discovery of the double scaling limit for HMM, Periwal and Shevitz [9,10] discovered a similar double scaling limit for the UMM. This is reached in a quite similar way by taking $N \to \infty$ and $\lambda \to \lambda_c$ and keeping $t = (1-\frac{n}{N})N^{\frac{2k}{2k+1}}$ and $y = (1-\frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ fixed. In this limit there exists a genus expansion of the free energy in the weak coupling limit in terms of the “string” coupling $\kappa^2 = x^{-2-\frac{1}{k}}$. It is tempting to try to relate the theory to a known CFT coupled to gravity. Comparison of the string susceptibility and the scaling part of microscopic and microscopic loops [11] point to $(4k, 2)$ superminimal CFT or $O(-2)$ models. Careful examination of the solutions to the string equation [56] rules out the latter and the lack of sufficient continuum results make the former inconclusive. Other possibilities could be a topological phase of quantum gravity or an open-closed string theory interpretation [57] but the correct interpretation, if any, is far from being well understood.

One difficulty with interpreting UMM as a statistical model coupled to gravity is the lack of a clear random surface description similar to the one found for HMM. Neuberger *e.g.* attempts to obtain such an interpretation by considering (4.1.1) written in terms of a hermitian matrix Φ such that $U = e^{i\Phi}$. In the case of the simplest potential $V(U + U^\dagger) = U + U^\dagger$, he introduced fermionic (grassmannian) $2N \times 2N$ hermitian matrices ψ and $\overline{\psi}$ in order to exponentiate the Haar measure dU. The result is

$$Z_U = \int d\Phi d\psi d\overline{\psi}\, \exp\{-\mathrm{tr}(\Phi^2 + \overline{\psi}\psi + 2\sum_{l=1}^{\infty} \frac{(-1)^l}{2l+1} (\frac{\lambda}{N})^l \sum_{k=0}^{2l} \frac{(-1)^k}{k!(2l-k)!} \Phi^k \overline{\psi} \Phi^{2l-k} \psi$$
$$+ 2\sum_{l=1}^{\infty} \frac{(-1)^l}{(2l+2)!} (\frac{\lambda}{N}) \Phi^{2l+2})\}\,.$$

This describes a surface with an infinite number of types of bosonic vertices. The bosonic parts of the surfaces have fermionic double lined boundaries. It is not clear however what types of surfaces are "creamed out" by the double scaling limit. In that respect the result of [14,11], that show that HMM have multicritical potentials that lead to a double scaling limit which is the same as the one obtained from UMM, is very interesting. This class of potentials look like a double well potential as opposed to the ones described in the previous section that look like a single well potential. The density of eigenvalues has support on two segments (cuts) of the real line and criticality is reached when the two ends of the segments meet with each other. In the potentials described in the previous section, criticality is reached when the potential is such that the eigenvalues reach the tip of the well and are just about to "spill out" on the whole (or half) real axis. The former case resembles the multicritical points of UMM discussed here. The eigenvalues of UMM in the weak coupling phase are distributed over part of the unit circle and multicriticality is observed when the eigenvalues are just covering the whole circle. The scaling of the eigenvalue distribution in the large N limit near the edge of its support is responsible for the scaling of the operators of the continuum theory and is similar for the two models. Therefore we expect the same universal behaviour from the two models and this is indeed the case.

Integrable hierarchies arise naturally in the continuum theory of UMM describing the -well defined- flows between multicritical points. The hierarchy related to UMM is the mKdV hierarchy. In the double cut HMM the relevant hierarchy is the NLS hierarchy [11] which reduces to the mKdV hierarchy when we consider only odd order perturbations. Most likely the same hierarchy arises from UMM by considering more general potentials -see discussion in section 6.4. These similarities with HMM are remarkable and give further motivation to search for a world sheet interpretation of UMM.

4.2. Critical Behaviour of UMM in the large N limit

The integral (4.1.1) can be reduced to an integral over the eigenvalues $z_j = \mathrm{e}^{i\alpha_j}$ of U by integrating out the angular components of U. We define $U = V^\dagger z V$ with $V \in U(2N)$ and $z = diag\,(z_j)$. Then $U \approx z + i[z, K]$ where $V = \mathrm{e}^{iK}$. By computing the Jacobian as in the case of HMM we obtain

$$dU = \prod_i dU_{ii} \prod_{i<j} d\mathrm{Re}U_{ij}\, d\mathrm{Im}U_{ij} = dK d\alpha\, \Delta(\alpha)\overline{\Delta}(\alpha) \tag{4.2.6}$$

where $|\Delta(z)|^2 = |\Delta(\alpha)|^2 = \prod_{k<j} |z_k - z_j|^2 = 4^{2N} \prod_{k<j} \sin^2\left(\frac{\alpha_i - \alpha_j}{2}\right)$ is the Vandermonde determinant. Then we can integrate dK out and obtain [8]

$$\begin{aligned} Z_U &= \int \{\prod_j \frac{dz_j}{2\pi i z_j}\}\, |\Delta(z)|^2 \exp\{-\frac{N}{\lambda} \sum_i V(z_i + z_i^*)\} \\ &= \int \{\prod_j d\alpha_j\}\, |\Delta(\alpha)|^2 \exp\{-\frac{N}{\lambda} \sum_i V(2\cos\alpha_i)\}\,. \end{aligned} \tag{4.2.7}$$

The large N limit is dominated by the saddle points determined by

$$\frac{2N}{\lambda} V'(2\cos\alpha_i)\sin\alpha_i + \sum_{\substack{j=1 \\ i\neq j}}^{2N} \cot\frac{\alpha_i - \alpha_j}{2} = 0 \,. \tag{4.2.8}$$

At these points the free energy of the model is

$$\lim_{N\to\infty}\frac{1}{N^2}\ln Z_U = \lim_{N\to\infty}\frac{1}{N^2}\{-\frac{N}{\lambda}\sum_{i=1}^{2N} V(2\cos\alpha_i) + \sum_{\substack{j=1 \\ i\neq j}}^{2N} \ln|\sin\frac{\alpha_j - \alpha_i}{2}|\} \tag{4.2.9}$$

where the α_i are solutions to (4.2.8).

In the limit $N \to \infty$ we may reach a continuum limit of the above equations by replacing $\alpha_i = \alpha(\frac{i}{2N}) = \alpha(x)$ where $i = 1, \ldots, 2N\,, x \in [0,1]$ and $\frac{1}{2N}\sum_{i\neq j} \to P\int_0^1 dx\,,\ \frac{1}{2N}\sum_i \to \int_0^1 dx$. We introduce the density of eigenvalues

$$\rho(\alpha) = \frac{dx}{d\alpha} \geq 0 \quad \text{such that} \quad \int_{\alpha_c}^{2\pi-\alpha_c} d\alpha\,\rho(\alpha) = 1. \tag{4.2.10}$$

Condition (4.2.8) and the free energy (4.2.9) in the continuum limit are given by

$$\frac{1}{\lambda}V'(2\cos\alpha(x))\sin\alpha(x) = -P\int_{\alpha_c}^{2\pi-\alpha_c} d\beta\,\rho(\beta)\cot\frac{\alpha-\beta}{2} \tag{4.2.11}$$

and

$$\ln Z_U = -\frac{1}{\lambda}\int_{\alpha_c}^{2\pi-\alpha_c} d\alpha\,\rho(\alpha)\cos\alpha + P\int d\alpha d\beta\,\rho(\alpha)\rho(\beta)\ln|\sin\frac{\alpha-\beta}{2}| + \text{const.}\,. \tag{4.2.12}$$

Therefore the dynamics of the model are described by a one dimensional gas of like electric charges living on the unit circle. The first term in (4.2.12) describes their interaction with an external electric field and the second one the mutual Coulomb repulsion. For example if $\lambda > 0$ and $V(z + z^*) = z + z^*$, the gas is subject to an external homogeneous electric field in the $-x$ direction. When $\lambda \to +\infty$ (the strong coupling limit), the electric field is weak compared to the Coulomb interaction and the eigenvalues tend to distribute homogeneously on the unit circle. As $\lambda \to 0$ (weak coupling limit) the electric field pushes all eigenvalues close to $\alpha = \pi$. Therefore at an intermediate value $\lambda = \lambda_c$, $\rho(\alpha)$ thins out around $\alpha = 0$ and develops a cut. For $\lambda < \lambda_c$ $\rho(\alpha)$ has support only on the interval $[\alpha_c, 2\pi - \alpha_c]$. I denote this part of the unit circle S^1 by $S^1_{\alpha_c}$. As $\lambda \to \lambda_c$, $\alpha_c \to 0$ and the scaling of $\rho(\alpha)$ near the edge of its support leads to the scaling of all thermodynamic quantities and to a third order phase transition. For more complicated potentials there can be multiple cuts created with the passage of a phase of n_1 cuts to a phase with n_2 cuts. These transitions were studied by [58]. He found multicritical behaviour from transitions from the one cut phase to the no cut phase with $\Gamma_{str} = -1/k$ with $k = 1, 2, \ldots$. For the case were we have a transition from the one gap phase to the n gap phase, he found only transitions characterized by $\Gamma_{str} = -1$ and multicritical behaviour is not known to exist. No critical behaviour is observed from the no gap phase to the n gap phase.

In order to solve (4.2.11) for $\rho(\alpha)$ we study the function

$$F(z) = \int_{\alpha_c}^{2\pi-\alpha_c} d\beta\, \rho(\beta) \cot \frac{z-\beta}{2} \,. \tag{4.2.13}$$

The function (4.2.13) has the following properties

(i) It is periodic $F(z) = F(z + 2\pi)$.

(ii) It is analytic for $e^z \notin S^1_{\alpha_c}$.

(iii) It is real for $e^z \notin S^1_{\alpha_c}$ and as $S^1_{\alpha_c}$ is approached we have

$$F(\alpha \pm i\epsilon) = -\frac{1}{\lambda} V'(\cos\alpha)\sin\alpha \mp 2\pi i \rho(\alpha) \,. \tag{4.2.14}$$

(iv)

$$F(z) \to \mp i \quad \text{as} \quad z \to z_1 \pm i\infty \,. \tag{4.2.15}$$

In order to prove (iii) we use the well known identity $\frac{1}{w'-w\mp i\epsilon} = P(\frac{1}{w'-w}) \pm \delta(w'-w)$. Since $\cot(z-\beta/2) \sim \frac{2}{z-\beta}$ as $z \to \beta$ we have that $\cot(\alpha - \beta \mp \epsilon) = P(\cot\frac{\alpha-\beta}{2}) \pm 2i\pi\delta(\alpha-\beta)$. Using (4.2.10) and (4.2.11) we easily obtain (4.2.14). For the proof of (iv) use (4.2.13) and the normalization (4.2.10).

Solutions to the above conditions are given by

$$F(z) = -\frac{1}{\lambda}V'(\cos z)\sin z - P(\sin^2\frac{z}{2})\sin\frac{z}{2}(\cos^2\frac{z}{2} - \cos^2\frac{\alpha_c}{2})^{\frac{1}{2}} \qquad (4.2.16)$$

The polynomial $P(z)$ is a polynomial of degree one less than $V(z)$. We can solve for its coefficients and $\cos\frac{\alpha_c}{2}$ as a function of the couplings using (4.2.15). One way to do this is by setting $Z = \sin\frac{z}{2}$. Then $\cos\frac{z}{2} = -\sqrt{1-Z^2} = -iZ + \frac{i}{2Z} + \frac{i}{8Z^3} + \ldots$ (the sign choice is made for $\mathrm{Re}z > 0$ so that $2i\sin\frac{z}{2} = \mathrm{e}^{i\frac{z}{2}} - \mathrm{e}^{-i\frac{z}{2}}$ and $2\cos\frac{z}{2} = \mathrm{e}^{i\frac{z}{2}} + \mathrm{e}^{-i\frac{z}{2}}$ as $z \to z_1 + i\infty$) and we can expand (4.2.16) in terms of Z for large Z. Then (4.2.15) gives a set of linear equations determining the coefficients in the potential and α_c as a function of the couplings g_k and λ (there are k couplings g_k in the potential and λ and k coefficients in $P(z)$ and α_c). Note that g_k turn out to be real, resulting in real potentials, which is not a priori true.

From (4.2.14) and (4.2.16) we obtain the density of eigenvalues

$$\rho(\alpha) = \frac{1}{2\pi}P(\sin^2\frac{\alpha}{2})\sin\frac{|\alpha|}{2}(\cos^2\frac{\alpha_c}{2} - \cos^2\frac{\alpha}{2})^{\frac{1}{2}}\,. \qquad (4.2.17)$$

The k^{th} multicritical point is reached by tuning the couplings in the potential so that $P(z) = a_k z^{k-1}$ and $\cos\frac{\alpha_c}{2} \to 1$. In this case the critical density of eigenvalues is given by

$$\rho_k(\alpha) \propto \sin^{2k}\frac{\alpha}{2}\,, \qquad (4.2.18)$$

which for α close to its critical value $\alpha_c = 0$ gives

$$\rho_k(\alpha) \sim \alpha^{2k}\,. \qquad (4.2.19)$$

We always normalize the critical potential so that $\lambda_c = 1$. In this case the k^{th} multicritical potential is given by

$$-V_k'(2\cos z)\sin z + c_k\sin^{2k-1}\frac{z}{2}(\cos^2\frac{z}{2} - 1)^{\frac{1}{2}} \sim -i\,.$$

Since the potential is a polynomial, it is fully determined by its large $Z \equiv \cos\frac{z}{2}$ asymptotics. We find

$$V_k'(4Z^2 - 2) = \frac{1}{2}c_k(1 - Z^2)^{k-1}(1 - \frac{1}{Z^2})_+^{\frac{1}{2}}\,, \qquad (4.2.20)$$

In the last equation we expand the right hand side in powers of Z and keep only positive powers coefficients. By equating equal powers of Z^2 we obtain the couplings g_k. In chapter 6 we will see that the above potentials result in a third order phase transition such that $\ln Z_U \sim (\lambda_c - \lambda)^{2+\frac{1}{k}}$.

The scaling operators of the theory σ_l are those that perturb the multicritical potentials $V_k \to V_k + t_{2l+1}\widetilde{V}_l$ so that they modify the scaling behaviour of $\rho(\alpha)$ near the edge of its support

$$\rho_k(\alpha) \to \rho_k(\alpha) + \widetilde{\rho}_l(\alpha)\,.$$

The normalization condition for $\rho(\alpha)$ is preserved if $\int_{\alpha_c}^{2\pi-\alpha_c} \widetilde{\rho}_l(\alpha)\, d\alpha = 0$. The perturbations $\widetilde{\rho}_l(\alpha)$ scale as

$$\widetilde{\rho}_l(\alpha) \propto \sin^{2l}\frac{\alpha}{2} \sim \alpha^{2m} \tag{4.2.21}$$

giving relevant operators for $l < k$, marginal for $m = k$ and irrelevant for $m > k$. For $l < k$ the operators are relevant because they modify the leading scaling behaviour of $\rho_k(\alpha)$. The solutions satisfying the normalization condition are given by $\widetilde{\rho}_m(\alpha) \propto \frac{d}{d\alpha}\sin^{2m}\alpha(1-\cos^2\frac{\alpha}{2})_+^{\frac{1}{2}}$ and correspond to multicritical potentials

$$\widetilde{V}_l \propto (1-Z^2)^l(1-\frac{1}{Z})_+^{\frac{1}{2}} \tag{4.2.22}$$

The operators σ_l are further discussed in chapter 6. Their scaling in the large N limit is found to be [10,4]

$$\langle \sigma_{m_1} \ldots \sigma_{m_p} \rangle \sim x^{[\Sigma+1-(p-2)k]/k}\,. \tag{4.2.23}$$

where $\Sigma = \sum_i m_i$ and $x = 1 - \frac{\lambda}{\lambda_c}$. Note that (4.2.23) is identical to (3.2.21) and HMM and UMM have the same universal behaviour in the spherical limit!

CHAPTER 5

The Method of Orthogonal Polynomials for UMM

In this chapter I present the method of orthogonal polynomials that has been used successfully for solving the HMM [42] and UMM [9,10] at the discrete level. For UMM this method was first used by Periwal and Shevitz. They defined a basis of orthogonal polynomials $\{P_n(z)\}$ (I call it the PS basis) that enabled them to solve UMM in the double scaling limit. UMM in this basis is discussed in section 5.1. A more convenient basis was defined by Myers and Periwal [59] and was used to solve UMM by the authors of [21]. This is called the trigonometric basis $\{c_n^{\pm}(z)\}$ and is discussed in section 5.2. In this basis the similarities between HMM and UMM are more obvious and it is easy to develop the operator formalism for UMM. The transformation between the two bases is used in order to obtain the double scaling limit in the trigonometric basis.

5.1. The Periwal-Shevitz (PS) Basis $\{P_n(z)\}$.

The PS basis is defined by orthogonal polynomials

$$P_n(z) = z^n + S_{n-1} + \sum_{k=1}^{n-1} \alpha_{n,k} z^k \,, \qquad \alpha_{n,k} \geq 0 \,, \qquad S_{n-1} \geq 0$$

such that

$$< P_n, P_m >= \int d\mu \, P_n(1/z) P_m(z) = h_n \delta_{n,m} \,. \tag{5.1.1}$$

Recall that $z = \mathrm{e}^{i\alpha}$ so that $P_n(1/z) = P_n^*(z)$. The measure

$$\int d\mu = \oint \frac{dz}{2\pi i z} \, \mathrm{e}^{-\frac{N}{\lambda} V(z+z^*)} = \int \frac{d\alpha}{2\pi} \, \mathrm{e}^{-\frac{N}{\lambda} V(2\cos\alpha)} \,. \tag{5.1.2}$$

The main property of the polynomials $P_n(z)$ used for solving the model is

$$P_{n+1}(z) = z P_n(z) + S_n P_n(1/z) \,, \tag{5.1.3}$$

where

$$\frac{h_{n+1}}{h_n} = 1 - S_n^2 \,. \tag{5.1.4}$$

Proof: Define the operator M_n and the polynomials $Q_n(z)$ by

$$\begin{aligned} M_n P_m(z) =& z^n P_m(1/z) \\ Q_n(z) \equiv& M_n P_n(z) = z^n P_n(1/z)\,. \end{aligned} \tag{5.1.5}$$

Use the fact that $z^n = P_n(z) + \sum_{k=0}^{n-1} c_{n,k} P_k(z)$ to prove that

$$< z^k | \frac{1}{h_n} Q_n >= \int d\mu\, z^{-k} \frac{1}{h_n} Q_n(z) = \delta_{k,0} \qquad k \le n\,. \tag{5.1.6}$$

Then

$$< P_m | \frac{1}{h_n} Q_n >= (< z^m | + S_{m-1} < z^0 | + \sum_{k=1}^{m-1} \alpha_{m,k} < z^k |) | \frac{1}{h_n} Q_n >= S_{m-1} \qquad m \le n\,,$$

so that

$$\begin{aligned} \frac{1}{h_n} Q_n(z) &= \sum_{m=0}^{n} \frac{1}{h_m} P_m(z) < P_m | \frac{1}{h_n} Q_n > \\ &= \sum_{m=o}^{n} \frac{S_{m-1}}{h_m} P_m(z)\,. \end{aligned} \tag{5.1.7}$$

Then

$$\frac{1}{h_n} Q_n(z) - \frac{1}{h_{n-1}} Q_{n-1}(z) = \frac{S_{n-1}}{h_n} P_n(z)\,. \tag{5.1.8}$$

Apply M_n on both sides and get

$$P_n(z) = \frac{h_n}{h_{n-1}} z P_{n-1}(z) + S_n Q_n(z)\,. \tag{5.1.9}$$

From (5.1.8), $Q_n(z) = S_{n-1} P_n(z) + \frac{h_n}{h_{n-1}} z^{n-1} P_{n-1}(1/z)$ so that

$$(1 - S_{n-1}^2) P_n(z) = \frac{h_n}{h_{n-1}} z P_{n-1}(z) + \frac{h_n}{h_{n-1}} S_{n-1} z^{n-1} P_{n-1}(1/z)\,. \tag{5.1.10}$$

Equating powers of z^n we obtain (5.1.4) and substituting back to (5.1.10) we obtain (5.1.3).

The partition function (4.2.7) is given in terms of the norms h_n of the orthogonal polynomials. By writing the Vandermonde determinant as

$$\begin{aligned}\Delta(z) =& \prod_{i<j}(z_i - z_j)\\ =& det\,(z_i^k)\\ =& det\,(P_k(z_i))\\ =& \sum_k \epsilon_{k_1 \ldots \kappa_{2N}} P_{k_1}(z_1)\ldots P_{k_{2N}}(z_{2N})\,,\end{aligned}$$

we obtain

$$\begin{aligned}Z_U =& \sum_{k,l}\int [\prod_j d\mu(z_j)]\epsilon_{k_1\ldots\kappa_{2N}}\epsilon_{l_1\ldots\lambda_{2N}} P^*_{k_1}(z_1)\ldots P^*_{k_{2N}}(z_{2N})P_{l_1}(z_1)\ldots P_{l_{2N}}(z_{2N})\\ =& \sum_{k,l}\epsilon_{k_1\ldots\kappa_{2N}}\epsilon_{l_1\ldots\lambda_{2N}} h_{k_1}\delta_{k_1,l_1}\ldots h_{k_{2n}}\delta_{k_{2N},l_{2N}}\\ =& (2N)!\prod_{i=0}^{2N} h_i = (2N)!\prod_{i=0}^{2N}(1-S_{2N-i})^i\,.\end{aligned} \tag{5.1.11}$$

Therefore by constructing the orthogonal polynomials - *e.g.* by the Gram-Schmidt method - we can compute the partition function. All the dynamical information is contained in the coefficients S_n. Alternatively one may solve directly for the coefficients S_n by solving a recursion relation called, the string equation. This is obtained by considering the trivial relation

$$\int d\mu\, P_n(1/z)\partial_z P_{n+1}(z) = -\int d\mu\, \partial_z P_n(1/z)P_{n+1}(z) + \int d\mu\, P_n(1/z)(\frac{1}{z}+\frac{N}{\lambda}\partial_z V)P_{n+1}(z)\,.$$

Using the fact that $\partial_z P_{n+1}(z) = (n+1)P_n(z) + \sum_{m=0}^{n-1}\gamma_{nm}P_m(z)$, that $\frac{1}{z}P_n(1/z) = P_{n+1}(1/z) + S_n z^{-n}P_n(z)$ and that $\partial_z V(z+\frac{1}{z}) = (1-\frac{1}{z^2})V'(z+\frac{1}{z})$ we obtain the string equation

$$\frac{\lambda}{N}(n+1)h_n\, S_n^2 = \int d\mu\, P_n(1/z)(1-\frac{1}{z^2})V'(z+\frac{1}{z})P_{n+1}(z)\,. \tag{5.1.12}$$

Using the recursion relation (5.1.3), (5.1.12) becomes a non linear functional relation for S_n. For example, when $V(z+\frac{1}{z}) = z+\frac{1}{z}$ we obtain

$$\frac{\lambda}{N}(n+1)S_n^2 = S_n(S_{n+1}+S_n)(1-S_n^2)\,. \tag{5.1.13}$$

It is instructive to study the action of the operators z, $z_\pm \equiv z \pm 1/z$ and $z\partial_z \equiv z\frac{d}{dz}$ on the polynomials $P_n(z)$.

Using (5.1.3) and (5.1.7) we derive

$$\begin{aligned} z_\pm P_n(z) =& P_{n+1}(z) \pm S_{n-1} P_1(1/z) + \sum_{k=0}^{n} (z_\pm)_{nk} P_k(z) \\ z\partial_z P_n(z) =& n P_n(z) + \sum_{k=1}^{n-1} (z\partial_z)_{nk} P_k(z) \end{aligned} \tag{5.1.14}$$

where, for example,

$$\begin{aligned} (z_\pm)_{nk} =& -\frac{h_n}{h_k} S_n S_{k-1} \mp \frac{h_n^2}{h_k^2} S_{n-1}\,, \qquad 1 < k < n-1 \\ (z\partial_z)_{nk} =& -\frac{N}{\lambda h_k} \int d\mu P_k(1/z) z_- V'(z_+) P_n(z)\,. \end{aligned}$$

Similarly we can obtain the action of the operators z^k and $1/z^k$ [10].

The lesson we get from this little exercise is that because z has an upper triangular action of z, the action of operators like $z_\pm$ and $z\partial_z$ do not give short term relation as in the case of HMM. This makes the operators non local when we take the continuum limit and the operator formalism is quite complicated. These problems are surpassed if we use the trigonometric basis $\{c_n^\pm(z)\}$.

The integrability of UMM makes itself manifest already from the discrete level. We may easily get

$$\begin{aligned} \frac{\lambda}{N}\frac{\partial \ln h_n}{\partial \tilde{g}_k} =& -\frac{1}{h_n}\int d\mu P_n(1/z) z_+ P_n(z) \\ =& 2 S_n S_{n-1} \end{aligned}$$

which gives using (5.1.4)

$$\frac{\lambda}{N}\frac{\partial S_n}{\partial \tilde{g}_1} = -(1-S_n^2)(S_{n+1} - S_{n-1})\,. \tag{5.1.15}$$

Similarly

$$\frac{\lambda}{N}\frac{\partial S_n}{\partial \tilde{g}_2} = -(1-S_n^2)\{S_{n+2}(1-S_{n+1}^2) - S_n(S_{n+1}^2 - S_{n-1}^2) - S_{n-2}(1-S_{n-1}^2)\} \tag{5.1.16}$$

This system of equations possesses an R-matrix structure and form an integrable hierarchy. This was studied in [60]. It was called the modified Voltera hierarchy and it was shown in [21] that its continuum limit is the mKdV hierarchy.

5.2. The Trigonometric Basis $\{c_n^{\pm}(z)\}$.

By taking appropriate linear combinations of $P_n(z)$ and $P_n(1/z)$ which preserve the measure factor $|\Delta(z)|^2$ we may construct an alternative basis of orthogonal polynomials of the form [59]

$$\begin{aligned} c_n^{\pm}(z) &= z^n \pm z^{-n} + \sum_{i=1}^{i_{max}} \alpha_{n,n-i}^{\pm}(z^{n-i} \pm z^{-n+i}) \\ &= \pm c_n^{\pm}(z^{-1}) \,, \end{aligned} \tag{5.2.17}$$

where for $U(2N+1)$ n is a non-negative integer and $i_{max} = n$ and for U(2N) n is a positive half-integer and $i_{max} = n - \frac{1}{2}$. The polynomials $c_n^{\pm}(z)$ are orthogonal with respect to the inner product

$$\begin{aligned} \langle c_n^+ , c_m^+ \rangle &= \oint \frac{dz}{2\pi i z} \exp\{-\frac{N}{\lambda} V(z+z^*)\} \, c_n^+(z)^* c_m^+(z) \\ &= \mathrm{e}^{\phi_n^+} \, \delta_{n,m} \,, \\ \langle c_n^- , c_m^- \rangle &= \mathrm{e}^{\phi_n^-} \, \delta_{n,m} \,, \\ \langle c_n^+ , c_m^- \rangle &= 0 \,. \end{aligned} \tag{5.2.18}$$

The importance of this basis lies in the fact that the Vandermonde determinant is

$$|\Delta(z)|^2 = \left| det \begin{pmatrix} c_i^-(z_j) \\ c_i^+(z_j) \end{pmatrix} \right|^2 \quad , \tag{5.2.19}$$

where $j = 1, \ldots, 2N$, $i = \frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}$ for $U(2N)$ and $j = 1, \ldots, 2N+1$, $i = 0, 1, \ldots, N$ for $U(2N+1)$ (where the line $c_0^-(z) \equiv 0$ is understood to be omitted). This can be proved by induction. Similarly to (5.1.11) it can be shown that [21]

$$Z_N^U = \prod_n \mathrm{e}^{\phi_n^+} \mathrm{e}^{\phi_n^-} = \tau_N^{(+)} \tau_N^{(-)} \,. \tag{5.2.20}$$

The transformation from the PS to the trigonometric basis can be computed recursively. Although it is not known in closed form because of its complexity, the authors of [21] used symbolic manipulation programs to recursively compute the relation between the norms of the polynomials in the two bases. The consistency of the results confirms their computation. The result is that the norms $e^{\phi_n^{\pm}}$ are related to the norms h_n of the $P_n(z)$ polynomials by

$$e^{\phi_n^{\pm}} = 2(1 \mp S_{2n-1})\, h_{2n-1}\,. \tag{5.2.21}$$

Observe that

$$\begin{aligned} z_+ c_n^{\pm}(z) =& (z + \frac{1}{z}) c_n^{\pm}(z) \\ =& z_{n+1} \pm z^{-(n+1)} + \sum(\alpha_{n,k+1}^{\pm} + \alpha_{n,k-1}^{\pm})(z^n \pm z^{-n}) \\ =& c_{n+1}^{\pm}(z) + \sum_{k=0}^{n} (z_+)_{nk}^{\pm} c_k^{\pm}(z)\,. \end{aligned}$$

Using the condition (5.2.18) and the hermiticity properties (5.2.17) we derive two important recursion relations

$$\begin{aligned} z_+\, c_n^{\pm}(z) =& c_{n+1}^{\pm}(z) - r_n^{\pm}\, c_n^{\pm}(z) + R_n^{\pm}\, c_{n-1}^{\pm}(z) \\ z_-\, c_n^{\pm}(z) =& c_{n+1}^{\mp}(z) - q_n^{\pm}\, c_n^{\mp}(z) - Q_n^{\pm}\, c_{n-1}^{\mp}(z) \end{aligned} \tag{5.2.22}$$

All the coefficients in (5.2.22) can be computed as a function of the norms $\mathrm{e}^{\phi_n^{\pm}}$ and, using (5.2.21), as function of the S_n's of the PS basis. It is trivial to compute $R_n^{\pm}$ and $Q_n^{\pm}$ using the relation $\int d\mu(c_n^{\pm}(z))^* z_+ c_{n-1}^{\pm}(z) = \mathrm{e}^{\phi_n^{\pm}}$ and $\int d\mu(c_n^{\pm}(z))^* z_- c_{n-1}^{\mp}(z) = \mathrm{e}^{\phi_n^{\pm}}$. The result is

$$\begin{aligned} R_n^{\pm} =& \mathrm{e}^{(\phi_n^{\pm} - \phi_{n-1}^{\pm})} = (1 \mp S_{2n-1})(1 - S_{2n-2}^2)(1 \pm S_{2n-3}) \\ Q_n^{\pm} =& \mathrm{e}^{(\phi_n^{\pm} - \phi_{n-1}^{\mp})} = (1 \mp S_{2n-1})(1 - S_{2n-2}^2)(1 \mp S_{2n-3})\,. \end{aligned} \tag{5.2.23}$$

The coefficients $r_n^{\pm}$ can be computed from

$$\begin{aligned} r_n^{\pm} =& \frac{\partial \phi_n^{\pm}}{\partial g_1} \\ =& \mathrm{e}^{-\phi_n^{\pm}} \frac{\partial}{\partial g_1} [2(1 \mp S_{2n-1}) h_{2n-1}] \\ =& \pm S_{2n}(1 \pm S_{2n-1}) \mp S_{2n-2}(1 \mp S_{2n-1})\,, \end{aligned} \tag{5.2.24}$$

where in the third line we used (5.1.15) ($g_1 = \tilde{g}_1$). The coefficients $q_n^{\pm}$ can be computed using the relation $[z_+, z_-] = 0$. One of the relations obtained this way is [22]

$$
\begin{aligned}
q_n^{\pm} &= \frac{(Q_{n+1}^{\pm} - Q_n^{\pm}) + (R_{n+1}^{\mp} - R_n^{\pm})}{r_n^{\pm} - r_n^{\mp}} \\
&= \mp(1 \mp S_{2n-1})(S_{2n} + S_{2n-2})\,.
\end{aligned}
\tag{5.2.25}
$$

The integrable flows analogous to the modified Volterra hierarchy are now those of the Toda chain on the half line [21]

$$
\frac{\partial^2 \phi_n^{\pm}}{\partial g_1^2} = e^{\phi_{n+1}^{\pm} - \phi_n^{\pm}} - e^{\phi_n^{\pm} - \phi_{n-1}^{\pm}}\,. \tag{5.2.26}
$$

The partition function (5.2.20) is therefore given by the product of two Toda τ-functions $\tau_n^{\pm}$.

The dynamics, contained in the potential $V(z_+)$ are given by the action of the operator $z\partial_z$ on the polynomials $c_n^{\pm}(z)$. It is easy to show that

$$
z\partial_z c_n^{\pm} = n\, c_n^{\mp} + \frac{N}{\lambda} \sum_{r=1}^{k} (\gamma_z^{\pm})_{n,n-r}\, c_{n-r}^{\mp}\ , \tag{5.2.27}
$$

where

$$
(\gamma_z^{\pm})_{n,n-r} = e^{-\phi_{n-r}^{\mp}} \int d\mu\, (c_{n-r}^{\mp})^* \,(z\partial_z V(z_+))\, c_n^{\pm}\,, \tag{5.2.28}
$$

and k is the highest power of z_+ in the potential. For $k = 1$, for example, the above relation becomes

$$
z\partial_z c_n^{\pm} = n\, c_n^{\mp} - \frac{N}{\lambda} Q_n^{\pm} c_{n-1}^{\mp} \quad . \tag{5.2.29}
$$

The operator $z\partial_z$ acting on $c_n^{\pm}$ is not hermitian and is not appropriate for taking the continuum limit. We need to compute the action of $z\partial_z$ on a basis of functions $\pi_n^{\pm}$ orthonormal with respect to the "flat" measure $\frac{dz}{2\pi i z}$. Therefore, we define

$$
\pi_n^{\pm}(z) = e^{-\phi_n^{\pm}/2} e^{-\frac{N}{2\lambda} V(z_+)} c_n^{\pm}(z) \tag{5.2.30}
$$

and find that

$$\langle \pi_n^{\pm}(z), \pi_m^{\pm}(z)\rangle = \oint \frac{dz}{2\pi i z}(\pi_n^{\pm}(z))^*(\pi_m^{\pm}(z)) = \delta_{n,m}^{\pm;\pm} \,. \tag{5.2.31}$$

The recursion relations (5.2.22) become

$$\begin{aligned} z_+\, \pi_n^{\pm}(z) =& \sqrt{R_{n+1}^{\pm}}\pi_{n+1}^{\pm}(z) - r_n^{\pm}\pi_n^{\pm}(z) + \sqrt{R_n^{\pm}}\pi_{n-1}^{\pm}(z) \ , \\ =& Q_{nm}^{(+)\pm\pm}\pi_m^{\pm}(z) \\ z_-\, \pi_n^{\pm}(z) =& \sqrt{Q_{n+1}^{\mp}}\pi_{n+1}^{\mp}(z) - q_n^{\pm}\sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}}\pi_n^{\mp}(z) - \sqrt{Q_n^{\pm}}\pi_{n-1}^{\mp}(z) \\ =& Q_{nm}^{(-)\pm\pm}\pi_m^{\pm}(z) \,. \end{aligned} \tag{5.2.32}$$

The action of the operator $z\partial_z$ on the $\pi_n^{\pm}(z)$ basis is found to be

$$\begin{aligned} z\partial_z \pi_n^{\pm}(z) =& -\frac{N}{2\lambda}\sum_{r=1}^{k}(v_z^{\pm})_{n,n+r}\pi_{n+r}^{\mp}(z) + \left\{ n\sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}} - \frac{N}{2\lambda}(v_z^{\pm})_{n,n}\right\}\pi_n^{\mp}(z) \\ & + \frac{N}{2\lambda}\sum_{r=1}^{k}(v_z^{\pm})_{n,n-r}\pi_{n-r}^{\mp}(z) \\ =& P_{nm}^{\pm\pm}\pi_m^{\pm}(z) \,, \end{aligned} \tag{5.2.33}$$

where

$$(v_z^{\pm})_{n,n-r} = \oint \frac{dz}{2\pi i z}(\pi_{n-r}^{\mp}(z))^* \left(z\partial_z V(z_+)\right) \pi_n^{\pm}(z) \ . \tag{5.2.34}$$

The $k=1$ case now becomes

$$\begin{aligned} z\partial_z \pi_n^{\pm}(z) =& -\frac{N}{2\lambda}\sqrt{Q_{n+1}^{\mp}}\pi_{n+1}^{\mp}(z) + (n + \frac{N}{2\lambda}q_n^{\pm})\sqrt{\frac{Q_n^{\mp}}{R_n^{\pm}}}\pi_n^{\mp}(z) \\ & - \frac{N}{2\lambda}\sqrt{Q_n^{\pm}}\pi_{n-1}^{\mp}(z) \ . \end{aligned} \tag{5.2.35}$$

It is easy to check that the above operator is hermitian. The string equation is derived from the relation $[z\partial_z, z_\pm] = z_\mp$ or $[P, Q^{\pm}]_{nm} = -Q_{nm}^{\mp}$ [59,61].

I close this section by briefly mentioning some results of [21]. The partition function is annihilated by a set of constraints L_n for $n \geq 1$ that form a centerless

Virasoro algebra $[L_n, L_m] = (n-m)L_{n+m}$. This is due to the requirement that the transformations

$$\delta_n U = \epsilon(U^{n+1} - U^{1-n}) \qquad n \geq 1\,,$$

which leave the UMM potential invariant, are also a symmetry of the quantum theory. Invariance of the partition function under the action of the above transformations implies that the partition function is annihilated by the Virasoro constraints

$$\mathrm{L}_n = \sum_{k=0}^{\infty} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{1}{2} \sum_{1\leq k\leq n} \frac{\partial^2}{\partial g_k \partial g_{n-k}}\,. \tag{5.2.36}$$

In [21] it was argued that the string equation can be viewed as a consistency condition of the integrable hierarchy and the Virasoro constraints. By taking an appropriate continuum limit of these constraints we can get the continuum Virasoro constraints obtained in the double scaling limit. These are discussed in chapter 7.

CHAPTER 6

The Double Scaling Limit

Inspired by the success in finding non perturbative solutions for HMM and the similarities between UMM and HMM, Periwal and Shevitz [9,10] solved UMM in the double scaling limit. The solutions have many similarities. To mention a few they have similar multicritical structure and scaling of operators in the planar limit. The double scaling limit is obtained in a quite similar way, and the solutions at the multicritical points are obtained from a single differential equation, the string equation. The solutions to the string equation are asymptotic to a weak coupling "genus" expansion. The flows between multicritical points are described by the mKdV hierarchy. This is less mysterious in the light of the discovery [11] that the double cut HMM has the same universal behaviour as the UMM. Despite these successes, a clear world sheet interpretation of the double scaling limit of UMM is still an open challenging problem. This becomes more difficult due to the lack of intuition that could have been provided by a random surface interpretation of the discrete UMM.

In this chapter I discuss the double scaling limit in the PS basis and use the transformation (5.2.21) to show that the string equation can be obtained in the operator formalism in the trigonometric basis in a similar way as in the HMM. I derive the mKdV flows between multicritical points in section 6.3 and discuss the double cut (CDM) HMM and its relation to UMM in the last section.

6.1. The Double Scaling Limit in the PS Basis.

Consider the string equation (5.1.13) for the $V(z_+) = z_+$ potential. In the large N limit $S_n \to S = S(\lambda) \equiv S_{2N}$ and (5.1.13) becomes

$$\lambda S^2 = S^2(1 - S^2)\,. \tag{6.1.1}$$

At $\lambda_c = 1$ the roots of (6.1.1) become degenerate and $S = 0$. Therefore we look for scaling solutions of the form

$$S_{2N} = S(\lambda) = -\frac{1}{2}(2N)^{-\mu} v((2N)^{\nu}(\lambda_c - \lambda)) \tag{6.1.2}$$

The square of the function v is proportional to the scaling part of the specific heat. To see this use (5.1.11) and $\frac{\partial^2 Z}{\partial \lambda^2} \sim \frac{Z_{2N+1}Z_{2N-1}}{(Z_{2N})^2} = \frac{h_{2N+1}}{h_{2N}} = 1 - S_{2N}^2 \sim v^2$. I will derive this more rigorously in section 6.3. Substituting $S_{2N+k} = S(\lambda \frac{2N+k}{2N}) = S(\lambda) + (\frac{\lambda k}{2N})S'(\lambda) + \frac{1}{2}(\frac{\lambda k}{2N})^2 S''(\lambda) + \ldots$ we see that by choosing $\mu = 1/3$ and $\nu = 2/3$ we obtain a non trivial equation for v if we scale $\frac{\lambda}{\lambda_c} = 1 - yN^{-2/3}$ and $\frac{n}{N} = 1 - tN^{-2/3}$. The equation in the variable $x = t + y$ is

$$\frac{1}{2}v''(x) - v(x)^3 = -4v(x)\,x\,, \tag{6.1.3}$$

which is the Painlevé II equation. It admits an asymptotic solution for $x \to \infty$

$$\begin{aligned} v &\sim x^{\frac{1}{2}}(1 - \frac{3}{48}x^{-3}) \\ &= x^{\frac{1}{2}}(1 + \sum_{h=1}^{\infty} f_h \kappa^{2g}) \end{aligned} \tag{6.1.4}$$

in terms of the "string coupling" $\kappa^2 = x^{-3}$ which is the "genus expansion" for the first multicritical point of the UMM. The generic real solution of (6.1.3) has movable first order poles on the real axis, which makes the specific heat singular at these values of x. Then the genus expansion is insufficient for determining the solution past the poles and introduces non perturbative parameters (the positions of the poles) in the theory as was the case for HMM. It was shown in [20], in the spirit of [46], that solutions with poles on the real axis are incompatible with the loop equations. The loop equations are the Schwinger-Dyson equations and have to be satisfied non perturbatively. They are a direct consequence of the Virasoro constraints discussed in section 5.2.

Physical solutions have to be real and pole-free on the real axis. The double scaling limit should approach smoothly the spherical limit [47]. In this limit ($\lambda < \lambda_c$ and $x \to +\infty$) the scaling function $v \sim x^{1/2}$. When we approach the critical point from the strong λ coupling limit ($\lambda > \lambda_c$ and $x \to -\infty$) we don't have scaling of the eigenvalue density and the scaling part of the specific heat should vanish. In [20,62] it was shown that there exist a *unique* real and finite on the real axis solution that for $x \to +\infty$ behaves like $v \sim x^{1/2}$ and vanishes exponentially for $x \to -\infty$. This result holds for the higher multicritical points as well. Therefore there exist physically acceptable solutions for every multicritical point, making the flows between them possible.

For potentials of order k the string equation (5.1.12) in the large N limit gives [10]

$$\lambda S^2 = S^2 W(S^2)\,, \tag{6.1.5}$$

where $W(S^2)$ is a polynomial is S^2. By tuning the couplings in the potential we can arrange $W(S^2)$ so that [10]

$$W(S^2) = 1 - S^{2k}\,. \tag{6.1.6}$$

Then the solutions to (6.1.5) degenerate to $S = 0$ when $\lambda \to 1$. Periwal and Shevitz [10] found that the potentials that result in this multicritical behaviour are given by

$$\widehat{V}_k(z) = -\int_0^t \frac{dt}{t}[(1-t)(1-tz)]^k\,, \tag{6.1.7}$$

where $V_k(z_+) = \widehat{V}_k(z) + \widehat{V}_k(1/z)$. The multicritical potentials $\widehat{V}_k(z)$ are the same as the one computed in section 4.2 up to an irrelevant z independent (infinite) constant. By taking the double scaling limit $N \to \infty$ and $\lambda \to \lambda_c$, with $t = (1-\frac{n}{N})N^{\frac{2k}{2k+1}}$, $y = (1-\frac{\lambda}{\lambda_c})N^{\frac{2k}{2k+1}}$ held fixed the ansatz

$$S_{2N} \to -\frac{1}{2}v(x)N^{-\frac{1}{2k+1}}\,, \tag{6.1.8}$$

where $x = t + y$, yields the string equation

$$2^{k-2}a_k v x = \partial^{-1}\,\mathcal{D}_{PS}^{k-1}\partial(\frac{1}{2}v'' - v^3) \tag{6.1.9}$$

where $\mathcal{D}_{PS} - \frac{1}{2}\partial^2 - 2v^2 - 2v'\partial^{-1}v$, $\partial \equiv \partial/\partial x$ and

$$a_k^{-1} = 2(2k+1)\sum_{l=1}^{k}(-1)^l\, l^{2k}\frac{B(k+1,k+1)}{\Gamma(k-l+1)\Gamma(k+l+1)}\,.$$

This equation is closely related the the mKdV hierarchy defined by

$$\frac{\partial v}{\partial t_{2k+1}} = -M^{mKdV}\,\mathrm{R}^{mKdV}_{k+1}[v]\,, \tag{6.1.10}$$

where $M^{mKdV} = \frac{1}{2}\partial^3 - (v^2\partial + \partial v^2) + 2v'\partial^{-1}v'$ and $\partial\,\mathrm{R}^{mKdV}_{k+1}[v] = M^{mKdV}\,\mathrm{R}^{mKdV}_{k}[v]$, $\mathrm{R}^{mKdV}_{1}[v] = v$. Since $\mathcal{D}_{PS} = M^{mKdV}\partial^{-1}$, we can write (6.1.9) as

$$2^{k-2}a_k v x = \mathrm{R}^{mKdV}_{k+1}[v]\,. \tag{6.1.11}$$

The mKdV hierarchy is related to the KdV hierarchy we found in HMM. I will elaborate on this point in the following chapter. It is because of this relation, however, that

$$\mathrm{R}^{mKdV}_{k+1}[v] = 2^{k-2}(\partial + 2v)\mathrm{R}_k[u] = 2^{k-2}\widehat{\mathcal{D}}\mathrm{R}_k[u] \tag{6.1.12}$$

where

$$u = v^2 - v' \tag{6.1.13}$$

$\widehat{\mathcal{D}} = \partial + 2v$ and $\mathrm{R}_k[u]$ are the Gelfand-Dikii potentials of the KdV hierarchy defined in (3.2.15). The function u flows according to the KdV hierarchy (3.3.35). (6.1.13) is called the Miura transformation. Using (6.1.12) we can write the string equation as

$$\widehat{\mathcal{D}}\mathrm{R}_k[u] = a_k v x\,, \qquad u = v^2 - v'\,. \tag{6.1.14}$$

The resulting equation is a differential equation of order $2k$. The genus expansion in terms of the string coupling $\kappa^2 = x^{-(2+\frac{1}{k})}$ is

$$\begin{aligned} v &\sim x^{\frac{1}{2k}}\Big(1 + \sum_{h=1}^{\infty} v_h x^{-h(2+\frac{1}{k})}\Big) \\ &= x^{\frac{1}{2k}}\Big(1 + \sum_{h=1}^{\infty} v_h \kappa^{2h}\Big)\,. \end{aligned} \tag{6.1.15}$$

As mentioned before, there is a unique real, pole free on the real axis solution that matches (6.1.15) at $x = +\infty$ and vanishes exponentially as $x \to -\infty$. In the spherical limit $v \sim x^{1/2k}$ as we can easily check from (6.1.11). In this limit we can ignore the derivatives of v and

$$\mathrm{R}_{k+1}^{mKdV} \sim v^{2k+1} \,. \tag{6.1.16}$$

The coefficients v_h grow as $(2h)!$. This makes the series (6.1.15) non Borel summable [44]. Therefore there are many functions that are asymptotic to (6.1.15) at $x = \infty$ that differ from each other by exponentially small terms. Such non perturbative corrections to (6.1.15) can be studied by linearizing (6.1.14) by substituting $v \sim x^{\frac{1}{2k}} + \epsilon(x)$ and keeping only linear terms. The ansatz $\epsilon \sim x^{\alpha} e^{\gamma x^{\beta}}$ gives as the leading contribution to v [10]

$$v \sim x^{-\frac{2k-1}{4k}} \exp\left(-\frac{4k}{2k+1} x^{\frac{2k+1}{2k}}\right) . \tag{6.1.17}$$

The local operators σ_l of the theory are defined by perturbing away from the k^{th} multicritical point

$$V_k \to V_k + \sum_l t^{(0)}_{2l+1} \widetilde{V}_{2l+1} \,.$$

Then the functions $W_k(S^2)$ in (6.1.5) change to $W_k(S^2) + \sum_l t^{(0)}_{2l+1} \widetilde{W}_l(S^2)$. Since $\widetilde{W}_l(S^2)$ scale as $N^{-\frac{2l+1}{2k+1}}$ and the $W_k(S^2)$ scale as N^{-1}, we renormalize the couplings to

$$t_{2l+1} \equiv t^{(0)}_{2l+1} N^{\frac{2k-2l}{2k+1}} \,.$$

Similarly $\lambda_c = 1 + \sum_l t^{(0)}_{2l+1} \lambda_{cm} = 1 + \sum_l t_{2l+1} N^{\frac{2l-2k}{2k+1}}$. For relevant operators $m < k$ and $\lambda_c = 1$. For marginal operators there is only a shift in the value of λ_c. For irrelevant operators we must set $t_{2l+1} = 0$ for finite λ_c.

Then the correlation functions are defined by

$$\langle \sigma_{l_1} \dots \sigma_{l_p} \rangle = \frac{\partial}{\partial t_{2l_1+1}} \cdots \frac{\partial}{\partial t_{2l_p+1}} \ln Z_U[t]|_{t=0} \,. \tag{6.1.18}$$

In particular the specific heat is given by

$$\langle \sigma_0 \sigma_0 \rangle = \partial^2 \ln Z_U = v^2 \,. \tag{6.1.19}$$

Then

$$\langle \sigma_{l_1} \dots \sigma_{l_p} \sigma_0 \sigma_0 \rangle = \frac{\partial}{t_{2l_1+1}} \dots \frac{\partial}{t_{2l_p+1}} v^2|_{t=0} \,. \tag{6.1.20}$$

The string equation describing the perturbed, massive model is

$$\sum_{l \geq 1} (2l+1) t_{2l+1} \widehat{\mathcal{D}} \mathrm{R}_l[u] = -vx \tag{6.1.21}$$

and gives $v = v(x,t)$. The k^{th} multicritical point is given by $t_{2k+1} = \frac{1}{(2k+1)a_k}$ and all other t's set to zero.

In order to calculate the scaling of the correlation functions of the operators σ_l in the spherical limit, we consider the spherical limit of (6.1.21) which is reached by neglecting the derivatives of the function v. We obtain

$$x = v^{2k} + \sum_l t_{2l+1} v^{2m} \,. \tag{6.1.22}$$

In order to solve for v^2 as a function of x and t we employ a method used in [4] (see appendix A of this reference). The result that we obtain is

$$\begin{aligned} \langle \sigma_{l_1} \dots \sigma_{l_p} \rangle \sim & (\frac{\partial}{\partial x})^{p-3} x^{\frac{\Sigma+1-k}{k}} \\ \sim & x^{\frac{\Sigma+1-(p-2)k}{k}} \,, \end{aligned} \tag{6.1.23}$$

where $\Sigma = \sum_i l_i$. In the following section I discuss in detail how to calculate some two and three point functions exactly in the double scaling limit. It is also straightforward, but tedious, to calculate corrections to (6.1.23) from surfaces of genus one, as well as the leading non perturbative corrections. This is done in detail in [10] and I refer the reader to this paper.

I close this section by discussing the possibility of an operator formalism in the PS basis similar to the one found for UMM [12]. As I already mentioned, it is not possible to obtain the continuum limit of the action of the operators z, $z_\pm$, $z\partial_z$ *etc.* on the PS basis as the action of differential operators. The reason is that the number of non zero off diagonal lines of the matrices representing those operators is infinite in the large N limit. Neuberger considered a different set of operators having a finite number of non zero off diagonal lines. The formalism is more complicated than in the case of HMM and the solutions for generic multicritical points are not known. In the next section I discuss a simpler formulation of the problem in the next section that will lead us to these solutions. In the following I omit most of the details, referring the reader to [12].

Consider the orthonormal polynomials $\widetilde{P}_n(z) = \frac{1}{\sqrt{h_n}} P_n(z)$. From (5.1.3) we have

$$z(\widetilde{P}_n(z) - \frac{S_n}{S_{n-1}}\sqrt{\frac{h_n}{h_{n-1}}}\widetilde{P}_{n-1}(z)) = \sqrt{\frac{h_{n+1}}{h_n}}\widetilde{P}_{n+1}(z) - \frac{S_n}{S_{n-1}}\widetilde{P}_n(z)\,. \qquad (6.1.24)$$

Define operators A and B by

$$z\sum_{m=0}^{\infty} A_{nm}\widetilde{P}_m(z) = \sum_{m=0}^{\infty} B_{nm}\widetilde{P}_m(z)\,. \qquad (6.1.25)$$

Then only $A_{j\,j}$, $A_{j\,j+1}$ $B_{j\,j}$ and $B_{j\,j+1}$ are non zero. Consider the space spanned by the vectors

$$\psi_x(z) = \sum_{m=0}^{\infty} x_m\widetilde{P}_m(z)\,, \qquad x_m \in \mathrm{C}\,,$$

such that $\int d\mu\overline{\psi}_x(1/z)\psi_y(z) = \sum_m x_m^* y_m$, and consider the operator

$$W : \psi_x(z) \to z\,\psi_x(z)\,. \qquad (6.1.26)$$

Then $W = A^{-1}B$ and W is unitary. Consider the derivative operator

$$L : \psi_x(z) \to \mathrm{e}^{-v(z)} z\partial_z \mathrm{e}^{v(z)}\,\psi_x(z)\,, \qquad (6.1.27)$$

where $\frac{N}{\lambda}V(z_+) = v(z) + v(1/z)$. The operator L is hermitian and has only k non vanishing subdiagonals, *i.e.* $L_{nm} = 0$ for $|n-m| \geq k+1$, for a potential of order k. Therefore we expect that L gives a well defined differential operator in the continuum limit. The string equation becomes

$$-ALA^\dagger + BLB^\dagger = AA^\dagger = BB^\dagger\,. \qquad (6.1.28)$$

In the continuum limit reached at the k^{th} multicritical point with the double scaling limit, $A \to A_0$ and $B \to A_0$ where

$$A_0 = N^{\frac{1}{2k+1}}(\partial^2 - v'/v)\,. \tag{6.1.29}$$

If $\epsilon = N^{-\frac{1}{2(2k+1)}}(\partial^2 - \frac{v'}{v}\partial - v^2)$ then the string equation (6.1.28) becomes

$$\epsilon[L, A_0^\dagger] + [A_0, L]\epsilon^\dagger + [\epsilon A_0^\dagger, L] = A_0^\dagger A_0\,. \tag{6.1.30}$$

For the first multicritical point the solution to (6.1.30) is

$$L = 1 + N^{1/3}(\partial^2 + w)\,, \tag{6.1.31}$$

where $w(x) = x - 2v^2(x)$, and v satisfies the Painlevé II equation. The solutions for higher multicritical points are not known.

6.2. The Operator Formalism

Taking the double scaling limit in the trigonometric basis has certain advantages. Because of the recursion relation (5.2.22), the formulation of the model in this basis resembles that of the HMM. In particular it is possible to construct an operator formalism by considering the continuum limits of (5.2.32) and (5.2.33). These relations result in two continuum operators $\mathcal{P}$ and $\mathcal{Q}_-$ which are 2×2 matrices of differential operators [22]. The string equation in this formalism takes the form $[\mathcal{P}, \mathcal{Q}_-] = 1$, which is similar to (3.3.32). The mKdV flows between multicritical points is also easier to compute in the trigonometric basis.

In this section I discuss the continuum limit of the operators $z_\pm$ and $z\partial_z$ as defined in (5.2.32) and (5.2.33). At the discrete level, the above-mentioned operators act on an infinite dimensional inner product space of complex functions on the unit circle, spanned by the functions $\pi_n^\pm$ defined in (5.2.30). Taking the continuum limit means letting $N \to \infty$. But N appears only as the limit of the product (5.2.20) . In the continuum limit, therefore, only the indices n in a small neighbourhood of N contribute to the singular part of Z_U. For the $k^{\rm th}$ multicritical point the relevant index space is described by the scaling variable

$$t = (1 - \frac{n}{N}) N^{\frac{2k}{2k+1}} . \tag{6.2.32}$$

Therefore in the continuum limit, the space of functions on which the operators act is spanned by the functions $\pi^{\pm}(t, y, z)$ which are the continuum limits of $\pi_n^{\pm}(z)$. Only the index range $\frac{n}{N} \sim 1 - tN^{-\frac{2k}{2k+1}}$ participates in this limit. The double scaling limit ansatz entails taking $\lambda \to \lambda_c$ according to the scaling relation

$$y = (1 - \frac{\lambda}{\lambda_c}) N^{\frac{2k}{2k+1}} , \tag{6.2.33}$$

and scaling the recursion coefficients S_n as

$$S_{2n} \to -\frac{1}{2} v(t, y) N^{-\frac{1}{2k+1}} , \tag{6.2.34}$$

where $v^2(0, y)$ is the specific heat of the unitary matrix model. Then the elements of the space spanned by the functions $\pi_n^{\pm}$ and all quantities defined in the previous chapter become functions of t and y. The operators (5.2.32)–(5.2.33) have nonzero matrix elements $Q_{m,n}^{(\pm)\pm\pm}$ and $P_{m,n}^{\pm\pm}$, only for $|m - n| \leq 1$ and $|m - n| \leq 2k$ respectively. Therefore in the continuum limit they become finite order differential operators. Using the scaling of equations (6.2.32)–(6.2.34), the Taylor expansions

$$\begin{aligned} -2S_{2n-m} \to & N^{-\frac{1}{2k+1}} v(t + \frac{m}{2N} N^{\frac{2k}{2k+1}}, y) = N^{-\frac{1}{2k+1}} v(t + \frac{m}{2} N^{-\frac{1}{2k+1}}, y) = \\ & N^{-\frac{1}{2k+1}} v(t, y) + \frac{m}{2} N^{-\frac{2}{2k+1}} v'(t, y) + \dots \\ & + (\frac{m}{2})^r \frac{1}{r!} N^{-\frac{r+1}{2k+1}} v^{(r)}(t, y) + \dots , \end{aligned} \tag{6.2.35}$$

and

$$\begin{aligned} N^{-\frac{1}{2k+1}} \pi_{n-m}^{\pm}(z) \to & \pi^{\pm}(t, y, z) + m N^{-\frac{1}{2k+1}} (\pi^{\pm}(t, y, z))' + \dots \\ & + \frac{m^r}{r!} N^{-\frac{r+1}{2k+1}} (\pi^{\pm}(t, y, z))^{(r)} + \dots , \end{aligned} \tag{6.2.36}$$

and equations (5.2.21) and (5.2.23)–(5.2.25), we find that

$$\begin{aligned}
Q_n^{\pm}(t,y) &= 1 \pm N^{-\frac{1}{2k+1}} v(t,y) \pm N^{-\frac{2}{2k+1}} v'(t,y) + \mathcal{O}(N^{-\frac{3}{2k+1}}) \\
R_n^{\pm}(t,y) &= 1 - \frac{1}{2} N^{-\frac{2}{2k+1}} (v^2(t,y) \pm v'(t,y)) + \mathcal{O}(N^{-\frac{3}{2k+1}}) \\
r_n^{\pm}(t,y) &= \frac{1}{2} N^{-\frac{2}{2k+1}} (v^2(t,y) \mp v'(t,y)) + \mathcal{O}(N^{-\frac{3}{2k+1}}) \\
q_n^{\pm}(t,y) &= \pm N^{-\frac{1}{2k+1}} v(t,y) + \frac{1}{2} N^{-\frac{2}{2k+1}} (v^2(t,y) \mp v'(t,y)) + \mathcal{O}(N^{-\frac{3}{2k+1}}) \, .
\end{aligned} \tag{6.2.37}$$

Substituting in Eq.(5.2.32) and keeping terms of order $N^{-\frac{2}{2k+1}}$ and $N^{-\frac{1}{2k+1}}$ respectively we obtain

$$Q_{nm}^{(+)\pm\pm} \to 2 + N^{-\frac{2}{2k+1}} \, \mathcal{Q}_+ \, , \qquad Q_{nm}^{(-)\pm\pm} \to -2N^{-\frac{1}{2k+1}} \, \mathcal{Q}_- \, , \tag{6.2.38}$$

where $\mathcal{Q}_\pm$ are given by

$$\begin{aligned}
\mathcal{Q}_+ &= \begin{pmatrix} (\partial + v)(\partial - v) & 0 \\ 0 & (\partial - v)(\partial + v) \end{pmatrix} , \\
\mathcal{Q}_- &= \begin{pmatrix} 0 & \partial + v \\ \partial - v & 0 \end{pmatrix} .
\end{aligned} \tag{6.2.39}$$

In the above formula $x = t + z$, $\partial \equiv \partial/\partial x$ and $z_\pm$ act on the column vector $\binom{\pi^+(x,z)}{\pi^-(x,z)}$. In the continuum limit the operator $z\partial_z$ becomes

$$P_{nm} \to N^{\frac{1}{2k+1}} \, \mathcal{P}_k \, . \tag{6.2.40}$$

The matrix operator $\mathcal{P}_k$ has the form

$$\mathcal{P}_k = \begin{pmatrix} 0 & \mathbf{P}_k \\ \mathbf{P}_k^\dagger & 0 \end{pmatrix} , \tag{6.2.41}$$

with

$$\mathbf{P}_k = a_k^{-1} [\partial^{2k} + p_{k,2k-1} \partial^{2k-1} + \ldots - a_k(t+z)] \quad . \tag{6.2.42}$$

The coefficient a_k may be calculated from the action of $z\partial_z$ given in Eq.(5.2.33) and the k-multicritical potentials found in (6.1.7). The result is

$$a_k^{-1} = 2(2k+1)\sum_{l=1}^{k}(-1)^l \, l^{2k}\frac{B(k+1,k+1)}{\Gamma(k-l+1)\Gamma(k+l+1)} \quad . \tag{6.2.43}$$

The computation of $\mathbf{P}_k$ is straightforward, but becomes quite tedious for high values of k. For $k=1$, for example, $a_1 = -2$ and $V(z_+) = z_+$ and the explicit form of $z\partial_z$ is

$$z\partial_z \to N^{\frac{1}{3}}\ \mathcal{P}_1 \ , \tag{6.2.44}$$

where $\mathcal{P}_1$ is given by

$$\mathcal{P}_1 = \begin{pmatrix} 0 & \mathbf{P}_1 \\ \mathbf{P}_1^\dagger & 0 \end{pmatrix} , \tag{6.2.45}$$

with

$$\mathbf{P}_1 = -\frac{1}{2}[\partial^2 + v\partial + \frac{1}{2}(v' - v^2) + 2(t+z)] \ . \tag{6.2.46}$$

The calculation is done by substituting Eqs.(6.2.35)–(6.2.37) in Eq.(5.2.35).

The string equation is computed from $[z\partial_z, z_\pm] = z_\mp$, or $[P, Q^{(\pm)}]_{nm}^{\pm\pm} = -Q_{nm}^{(\mp)\pm\pm}$. As expected, we find that v obeys

$$\frac{1}{2}v''(x) - v(x)^3 = -4v(x)\,x\,, \tag{6.2.47}$$

which is the Painlevé II equation.

I will discuss now the form of the operator $\mathcal{P}_k$ of Eqs.(6.2.44) and (6.2.45) and of the string equation (6.2.47) for general k. I will show that [24] $\mathcal{P}_k$ is given as the positive part of a pseudo-differential operator as in the case of HMM and that the string equation is closely related to the mKdV hierarchy as in [10].

The string equation $[z\partial_z, z_\pm] = z_\mp$ in terms of the operators $\mathcal{P}_k$, $\mathcal{Q}_\pm$ is given by

$$\begin{aligned} &[P, Q^{(+)}]_{nm}^{\pm\pm} = -Q_{nm}^{(-)\pm\pm} \Rightarrow [\mathcal{P}_k, \mathcal{Q}_+] = 2\mathcal{Q}_- \Rightarrow \\ &\mathbf{P}_k(\partial - v)(\partial + v) - (\partial + v)(\partial - v)\mathbf{P}_k = 2(\partial + v) \end{aligned} \tag{6.2.48}$$

and

$$\begin{aligned} &[P, Q^{(-)}]_{nm}^{\pm\pm} = -Q_{nm}^{(+)\pm\pm} \Rightarrow [\mathcal{P}_k, \mathcal{Q}_-] = 1 \Rightarrow \\ &\mathbf{P}_k(\partial - v) - (\partial + v)\mathbf{P}_k^\dagger = 1 \\ &\mathbf{P}_k^\dagger(\partial + v) - (\partial - v)\mathbf{P}_k = 1\,. \end{aligned} \tag{6.2.49}$$

It is convenient to write the above equations in terms of

$$\widetilde{\mathcal{P}} = \mathcal{P} + \mathcal{X} \tag{6.2.50}$$

where

$$\widetilde{\mathcal{P}} = \begin{pmatrix} 0 & \widetilde{\mathbf{P}}_k \\ \widetilde{\mathbf{P}}_k^\dagger & 0 \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} . \tag{6.2.51}$$

Then equations (6.2.48) and (6.2.49) become

$$\widetilde{\mathbf{P}}_k(\partial - v)(\partial + v) - (\partial + v)(\partial - v)\widetilde{\mathbf{P}}_k = 2(vx)' \tag{6.2.52}$$

and

$$\begin{aligned} \widetilde{\mathbf{P}}_k(\partial - v) - (\partial + v)\widetilde{\mathbf{P}}_k^\dagger &= -2vx \\ \widetilde{\mathbf{P}}_k^\dagger(\partial + v) - (\partial - v)\widetilde{\mathbf{P}}_k &= 2vx \end{aligned} \tag{6.2.53}$$

Eliminating $\widetilde{\mathbf{P}}_k^\dagger(\widetilde{\mathbf{P}}_k)$ yields Eq.(6.2.52) and its hermitian conjugate respectively. The LHS of Eqs.(6.2.53) are differential operators of order $2k$. We get, therefore, a total of $4k+2$ equations, which is an overdetermined system of differential equations for the $2k+1$ functions $p_{k,i}$ and v. By checking the first few values of k we find that, remarkably, only $2k+1$ of them are independent. It seems that this is true for all k, although we have no general proof. If this is the case, Eq.(6.2.53) uniquely determines the operator $\widetilde{\mathbf{P}}_k$ and the string equation.

It is instructive to examine the $k=1$ case in this formalism. First note that in this case Eqs.(6.2.46) and (6.2.50) give

$$\begin{aligned} -2\widetilde{\mathbf{P}}_1 &= \partial^2 + v\partial + \frac{1}{2}(v' - v^2) \\ &= \{(\partial + v)A_1\}_+ , \end{aligned} \tag{6.2.54}$$

where $A_1 = [(\partial - v)(\partial + v)]^{1/2}$, and as usual $\{\ldots\}_+$ denotes the differential part of the pseudo-differential operator in the brackets. An obvious generalization of eq.(6.2.54) for the k^{th} multicritical point is [22]

$$a_k\widetilde{\mathbf{P}}_k = \{(\partial + v)A_k\}_+ \tag{6.2.55}$$

where

$$A_k = [(\partial - v)(\partial + v)]^{k-1/2} = \\ = \partial^{2k-1} + g_{k,2k-2}\partial^{2k-2} + \ldots + g_{k,0} + f_{k,1}\partial^{-1} + f_{k,2}\partial^{-2} + \ldots . \tag{6.2.56}$$

$\widetilde{\mathbf{P}}_k$ is then a differential operator of order $2k$ as in eq.(6.2.42). Then, eq.(6.2.56) determines the coefficients $p_{k,i}$ and eq.(6.2.53) gives two copies of the string equation for the function v. The latter is found to be

$$(\text{Res}\, A_k)' + 2(\text{Res}\, A_k)\, v = 2a_k v x \tag{6.2.57}$$

where $\text{Res}\, A_k = f_{k,1}$. Note that because $\text{Res}\, A_1 = \frac{1}{2}(v' - v^2)$, eq.(6.2.57) trivially gives eq. (6.2.47).

For the derivation of Eq.(6.2.57), we observe that the trivial equations

$$(\partial + v)A_k(\partial - v) - (\partial + v)A_k(\partial - v) = 0$$

and

$$\mathcal{O} = \mathcal{O}_+ + \mathcal{O}_-$$

for any pseudo-differential operator $\mathcal{O}$ give

$$\widetilde{\mathbf{P}}_k(\partial - v) - (\partial + v)\widetilde{\mathbf{P}}_k^\dagger = -\{(\partial + v)A_k\}_-(\partial - v) + (\partial + v)\{A_k(\partial - v)\}_- \ . \tag{6.2.58}$$

Since the only overlap of the pseudo-differential operators on each side of Eq.(6.2.58) is the constant part this establishes that the LHS of Eq.(6.2.58) is a purely multiplicative operator in the ring of pseudo-differential operators. Computing the RHS of Eq.(6.2.58) and equating it to the RHS of Eq.(6.2.53) we obtain the string equation (6.2.57).

Observe that

$$(\partial - v)(\partial + v) = \partial^2 + (v' - v^2) \equiv \partial^2 - u \tag{6.2.59}$$

where u is related to v by the Miura transformation (6.1.13) $u = v^2 - v'$. It is a standard result that

$$A_k = (\partial^2 - u)^{k-1/2} = \sum_{i=-\infty}^{k} \{e_{2i-1}, \partial^{2i-1}\} = \\ = \partial^{2k-1} - \frac{2k-1}{4}\{u, \partial^{2k-3}\} + \ldots + \{\mathrm{R}_k[u], \partial^{-1}\} + \ldots . \tag{6.2.60}$$

Therefore

$$\mathrm{Res}\, A_k = 2\,\mathrm{R}_k[u] . \tag{6.2.61}$$

Equation (6.2.57) and (6.2.61) gives the string equation (6.1.14). Substituting back in (6.2.55) we obtain another useful form of the operator $\widetilde{\mathbf{P}}_k$

$$a_k \widetilde{\mathbf{P}}_k = (\partial + v)(\partial^2 - u)_+^{k-1/2} + 2\mathrm{R}_k[u] . \tag{6.2.62}$$

By inserting the operators σ_l, we perturb the potential $V_k \to -\sum t_{2l+1}^{(0)} V_l$ and the derivative operator now becomes

$$\mathbf{P} = -\sum_{l \geq 1} (2l+1) t_{2l+1} \widetilde{\mathbf{P}}_l - x , \tag{6.2.63}$$

which obeys the string equation

$$[\mathcal{P}, \mathcal{Q}_-] = 1 . \tag{6.2.64}$$

The differential equation that is obtained from (6.2.64) is

$$\sum_{l \geq 1} (2l+1) t_{2l+1} \widehat{\mathcal{D}} \mathrm{R}_l[u] = -vx . \tag{6.2.65}$$

I close this section by mentioning that the string equation can be derived from an action principle similar to the one that exists for HMM [63]. Using the relation

$$\frac{\delta}{\delta u} \int dx\, \mathrm{R}_{k+1}[u] = -(k + \frac{1}{2}) \mathrm{R}_k[u] , \tag{6.2.66}$$

we find that by minimizing the action

$$I = \int dx \, \{\text{Res}\, A_{k+1} + a_k(k+\frac{1}{2})v^2 x\}\,, \tag{6.2.67}$$

we obtain the string equation (6.2.57). Indeed using (6.1.13), (6.2.61),(6.2.66) we get

$$\begin{aligned} \delta I &= -(2k+1)\int dx\,(\mathrm{R}_k[u]\delta u - a_k v x \delta v) = \\ &= -(2k+1)\int dx\,(\mathrm{R}_k[u](2v\delta v - \delta v') - a_k v x \delta v) = \\ &= -(2k+1)\int dx\,(2v\mathrm{R}_k[u] + D\,\mathrm{R}_k[u] - a_k v x)\delta v = \\ &= 0\,. \end{aligned} \tag{6.2.68}$$

6.3. The relation of UMM to the mKdV Hierarchy

The mKdV hierarchy is a system of differential equations for the function $v(x,t)$ defined by (6.1.10)

$$\begin{aligned} \frac{\partial v}{\partial t_{2k+1}} &= -\, M^{mKdV}\,\mathrm{R}_k^{mKdV}[v] \\ &= -\,\partial \mathrm{R}_{k+1}^{mKdV}[v] \\ &= -\,\partial \widehat{\mathcal{D}}\mathrm{R}_k[u_2]\,, \end{aligned} \tag{6.3.69}$$

where $M^{mKdV} = \frac{1}{2}\partial^3 - (v^2\partial + \partial v^2) + 2v'\partial^{-1}v'$, $u_2 \equiv u = v^2 - v'$ and $t_1 \equiv x$. The mKdV hierarchy is closely related to the KdV hierarchy. The functions $u_1 = v^2 + v'$ and $u_2 = v^2 - v'$ satisfy the KdV hierarchy

$$\begin{aligned} \frac{\partial u_i}{\partial t_{2k+1}} &= -\,\partial \mathrm{R}_k[u_i] \\ &= -\,M^{KdV}\mathrm{R}_{k-1}[u_i]\,, \end{aligned} \tag{6.3.70}$$

where $M^{KdV} = \frac{1}{4}\partial^3 - (u_i\partial + \partial u_i)$. The integrability of (6.3.69) and (6.3.70) has its heart in the existence of two compatible hamiltonian structures $\{\,,\,\}_1$ and $\{\,,\,\}_2$ [43] and an infinite set of conserved (mutually commuting) hamiltonians H_k^{mKdV} and H_k^{KdV} respectively. Then

$$\begin{aligned} \frac{\partial v}{\partial t_{2k+1}} &= \{H_k^{mKdV}, v\}_1 = \{H_{k+1}^{mKdV}, v\}_2 \\ \frac{\partial u_i}{\partial t_{2k+1}} &= \{H_k^{KdV}, u_i\}_1 = \{H_{k+1}^{KdV}, u_i\}_2\,, \end{aligned} \tag{6.3.71}$$

and $\{H_k, H_l\} = 0$. The Poisson brackets are given by

$$\begin{aligned}\{u(x), u(y)\}_2 =& \partial\, \delta(x-y) \\ \{u(x), u(y)\}_1 =& M^{KdV} \delta(x-y)\,,\end{aligned} \tag{6.3.72}$$

for the KdV hierarchy and by

$$\begin{aligned}\{v(x), v(y)\}_2 =& \partial^3 \delta(x-y) \\ \{v(x), v(y)\}_1 =& M^{mKdV} \delta(x-y)\,,\end{aligned} \tag{6.3.73}$$

for the mKdV hierarchy. In (6.3.73) the operator M^{mKdV} is understood as being equal to $M^{mKdV} = \frac{1}{2}\partial^3\delta(x-y) - (v^2\partial + \partial v^2)\delta(x-y) + 2v'\epsilon(x-y)v'$. Then the Gelfand-Dikii potentials for the two hierarchies are given by

$$\begin{aligned}\mathrm{R}_k^{mKdV}[v] =& \frac{\delta}{\delta v} H_k^{mKdV} \\ \mathrm{R}_k^{KdV}[u_i] \equiv& \mathrm{R}_k[u_i] = \frac{\delta}{\delta v} H_k^{KdV}\,.\end{aligned} \tag{6.3.74}$$

The relation between the two hierarchies is summarized in the relation

$$H_{k+1}^{mKdV}[v] = H_k^{KdV}[u_i] \tag{6.3.75}$$

The difference between $H_k^{KdV}[u_1]$ and $H_k^{KdV}[u_2]$ is only the integral of a total derivative and vanishes. Then, since $\mathrm{R}_{k+1}^{mKdV}[v] = \frac{\delta}{\delta v} H_{k+1}^{mKdV} = \frac{\delta}{\delta v} H_k^{KdV} = \frac{\delta u_i}{\delta v}\frac{\delta}{\delta u_i} H_{k+1}^{mKdV}$ and $\frac{\delta u_1}{\delta v} = 2v - \partial = \widehat{\mathcal{D}}^\dagger$ and $\frac{\delta u_2}{\delta v} = 2v + \partial = \widehat{\mathcal{D}}$, we can easily prove that

$$\mathrm{R}_{k+1}^{mKdV}[v] = \widehat{\mathcal{D}}^\dagger \mathrm{R}_k[u_1] = \widehat{\mathcal{D}} \mathrm{R}_k[u_2]\,. \tag{6.3.76}$$

We can also show that if v satisfies (6.3.69) then u_i satisfy (6.3.70)using $\frac{\partial}{\partial t}(v^2 \pm v') = (2v \pm \partial)\frac{\partial v}{\partial t}$. Two useful identities are

$$\begin{aligned}\widehat{\mathcal{D}}(-\partial)\widehat{\mathcal{D}}^\dagger =& 4M^{KdV} \\ \widehat{\mathcal{D}}^\dagger(-\partial)\widehat{\mathcal{D}} =& 4M^{KdV}\,.\end{aligned} \tag{6.3.77}$$

A consequence of (6.3.76) and (6.3.77) is that

$$\begin{aligned}\partial \mathrm{R}_k[u_1] + \partial \mathrm{R}_k[u_2] =& M^{KdV}\mathrm{R}_k[u_1] + M^{KdV}\mathrm{R}_k[u_2]\\ =& - v\partial\widehat{\mathcal{D}}^{\dagger}\mathrm{R}_k[u_1]\\ =& - v\partial\widehat{\mathcal{D}}\mathrm{R}_k[u_2]\end{aligned} \tag{6.3.78}$$

In order to proceed with the calculation of one and two point (connected) correlation functions we need the following formulas [4]

$$\begin{aligned}\langle \mathrm{tr}F_1(U)\rangle =& \mathrm{Tr}[\widehat{F}_1(\widehat{z})\Pi_N]\\ \langle \mathrm{tr}F_1(U)\mathrm{tr}F_2(U)\rangle =& \mathrm{Tr}[\Pi_N\widehat{F}_1(\widehat{z})(1-\Pi_N)\widehat{F}_2(\widehat{z})]\end{aligned} \tag{6.3.79}$$

for a $2N \times 2N$ or a $(2N+1)\times(2N+1)$ UMM. The $F_i(U)$ are real functions of the matrix U and

$$\Pi_N = \sum_{n=0,\pm}^{N} |n\pm><\pm n| \tag{6.3.80}$$

is the projection operator on the subspace spanned by the functions $|n\pm>=\pi_n^{\pm}(z)$ for $n = 0, 1, \ldots, N$. tr is the matrix trace and Tr is the trace over *all* the states $|n\pm>$. The $\widehat{F}_i(\widehat{z})$ are operators, functions of the operator $\widehat{z}$ acting on the states $|n\pm>$.

In order to prove this formula consider the generating function $G(t_1,t_2)$ of the above *connected* correlation functions where

$$G(t_1,t_2) = \langle \mathrm{e}^{t_1\mathrm{tr}F_1(U)+t_2\mathrm{tr}F_2(U)}\rangle$$

and compute

$$\begin{aligned}\exp\{G(t_1,t_2)\} =& \frac{1}{Z}\int dU\ \exp\{-\frac{N}{\lambda}\mathrm{tr}V\}\mathrm{e}^{\sum_\alpha t_\alpha \mathrm{tr}F_\alpha(U)}\\ =& \frac{1}{Z}\int[\frac{dz}{2\pi i z}]|\Delta(z)|^2\exp\{-\frac{N}{\lambda}\mathrm{tr}V + \sum_\alpha t_\alpha\mathrm{tr}F_\alpha(U)\}\\ =& det_N\ <\pm n|\mathrm{e}^{G(t_1,t_2)}|m\pm>\\ =& \exp\sum_{n,\pm}^{N}\ln<\pm n|\mathrm{e}^{G(t_1,t_2)}|n\pm>,\end{aligned}$$

where det is over only the first N states $|n\pm>$. In the last line I used the identity $det\,A = e^{\mathrm{tr}\ln A}$. Expand $G(t_1,t_2)$ for small t_1 and t_2 and find after some algebra that

$$G(t_1,t_2) = \sum_\alpha t_\alpha \mathrm{Tr}[\widehat{F}_\alpha(\widehat{z})\Pi_N] + \\ + \frac{1}{2}\sum_{\alpha,\beta} t_\alpha t_\beta (\mathrm{Tr}[\widehat{F}_\alpha(\widehat{z})\widehat{F}_\beta(\widehat{z})\Pi_N] - \mathrm{Tr}[\widehat{F}_\alpha(\widehat{z})\Pi_N\widehat{F}_\beta(\widehat{z})\Pi_N]) + \mathcal{O}(t^3)\,.$$

Then $\langle \mathrm{tr}F_\alpha(U)\rangle = \frac{\partial}{\partial t_\alpha}G(t_1,t_2)|_{t=0}$ and $\langle \mathrm{tr}F_\alpha(U)\mathrm{tr}F_\beta(U)\rangle = \frac{\partial^2}{\partial t_\alpha t_\beta}G(t_1,t_2)|_{t=0}$ which gives (6.3.79).

Using these formulas we can derive the correlation functions of the operators σ_l

$$\langle\sigma_k\rangle = \mathrm{Tr}\{\widetilde{V}_k(z_+)\Pi_N\} \\ = \oint \frac{dz_+}{2\pi i z_+}\widetilde{V}_k(z_+)\mathrm{Tr}\{\frac{1}{z_+ - Q^{(+)}}\Pi_N\}\,. \qquad (6.3.81)$$

Similarly the two point function $\langle\sigma_k\sigma_1\rangle = \partial\langle\sigma_k\rangle$ is given by

$$\langle\sigma_k\sigma_0\rangle = \mathrm{Tr}\{\Pi_N\widetilde{V}_k(\widehat{z}_+)(1-\Pi_N)\widehat{z}_+\} \\ = -\oint \frac{dz_+}{2\pi i z_+}\widetilde{V}_k(z_+)\mathrm{Tr}\{\Pi_N\frac{1}{z_+ - Q^{(+)}}(1-\Pi_N)(z_+ - Q^{(+)})\} \\ \propto \oint \frac{dz_+}{2\pi i z_+}(1-Z^2)^k(1-\frac{1}{Z})^{\frac{1}{2}}\{\sqrt{R^+_{N+1}}(\frac{1}{z_+ - Q^{(+)}})^{++}_{N\,N+1} \\ + \sqrt{R^-_{N+1}}(\frac{1}{z_+ - Q^{(+)}})^{--}_{N\,N+1}\}\,. \qquad (6.3.82)$$

In the third line above I used the expression for the operator z_+ given in (5.2.32).

In the double scaling limit $z_+ = 2\cos\alpha \to 2-\alpha^2$ where $\alpha = N^{-\frac{1}{2k+1}}\nu$, $Q^{(+)} \to 2 + N^{-\frac{2}{2k+1}}\mathcal{Q}_+$, $\sqrt{R^\pm_{N+1}} \to 1 + \frac{1}{4}N^{-\frac{2}{2k+1}}(\mp v' - v^2)$ and $|N\pm> \to N^{\frac{1}{2(2k+1)}}|x\pm>$ and (6.3.82) becomes [64]

$$\langle\sigma_k\sigma_0\rangle \propto \oint \frac{d\nu}{2\pi i\nu}\nu^{2k+3}(<+x|\frac{1}{-\nu^2-\partial^2+u_1}|x+> + <-x|\frac{1}{-\nu^2-\partial^2+u_2}|x->) \\ \propto \oint \frac{d\nu}{2\pi i\nu}\nu^{2k+3}(\sum_l \frac{\mathrm{R}_l[u_1]}{\nu^{2l+1}} + \sum_l \frac{\mathrm{R}_l[u_2]}{\nu^{2l+1}}) \\ \propto \mathrm{R}_k[u_1] + \mathrm{R}_k[u_2]\,. \qquad (6.3.83)$$

In the second line I used the famous result (3.2.14) of Gelfand and Dikii on the asymptotic expansion in ν of the resolvent of the operator $-\partial^2+(u-\nu^2)$. Therefore

$$\langle\sigma_k\sigma_0\sigma_0\rangle \propto \partial \mathrm{R}_k[u_1] + \partial \mathrm{R}_k[u_2] = -v\widehat{\mathcal{D}}\mathrm{R}_k[u_2] \,.$$

Then using

$$\langle\sigma_k\sigma_0\sigma_0\rangle = \frac{\partial}{\partial t_{2k+1}} < \sigma_0\sigma_0 >= 2v\frac{\partial v}{\partial t_{2k+1}}$$

we obtain

$$\frac{\partial v}{\partial t_{2k+1}} = -\partial\widehat{\mathcal{D}}\mathrm{R}_k[u_2] \tag{6.3.84}$$

which are the mKdV flows. Note that (6.3.83) implies that

$$\langle\sigma_0\sigma_0\rangle \propto u_1 + u_2 \propto v^2 \,. \tag{6.3.85}$$

This proves the earlier assertion that the specific heat of UMM is given by the square of the scaling function v. This result can be calculated by using directly the relation $\langle\sigma_0\sigma_0\rangle = \mathrm{Tr}[\Pi_N\widehat{z}_+(1-\Pi_N)\widehat{z}_+]$ as well.

The relation between the string equation and the mKdV flows is further explored in chapter 7. An important result for the consistency of UMM of the analysis presented there, is the compatibility of the string equation with the flows mentioned before. This is directly related to the Virasoro constraints annihilating the partition function and it is discussed in chapter 7 using the Grassmannian formulation of the string equation and the flows. Hollowood et al [19] discuss this question in the formalism presented here.

6.4. The Double Cut HMM.

It is possible to obtain the same multicritical behaviour and double scaling limit as the ones I discussed in the previous section for the UMM by a certain class of multicritical potentials $V_k(\Phi)$ of a HMM [14,11]. These potentials have the shape of a double well as opposed to the multicritical potentials of HMM discussed in chapter 3 which have the shape of an inverted double well. In one phase the eigenvalue distribution has support in the two wells only and we have a two cut distribution of eigenvalues. As we lower the local maximum of these potentials, the eigenvalues tend to spread over the whole well. Multicritical behaviour is observed exactly when the two cuts merge into one. This is similar to the case of UMM where the two ends of the support of the eigenvalue distribution meet at the point $z = 1$. The similarity is stronger, since the scaling of the eigenvalue density near the edge of its support for the two models is identical.

The multicritical potentials resulting in the critical two cut distribution for the HMM are

$$V'_{2k}(\varphi) \propto \phi^{2k+1}(1 - \frac{1}{\varphi^2})^{\frac{1}{2}}|_+ \,, \tag{6.4.86}$$

where the index + means that we keep the polynomial part in an expansion of the square root around ∞ in the complex φ plane. These potentials result in an eigenvalue distribution that behaves like

$$\rho_{2k+1}(\varphi) \begin{cases} = 0 & |\varphi| > 1 \\ \sim \varphi^{2k} & \varphi \to 0 \\ \sim \sqrt{1 \mp \varphi} & \varphi \to \pm 1^{\mp} \,. \end{cases} \tag{6.4.87}$$

The perturbations giving the local operators σ_k are defined by the multicritical potentials

$$\widetilde{V}_m = \varphi^m (\varphi^2 - 1)^{\frac{3}{2}}|_+ \,, \tag{6.4.88}$$

which perturb the density of eigenvalues by $\rho_{2k+1}(\varphi) \to \rho_{2k+1}(\varphi) + \widetilde{\rho}_m(\varphi)$ where $\widetilde{\rho}_m(\varphi)$ scales when $\varphi \to 0$ as

$$\widetilde{\rho}_m(\varphi) \sim \varphi^{m-1} \,, \qquad m = 1, 2, \ldots \tag{6.4.89}$$

and the same as $\rho_{2k+1}(\varphi)$ when $\varphi \to \pm 1^{\mp}$.

Note that the sign of the φ^2 term in (6.4.86) is negative. This makes the random surface interpretation of chapter 3 problematic since the regularized 2-d gravity partition function would have complex cosmological constant ($N = -\mathrm{e}^{-\mu}$). The multicritical scaling (6.4.87) of $\rho_{2k+1}(\varphi)$ is the same as the multicritical behaviour of $\rho_k(\alpha)$ of the UMM. Therefore we expect a correspondence between the scaling operators of the two theories

$$\sigma_m^{UMM} \leftrightarrow \sigma_{2m}^{HMM} \,. \tag{6.4.90}$$

This is indeed the case. What about the odd order operators σ_{2m-1}^{HMM}? Are there any real UMM potentials that result in an eigenvalue distribution as (6.4.89) as $\alpha \to 0$? The answer is yes and it is discussed briefly at the end of this section.

In the orthogonal polynomial method, the coefficients R_n and S_n of the recursion relation

$$\varphi\, P_n(\varphi) = \sqrt{R_{n+1}} P_{n+1}(\varphi) - S_n P_n(\varphi) + \sqrt{R_n} P_{n-1}(\varphi) \tag{6.4.91}$$

have a double scaling limit

$$\begin{aligned} R_n =& r_c + (-1)^n N^{-\frac{1}{2k+1}} f(x) + \dots \\ S_n =& s_c + (-1)^n N^{-\frac{1}{2k+1}} i\, g(x) + \dots . \end{aligned} \tag{6.4.92}$$

The orthonormal polynomials tend to two scaling functions

$$\begin{aligned} (-1)^n P_{2n}(\varphi) \to& N^{-\frac{1}{2(2k+1)}} p_+(x,\varphi) \\ (-1)^n P_{2n+1}(\varphi) \to& N^{-\frac{1}{2(2k+1)}} p_-(x,\varphi) . \end{aligned}$$

The limit of (6.4.91) is represented by the action of an operator $\widetilde{\varphi} = N^{\frac{1}{2k+1}} \varphi$ on the vector

$$\Psi = i\sigma_2 \begin{pmatrix} p_+ \\ p_- \end{pmatrix} = \begin{pmatrix} p_- \\ -p_+ \end{pmatrix}$$

which is

$$\begin{aligned} \widetilde{\varphi}\Psi =& 4(\partial + f(x)\mathcal{J}_1 + ig(x)\mathcal{J}_2)\mathcal{J}_3\Psi \\ =& 4(\partial + q(x))\mathcal{J}_3\Psi , \end{aligned} \tag{6.4.93}$$

where $\mathcal{J}_1 = \frac{1}{2}\sigma_3$, $\mathcal{J}_2 = \frac{1}{2}\sigma_1$ and $\mathcal{J}_3 = \frac{1}{2}\sigma_2$ are generators of the algebra $[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k$ and σ_i are the Pauli matrices. The resolvent

$$\mathcal{R} = \frac{1}{(\partial + q)\mathcal{J}_3 - \frac{\varphi}{4}}$$

has an asymptotic expansion [65]

$$\mathcal{J}_3\mathcal{R} = \sum_{k=0}^{\infty} \operatorname{grad} \widehat{H}_{k+1} \varphi^{-k}$$

where grad $\widehat{H}_{k+1} = -\mathcal{J}_1 G_k - i\mathcal{J}_2 F_k + \mathcal{J}_3 H_k$, and satisfies the relation

$$\mathcal{J}_3 \partial \mathcal{R} = [\mathcal{J}_3 \mathcal{R}, q - \varphi \mathcal{J}_3] \,.$$

The last relation gives

$$\begin{aligned} F_{k+1} =& G_k' + g H_k \\ G_{k+1} =& F_k' + f H_k \\ H_k' =& g G_k - f F_k \,, \end{aligned} \tag{6.4.94}$$

with $G_{-1} = F_{-1} = 0$ and H_{-1}.

The string equation is

$$\begin{aligned} \sum_{k\geq 1} k t_k G_{k-1} =& 0 \\ \sum_{k\geq 1} k t_k F_{k-1} =& 0 \,. \end{aligned} \tag{6.4.95}$$

Correlation functions in the double scaling limit can be obtained in a similar way to the one described in the previous section. Then

$$\langle \sigma_0 \sigma_0 \rangle = \frac{1}{4}(g^2 - f^2) \tag{6.4.96}$$

is the scaling part of the specific heat and

$$\langle \sigma_m \sigma_0 \rangle = \frac{1}{2} H_m \,. \tag{6.4.97}$$

The flow equations derive from the relations $\langle \sigma_m \sigma_0 \sigma_0 \rangle = \frac{\partial}{\partial t_m} \langle \sigma_m \sigma_0 \rangle = \partial \langle \sigma_m \sigma_0 \rangle$. They are

$$g \frac{\partial g}{\partial t_m} - f \frac{\partial f}{\partial t_m} = \partial H_m \,. \tag{6.4.98}$$

These equations are closely related to the NLS (Non Linear Schrödinger) hierarchy [19]

$$\frac{\partial f}{\partial t_m} = F_m \qquad\qquad \frac{\partial g}{\partial t_m} = G_m \,. \tag{6.4.99}$$

We obtain the mKdV hierarchy by setting $g = 0$. Then we have non trivial flows only for F_{2k+1}. Using (6.4.94) with $g = 0$ we can show that

$$\begin{aligned} F_{2k+1} =& 2(M^{mKdV} \partial^{-1}) F^{2k-1} \\ =& 2(\frac{1}{2}\partial^2 - 2f^2 - 2f' \partial^{-1} f) F^{2k-1} \end{aligned}$$

which proves that

$$\begin{aligned}F_{2k+1} =& - \partial \mathrm{R}^{mKdV}[f] \\ =& - \partial \widehat{\mathcal{D}} \mathrm{R}_k [f^2 - f'] \,.\end{aligned}$$

Substituting back into (6.4.98) we obtain the mKdV flows for the function f. Similarly (6.4.95) becomes the UMM string equation

$$\sum_{k\geq 0} t_{2k+1} \widehat{\mathcal{D}} \mathrm{R}_k [f^2 - f'] = 0 \,.$$

Therefore the multicritical points of the UMM considered in the previous section correspond to the odd order multicritical points of the full double cut HMM. This is not a mystery since by simply transforming the UMM operator $\mathcal{Q}_-$ using the matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we can put it in the form

$$\begin{aligned}\mathcal{Q}'_- =& S^{-1} \mathcal{Q}_- S \\ =& (\partial + v\sigma_1)\sigma_3 \,.\end{aligned}$$

This is the same operator as in (6.4.93) acting on the vectors $\Psi = S \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}$ when $g = 0$. The question that naturally arises is whether one can obtain the even operators too from the UMM by considering more general *real* potentials than the ones of the symmetric UMM studied so far. The answer is no for polynomial potentials in U and $U^\dagger$. If one considers

$$\begin{aligned}V(z) =& V^{(+)}(z_+) + V^{(-)}(-iz_-) \\ =& V^{(+)}(2\cos\alpha) + V^{(-)}(2\sin\alpha) \\ =& \sum g_k^+ (2\cos\alpha)^k + \sum g_k^- (2\sin\alpha)^k \,,\end{aligned} \tag{6.4.100}$$

the saddle point equation gives imaginary g_k^- for even k. One has to consider potentials of the form

$$\begin{aligned}V(z) =& V^{(+)}(\tilde{z}_+) + V^{(-)}(-i\tilde{z}_-) \\ =& V^{(+)}(2\cos\frac{\alpha}{2}) + V^{(-)}(2\sin\frac{\alpha}{2}) \\ =& \sum t_k^+ (2\cos\frac{\alpha}{2})^k + \sum t_k^- (2\sin\frac{\alpha}{2})^k \,,\end{aligned} \tag{6.4.101}$$

where $\tilde{z}_\pm = z^{\frac{1}{2}} \pm z^{-\frac{1}{2}}$. Then (4.2.14) and (4.2.15) give perturbations to the density of eigenvalues that scale as (6.4.89) for $\alpha \to 0$. The operator $V^{(+)}(\tilde{z}_+)$ results in the familiar odd order multicritical points and $V^{(-)}(-i\tilde{z}_-)$ gives the even order operators. The details of the construction of the double scaling limit are presently under investigation.

Finally the authors of [11] calculate the loop operators in the WKB (planar) approximation:

$$\begin{aligned} \langle w(l)\rangle =& \int_{t_1} dx \ < x|\mathrm{tr}\, e^{-l\begin{pmatrix} \partial & -f-g \\ f-g & -\partial \end{pmatrix}}|x> \\ =& -\pi \int_{t_1} dx\, h_i^{\frac{1}{2}} J_1(l h_1^{\frac{1}{2}})\,, \end{aligned}$$

where $h_1 = -H_1 = \frac{1}{2}(f^2 - g^2)$ and the operators

$$\langle \sigma_m w(l)\rangle = h_1^{\frac{m}{2}} J_m(l h_1^{\frac{1}{2}})\,, \qquad m = 2k+1\,,$$

and $\langle \sigma_m w(l)\rangle = 0$ for m even. If we interpret these correlators as the wave functions $\psi_m(l)$ of a minisuperspace formulation of 2-d gravity in the sense discussed in chapter 3, the Wheeler-deWitt equation for $\psi_m(l)$ is a Bessel equation of the form

$$[(l\frac{\partial}{\partial l})^2 - \frac{\mu}{\gamma^4}l^2 - \frac{8}{\gamma^2}(\frac{Q^2}{8} + \Delta - 1)]\psi_m(l) = 0\,,$$

where

$$\begin{aligned} m =& 2k - 1 = \frac{8}{\gamma^2}(\frac{1-c}{24} + \Delta) \\ \gamma^2 =& \frac{1}{6}(13 - c - \sqrt{(1-c)(25-c)})\,. \end{aligned}$$

Under this assumption, we obtain a series of matter theories coupled to gravity with

$$c = 1 - \frac{6m^2}{m+1}\,, \qquad m = 2k+1$$

and $\psi_n(l) = 0$ being the wavefunctions of operators with scaling dimensions

$$\Delta_n = \frac{n^2 - m^2}{4(m+1)}\,.$$

The series obtained can be associated to $(4k, 2)$ superminimal models coupled to supergravity [11]. Unfortunately the evidence is inconclusive since we don't have correlators in the Ramond (odd) sector even at the tree level and the agreement is only with tree level correlators in the Neveu-Schwarz sector. Moreover the one dimensional two-cut model is the same as the one cut model and this is not expected for $\hat{c} = 1$ superminimal matter coupled to supergravity. The authors of [11] attempt to associate the double cut HMM with an $O(-2)$ model but this has been ruled out by [56].

CHAPTER 7

The Space of Solutions the String Equation

In this chapter I discuss how one can compute the space of solutions to the string equation. The latter was written in a simple form $[\mathcal{P}, \mathcal{Q}_-] = 1$ in the operator formalism in chapter 6 [22]. The operators $\mathcal{P}$ and $\mathcal{Q}_-$ at the k^{th} multicritical point are 2×2 matrices of differential operators of order $2k$ and 1 respectively. The form of the string equation makes the formulation of the problem in a Grassmannian representation simple and instructive. In this formalism it is possible to compute the space of solutions to the string equation and understand the emergence of the Virasoro constraints and their relation to the compatibility of the mKdV flows and the string equation [24].

In order to understand the connection between the Grassmannian formalism with the formalism presented in chapter 6, it is necessary to understand the τ-function formalism of the mKdV hierarchy. The mKdV equations are given in this formalism by the Hirota bilinear equations that the τ-functions must satisfy. The relation between Grassmannians and τ-functions is understood via the Plücker embedding of the Grassmannian in a free fermion Fock space. The τ-functions are viewed as bosonic states and the connection is established by the fermion boson equivalence in two dimensions. The big contribution of the Japanese school (see [66] and references therein for a review) is the discovery that the solutions to the mKdV hierarchy (more generally to the mKP hierarchy from which the mKdV is derived by reduction) are τ-functions that belong to the GL(∞) orbit of the vacuum, where GL(∞) acts on the states of the bosonic or fermionic Fock space. In the following two sections I give a brief introduction to these concepts (for a review see also [67]).

7.1. The τ-function Formalism and the Sato Grassmannian

In this section I discuss the τ-function formalism and its relation to the (big cell of the) Sato Grassmannian $Gr^{(0)}$ defined in chapter 3. First I discuss the mapping of the points $V \in Gr^{(0)}$ to states $|v>$ in a fermionic Fock space F of a free fermion theory. For simplicity the concepts are discussed originally in the case of finite Grassmannians $Gr(k,N)$. There is a natural action of the group GL(N) on F and the states $|v>$ belong to the GL(N) orbit of fermion filled states. Using the boson fermion mapping one can associate to $|v>$ a bosonic state τ which is called a τ-function. This establishes a one to one correspondence between τ-functions and points $V \in Gr(k,N)$.

The same results hold in the infinite dimensional case. The solutions to the mKdV equations correspond to two τ-functions τ_1 and τ_2 such that

$$u_i = -2\partial^2 \ln \tau_i\,, \quad i=1,2 \qquad \text{and} \qquad v = \partial \ln \frac{\tau_2}{\tau_1}\,.$$

v is the mKdV function (6.3.69) and $u_i = v^2 \pm v'$. To these τ-functions we associate two points V_1 and V_2 in $Gr^{(0)}$ restricted by simple conditions whose evolution with the times t_{2k+1} are given by the mKdV flows.

Since the Sato Grassmannian is an infinite dimensional generalization of finite dimensional Grassmannians, we start by reviewing the relevant concepts in the finite dimensional case. For a nice review along these lines see [68]. The Grassmannian $Gr(k,N)$ consists of all k-dimensional linear subspaces of C^N. A point $V \in Gr(k,N)$ is described by a basis $\{v_i\}$ with $i=1,\ldots,k$ and a basis of the orthogonal complement of V $\{w_i\}$ with $i=k+1,\ldots,N$. Then the pair (v,w) specifies a point in $Gr(k,N)$. A pair (v',w'), however, gives the same point if

$$(v',w') = (v,w)\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\,.$$

Then

$$Gr(k,N) \simeq \mathrm{GL}(N)/P$$

with $P = \left\{\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right\}$.

The relation between $Gr(k,N)$ and fermions is established by considering the GL(N) representation on a fermionic Fock space F defined by the vacua

$$|k>= e_1 \wedge \ldots \wedge e_k \qquad <k| = i_{e_k}\ldots i_{e_1} \qquad <i|k>= \delta_{ik}\,, \tag{7.1.1}$$

where $\{e_i\}$ is a basis of C^N and $i_{e_i}(e_j) = \delta_{ij}$ is the inner product operator. The fermionic operators are defined by

$$\psi_i^\dagger = e_i \wedge |\chi> \qquad \psi_i = i_{e_i}|\chi>, \tag{7.1.2}$$

and satisfy canonical anticommutation relations

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0. \tag{7.1.3}$$

The vacua $|k>$ carry charge k and $\psi_i^\dagger(\psi_i)$ create a charge $+1(-1)$. Then

$$\begin{aligned} &\psi_i^\dagger|k>=0 \quad i=1,\ldots,k \qquad \psi_i|k>=0 \quad i=k+1,\ldots,N \\ &<k|\psi_i^\dagger=0 \quad i=k+1,\ldots,N \qquad <k|\psi_i=0 \quad i=1,\ldots,k, \end{aligned} \tag{7.1.4}$$

The Plücker embedding is defined by assigning to every point $V \in Gr(k,N)$ a state

$$|v>= c\; v_1 \wedge \ldots \wedge v_k \quad \text{with} \quad v_i = \sum v_{ij} e_j, \tag{7.1.5}$$

where $\{v_i\}$ is a basis of V and c is an arbitrary constant. A change of basis $v_i \to a_{ij} v_j$ corresponds to $c \to (\det a)\, c$ and the state $|v>$ is well defined. The condition

$$\psi^\dagger[v_i]|v>=0 \quad \forall i, \tag{7.1.6}$$

with $\psi^\dagger[v_i] = \sum v_{ij}\psi_i^\dagger$ defines equivalently the state $|v>$ up to the constant c.

Then $a \in \mathrm{gl}(N)$ acts on F by

$$\hat{a}|\chi>= \sum \psi_i^\dagger a_{ij} \psi_j |\chi> \qquad |\chi>\in \mathrm{F}, \tag{7.1.7}$$

and on the space of operators on F by

$$[\psi_i, \hat{a}] = \sum_k a_{ik}\psi_k, \quad [\hat{a}, \psi_i^\dagger] = \sum_k \psi_k^\dagger a_{ki}. \tag{7.1.8}$$

The action of $g \in \mathrm{GL}(N)$ is defined by exponentiation of (7.1.7). For example

$$\hat{g}\psi_{i_1}^\dagger \psi_{i_2}^\dagger \ldots \psi_{i_1}\psi_{i_2}\ldots|0>= (\psi^\dagger g)_{i_1}(\psi^\dagger g)_{i_2}\ldots(g\psi)_{i_1}(g\psi)_{i_2}\ldots|0> \tag{7.1.9}$$

with $(\psi^\dagger g)_i \equiv \psi_j^\dagger g_{ji}$ and $(g\psi)_i \equiv g_{ij}\psi_j$. Then a gl$(N)$ operator a acting on $V \in Gr(k,N)$ by $a\,v = \sum (a_{ij}v_j)e_i$ corresponds to a fermionic operator $\widehat{a} = \sum \psi_i^\dagger a_{ij}\psi_j$. Then if $\widehat{a}_1 \leftrightarrow a_1$ and $\widehat{a}_2 \leftrightarrow a_2$, equations (7.1.8) give

$$[\widehat{a}_1, \widehat{a}_2] \leftrightarrow [a_1, a_2]\,. \tag{7.1.10}$$

Moreover note that if

$$\widehat{a}|v> = \text{const.}|v> \Leftrightarrow a\,V \subset V\,. \tag{7.1.11}$$

The state $|v>$ belongs to the GL(N) orbit of the state $|k>$. Since for $|v> = v_1 \wedge \ldots \wedge v_k$ every vector v_i can be written in the form $v_i = g\,e_i$ for some fixed $g \in$ GL(N), we have that $|v> = \widehat{g}|k>$ as defined in (7.1.9). Therefore the image of $Gr(k,N)$ under the Plücker embedding can be identified with the orbit GL$(N)|k>$.

The τ-functions are given by fermion correlators

$$\tau_V^{\mathcal{O}} = <\mathcal{O}>_V = <k|\mathcal{O}|v>\,, \tag{7.1.12}$$

with $\mathcal{O}$ a zero charge operator. Since the topology of $Gr(k,N)$ is non-trivial, we divide it into cells $(\mathcal{U}_a, a \in I)$. A point $V \in \mathcal{U}_a$ is represented by a basis $\{v_i^{(a)}\}$ and the state $|v>^{(a)} = v_1^{(a)} \wedge \ldots \wedge v_k^{(a)}$. Then if $V \in \mathcal{U}_a \cap \mathcal{U}_b$ we have $v_k^{(a)} = a_{ki}^{(ab)}\,v_i^{(b)}$ and

$$\tau_V^{\mathcal{O}(a)} = \det a^{(ab)}\,\tau_V^{\mathcal{O}(b)}$$

Therefore the τ-functions are really sections of a determinant line bundle over $Gr(k,N)$ whose transition functions are given by $\det a^{(ab)}$.

Most of the results carry over almost unchanged to the infinite dimensional case. For the infinite dimensional vector space we consider the space of formal Laurent series

$$H = \{\sum_n a_n z^n\,, \quad a_n = 0 \quad \text{for} \quad n \gg 0\}$$

and its decomposition

$$H = H_+ \oplus H_- ,$$

where $H_+ = \{ \sum_{n\geq 0} a_n z^n , \quad a_n = 0 \quad \text{for} \quad n \gg 0 \}$. Then the big cell of the Sato Grassmannian $Gr^{(0)}$ consists of all subspaces $V \subset H$ comparable to H_+, in the sense that the natural projection $\pi_+ : V \to H_+$ is an isomorphism. Then V admits a basis of the form $\{\phi_i(z)\}_{i\geq 0}$ where $\phi_i(z) = z^i +$ lower order terms. The Plücker embedding (7.1.5) is defined by the semi-infinite wedge product

$$|v >= c\,\phi_1(z) \wedge \phi_2(z) \wedge \ldots . \tag{7.1.13}$$

Care has to be taken so that a GL(∞) change of basis $\phi_i(z) \to a_{ij}\phi_j(z)$ does not introduce infinities, since det a can be infinite. We choose a set of admissible bases for $V \in Gr$ to be those whose matrix relating $\{\pi_+(\phi_i(z))\}_{i\geq 0}$ to $\{z^i\}_{i\geq 0}$ differs from the identity by an operator of trace class. Then the fermionic representation is defined on the Fock space built on the vacuum state of zero charge

$$|0 >= 1 \wedge z \wedge z^2 \wedge \ldots , \tag{7.1.14}$$

by fermions $\psi_i^\dagger$ and ψ_i defined as in (7.1.2). The states ($m > 0$)

$$|m >= \psi_m^\dagger \ldots \psi_1^\dagger|0 > , \quad | - m >= \psi_{-m+1} \ldots \psi_0|0 > \tag{7.1.15}$$

are the filled states with charge m and $-m$ respectively. The generalization of gl(N) is given by gl(∞) and is represented on F by its central extension $gl^*(\infty)$ with

$$\hat{a} = \sum_{i,j} : \psi_i^\dagger a_{ij} \psi_j : \tag{7.1.16}$$

where

$$: \psi_i^\dagger \psi_j : = \psi_i^\dagger \psi_j - < \psi_i^\dagger \psi_j >= \begin{cases} \psi_i^\dagger \psi_j & i > 0 \\ -\psi_j \psi_i^\dagger & i \leq 0 \end{cases} \tag{7.1.17}$$

is the normal ordering. The reason for introducing normal ordering is that the naive operator $\sum_{i,j} \psi_i^\dagger a_{ij} \psi_j$ maps an admissible basis to a non-admissible one.

Then the fermionic representation of the algebra $\mathrm{gl}(\infty)$ is defined by

$$r_F(a)|\chi>=\sum_{i,j}:\psi_i^\dagger a_{ij}\psi_j:|\chi>\qquad a\in \mathrm{gl}(\infty)\quad |\chi>\in \mathrm{F} \tag{7.1.18}$$

and of the group $\mathrm{GL}(\infty)$ by

$$\begin{aligned} &R_F(g)\Big(\psi_{i_1}^\dagger\psi_{i_2}^\dagger\ldots\psi_{i_1}\psi_{i_2}\ldots\Big)|-m>= \\ &\quad\Big((\psi^\dagger g)_{i_1}(\psi^\dagger g)_{i_2}\ldots(g\psi)_{i_1}(g\psi)_{i_2}\ldots\Big)|-m> \end{aligned} \tag{7.1.19}$$

for $m\gg 0$ such that $(\psi^\dagger g)_{-j}=\psi_{-j}^\dagger$ for $j>m$. In (7.1.19), $g\in\mathrm{GL}(\infty)$ and $(\psi^\dagger g)_i\equiv\psi_j^\dagger g_{ji}$ and $(g\psi)_i\equiv g_{ij}\psi_j$. The above representation conserves the charge and therefore preserves the decomposition

$$\mathrm{F}=\oplus_{m\in \mathrm{Z}}\mathrm{F}^{(m)}$$

where $\mathrm{F}^{(m)}$ is the space of states with charge m.

The connection of the fermion representation of $Gr^{(0)}$ and the KP and mKP hierarchies is made explicit by making use of the boson-fermion equivalence in two dimensions. The fermionic currents

$$J_n=\sum_{r\in \mathrm{Z}}:\psi_{n-r}^\dagger\psi_r:\qquad n\in \mathrm{Z} \tag{7.1.20}$$

satisfy the bosonic commutation relations

$$[J_m,J_n]=m\delta_{m,-n}\,. \tag{7.1.21}$$

By representing the bosonic Fock space by $\mathrm{B}\cong \mathrm{C}[t_1,t_2,\ldots,;z,z^{-1}]$, the space of polynomials in $t_1,t_2,\ldots,;z,z^{-1}$, $\frac{\partial}{\partial t_n}$ and $-nt_{-n}$ (with $n\geq 0$) act as creation and annihilation operators on B satisfying the algebra (7.1.21).

The operators J_n are mapped to $\frac{\partial}{\partial t_n}$ for $n\geq 0$ and to $-nt_{-n}$ for $n<0$. The filled state of charge m $|m>$ is mapped to z^m. Therefore we define an isomorphism

$$\sigma:\mathrm{F}\to \mathrm{B}$$

by

$$\sigma(|m>) = z^m\,, \quad \sigma J_n \sigma^{-1} = \frac{\partial}{\partial t_n}\ (n \geq 0) \quad \sigma J_n \sigma^{-1} = -nt_{-n}\ (n < 0)\,. \quad (7.1.22)$$

The Fock space $B \cong C[t_1, t_2, \ldots, ; u, u^{-1}]$ can be decomposed into (fixed charge m) subspaces $B^{(m)} \cong z^m C[t_1, t_2, \ldots,]$ such that

$$B = \oplus_{m \in Z} B^{(m)}$$

and the map $\sigma = \oplus_{m \in Z} \sigma^{(m)}$ into charge preserving maps

$$\sigma^{(m)} : F^{(m)} \rightarrow B^{(m)}\,.$$

The inner product in the fermion space $F^{(m)}$ carries over to the bosonic space $B^{(m)}$ and is given by the formula

$$\langle t_1^{k_1} t_1^{k_2} \ldots, t_1^{k_1} t_1^{k_2} \ldots \rangle = \frac{k_1!}{1} \frac{k_2!}{2^{k_2}} \frac{k_3!}{3^{k_3}} \ldots\,.$$

The τ-function $\tau_\chi(t)$ associated to the state $|\chi>$ is given by the boson fermion equivalence:

$$\tau_\chi(t) = \oplus_{m \in Z} \tau_\chi^{(m)}(t)\,,$$

where

$$\tau^\chi(t; z, z^{-1}) = \sum_{m \in Z} z^m < m| e^{\sum_{p \geq 1} t_p J_p} |\chi> \equiv \sum_{m \in Z} z^m \tau_m^\chi(t)\,. \quad (7.1.23)$$

In order to prove this relation use the identities $e^{\sum_{p \geq 1} t_p J_p} J_n = \frac{\partial}{\partial t_n} e^{\sum_{p \geq 1} t_p J_p}$ for $n > 0$ and $e^{\sum_{p \geq 1} t_p J_p} J_n = J_n e^{\sum_{p \geq 1} t_p J_p} - nt_{-n}$ for $n < 0$.

The fermion operators

$$\psi(z) = \sum_{n \in Z} \psi_n z^{-n} \qquad \text{and} \qquad \psi^\dagger(z) = \sum_{n \in Z} \psi_n^\dagger z^{-n}$$

act on the τ-functions by

$$
\begin{aligned}
(\sigma^{(m+1)}\psi(z)(\sigma^{(m)})^{-1})\tau^{(m)}(t) =& X(z,t)\tau^{(m)}(t) = z^{m+1}\Gamma_-(z)\Gamma_+(z)\tau^{(m)}(t) \quad \in \mathrm{B}^{m+1} \\
(\sigma^{(m-1)}\psi^\dagger(z)(\sigma^{(m)})^{-1})\tau^{(m)}(t) =& X^\dagger(z,t)\tau^{(m)}(t) = z^{-m}\Gamma_-^{-1}(z)\Gamma_+^{-1}(z)\tau^{(m)}(t) \quad \in \mathrm{B}^{m-1}
\end{aligned}
\tag{7.1.24}
$$

where

$$
\begin{aligned}
\Gamma_+(z) =& \exp\{-\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\} \\
\Gamma_-(z) =& \exp\{\sum_{n=1}^{\infty} z^n t_n\} \,.
\end{aligned}
\tag{7.1.25}
$$

These are called the vertex operators. In order to prove (7.1.24) note that $X(z,t) = z^{m+1}\Gamma_-(z)\Gamma_+(z)$ and $X^\dagger(z,t) = z^{-m}\Gamma_-^{-1}(z)\Gamma_+^{-1}(z)$ are the unique operators preserving the commutation relations $[J_m, \psi(z)] = -z^{-m}\psi(z)$ for $n > 0$ and $[J_m, \psi^\dagger(z)] = z^m\psi^\dagger(z)$ *i.e.* for example $[\frac{\partial}{\partial t_m}, X^\dagger(z,t)] = z^m X^\dagger(z,t)$ for $n > 0$ and $[t_m, X^\dagger(z,t)] = \frac{z^{-m}}{m}X^\dagger(z,t)$ for $n < 0$.

The important contribution of the Japanese school (see [66,69–71] and references therein) is that they found that the k^{th} modified KP (mKP) hierarchy is equivalent to the simple relations

$$
\begin{aligned}
z^0 \text{ term of} \qquad & \psi^\dagger(z)|g>_m \otimes \psi(z)|g>_0 = 0 \Rightarrow \\
& \sum_{j\in \mathrm{Z}} \psi_j^\dagger|g>_m \otimes \psi_j|g>_0 = 0 \,,
\end{aligned}
\tag{7.1.26}
$$

for all $m = 0, 1, \ldots, k-1$. The vector $|g>_m \equiv g|m> \in \mathrm{GL}(\infty)$ belongs to the $\mathrm{GL}(\infty)$ orbit of the filled state $|m>$. (7.1.26) is trivially true for $g = 1$ since either ψ_j or $\psi_j^\dagger$ annihilate $|m>$ or $|0>$. By acting with the Lie algebra element $\widehat{a} \in \mathrm{gl}(\infty)$, where the action is given by (7.1.8) we obtain

$$
\sum_{j\in \mathrm{Z}}[\widehat{a}, \psi_j^\dagger]|m> \otimes \psi_j|0> + \sum_{j\in \mathrm{Z}} \psi_j^\dagger|m> \otimes [\psi_j, \widehat{a}]|0> = 0 \,.
$$

By exponentiating the action of $\widehat{a}$ we obtain (7.1.26). The importance of this result lies in the fact that the converse is also true [69–71]. In particular for $k = 0$ we obtain the KP hierarchy and for $k = 1$ the second mKP hierarchy which is related to the UMM. I refer to the latter simply as the mKP hierarchy.

The relevant τ-functions of the mKP hierarchy are

$$\tau_i(t) =< i-1| \exp\{\sum_{p\geq 1} t_p J_p\} g|i-1> \quad (i=1,2)\,. \tag{7.1.27}$$

The conditions (7.1.26) on the τ-functions $\tau_i(t)$ are lead to the so called Hirota equations. These are bilinear identities derived from (7.1.24)

$$\begin{aligned}
&\oint X^\dagger(t',z)\tau_{m+1}(t',z)\otimes X(t'',z)\tau_1(t'',z)\frac{dz}{z}=0\,, \quad m=0,1\\
\Rightarrow &\oint \exp\{\sum_{n\geq 1} z^n(t'_n-t''_n)\}\exp\{-\sum_{n\geq 1}\frac{z^{-n}}{n}(\frac{\partial}{\partial t'_n}-\frac{\partial}{\partial t''_n})\}\tau_{m+1}(t',z)\tau_1(t'',z)z^m dz=0\\
\Rightarrow &\oint \exp\{\sum_{n\geq 1} 2z^n y_n\}\exp\{-\sum_{n\geq 1}\frac{z^{-n}}{n}(\frac{\partial}{\partial y_n})\}\tau_{m+1}(t+y,z)\tau_1(t-y,z)z^m dz=0\\
\Rightarrow &\sum_{n\geq 0} p_n(2y)p_{n+1+m}(-\frac{\tilde\partial}{\partial y})\tau_{m+1}(t-y,z)\tau_1(t+y,z)=0\,.
\end{aligned} \tag{7.1.28}$$

The polynomials $p_n(x)$ are the elementary Schur polynomials defined by

$$\sum_{k\in Z} p_k(t)y^k = \exp\sum_{k\in Z}\{z^k y_k\}\,. \tag{7.1.29}$$

I use the standard notation $\tilde y=(y_1,\frac12 y_2,\frac13 y_3,\ldots)$ and $\frac{\tilde\partial}{\partial y}=(\frac{\partial}{\partial y_1},\frac12\frac{\partial}{\partial y_2},\frac13\frac{\partial}{\partial y_3},\ldots)$. By Taylor expanding the last line we obtain the Hirota equations

$$\sum_{n\geq 0} p_j(2y)p_{j+k+1}(-\widetilde{D})\mathrm{e}^{\sum_{p=1}^{\infty} y_p D_p}\tau_{m+1}(t)\tau_1(t)=0\,. \tag{7.1.30}$$

D_p is the Hirota derivative

$$P(D_p)\,f(t)g(t)=P(\frac{\partial}{\partial y_p})\,f(t+y)g(t-y)|_{y=0}\,. \tag{7.1.31}$$

(7.1.30) is understood as a generating series, with $y_1,y_2,\ldots$ as free parameters. For example the simplest Hirota equation obtained for $k=0$ from $p_3(y)=\frac{1}{12}[(y_1^4-4y_1y_3+3y_2^2)-6(y_1^2y_2+y_4)]$ is

$$(D_1^4-4D_1D_3+3D_2)\,\tau_i\tau_i=0\,, \quad i=1,2\,.$$

Putting $x = t_1$, $y = t_2$ and $z = t_3$, as well as $u_i = -2\partial^2 \ln \tau_i$, the last equation gives the Kadomtsev-Petviashvilli equation for u:

$$\frac{3}{4}\partial_y^2 u_i = \partial_x(\partial_z u_i - \frac{3}{2}u_i \partial_x u_i - \frac{1}{4}\partial_x^3 u_i) .$$

One can go further and observe that the Kac-Moody algebra of sl_n (thought of as $\widehat{sl_n}(n, \mathrm{C}[u, u^{-1}])$) when embedded in $\mathrm{gl}(\infty)$ has irreducible highest weight representations on the space $\mathrm{B}_{(n)} = \oplus_{m=1}^{n-1} \mathrm{B}_{(n)}^{(m)}$ where $\mathrm{B}_{(n)}^{(m)} = \mathrm{C}[t_j | j \neq 0 \bmod n] \subset \mathrm{B}^{(m)}$. Therefore one can reduce the mKP(resp. KP) hierarchies and obtain the so called n-reduced mKP(resp. KP) hierarchies. Then one can show [70,71] that the τ-function $\tau_{(n)} = \oplus_{k=0}^{n-1} \tau_k$ belongs to the $\widehat{SL_n}$ orbit of the sum of the highest weight vectors $\oplus_{m=0}^{n-1} 1_m$. We are mainly interested in the second reduced mKP hierarchies.

The equation for the second reduced mKP hierarchy are simply given by (7.1.30) by deleting the dependence on even times. In particular the second reduced KP hierarchy is the KdV hierarchy and the second reduced mKP hierarchy is the mKdV hierarchy. In the latter case the simplest Hirota equations give

$$\begin{aligned} (D_1^4 - 4D_1 D_3)\, \tau_i \tau_i &= 0 , \quad i = 1, 2 \\ D_1^2\, \tau_2 \tau_1 &= 0 . \end{aligned} \tag{7.1.32}$$

Upon substituting

$$u_i = -2\partial^2 \ln \tau_i \qquad \text{and} \qquad v = \partial \ln \frac{\tau_2}{\tau_1} \tag{7.1.33}$$

we obtain the classical KdV equation

$$\partial_t u_i = -\frac{3}{2} u_i \partial_x u_i + \frac{1}{4}\partial_x^3 u_i ,$$

where $x = t_1$, $t = t_3$ and $u_i = v^2 \pm v'$. Therefore v satisfies the first equations of the mKdV hierarchy, *i.e.* the classical mKdV equation. As we will see in the next section, the rest of the Hirota equations give the full hierarchy.

Note that (7.1.33) implies that

$$v^2 = -\partial^2 \ln(\tau_1 \tau_2) \tag{7.1.34}$$

so that the partition function of the UMM is given by the product of the two mKdV τ-functions [19]:

$$Z = \tau_1\,\tau_2\,. \tag{7.1.35}$$

Therefore every solution $\tau_1(t)$ and $\tau_2(t)$ of the mKdV hierarchy corresponds to points $V_1(t)$ and $V_2(t)$ in $Gr^{(0)}$ given by the states $|v_i(t)>= \exp\{\sum_{p\geq 1} t_p J_p\} g|i-1>$. Then the time dependence of $V_i(t)$ is given by

$$\frac{\partial}{\partial t_{2k+1}}|v_i(t)>= J_{2k+1}|v_i(t)> \quad \text{and} \quad J_{2k}|v_i(t)>= 0\,, \tag{7.1.36}$$

or by using the correspondence (7.1.10)

$$\frac{\partial}{\partial t_{2k+1}} V_i(t) = z^{2k+1}\,V_i(t) \quad \text{and} \quad z^{2k}\,V_i(t) \subset V_i(t)\,. \tag{7.1.37}$$

Then

$$V_i(t) = \exp\{\sum_k t_{2k+1} z^{2k+1}\} V_i \equiv \gamma(t,z) V_i\,. \tag{7.1.38}$$

7.2. More on the mKP and KP hierarchies

In this section I give a brief description of the relation between the τ-function formalism of the previous section and the formalism for the KdV and mKdV hierarchies of chapters 3 and 6. For an excellent review see [66].

The KdV and mKdV hierarchies admit a representation in terms of differential operators derived from pseudo-differential operators ΨDO . For example for the simple KdV hierarchies (3.3.35) consider the operators

$$\begin{aligned} Q =&\partial^2 - u \\ Q_+^{k-\frac{1}{2}} =&(\partial^2 - u)_+^{k-\frac{1}{2}}\,. \end{aligned} \tag{7.2.39}$$

These obey $[Q_+^{k-\frac{1}{2}}, Q] = -[Q_-^{k-\frac{1}{2}}, Q] = -\partial Res\, Q_+^{k-\frac{1}{2}} = -2\partial \mathrm{R}_k[u]$. The first equation follows from the trivial relation $[Q^{k-\frac{1}{2}}, Q] = 0$ and the second one by noting that $[Q_+^{k-\frac{1}{2}}, Q]$ must be a differential operator so that only the differential part of $[Q_-^{k-\frac{1}{2}}, Q]$ can be non-zero. The third one comes from the relation $Res\, Q^{k-\frac{1}{2}} = 2\mathrm{R}_k[u]$ discussed in section 6.2. Obviously $\frac{\partial}{\partial t_{2k+1}} Q = -\frac{\partial u}{\partial t_{2k+1}}$, and the KdV hierarchy can be written in the form

$$\frac{\partial}{\partial t_{2k+1}} Q = [Q_+^{k-\frac{1}{2}}, Q] \,. \tag{7.2.40}$$

The general KdV hierarchy is defined by (3.4.56) [72]

$$\frac{\partial}{\partial t_p} Q = [Q_+^{\frac{p}{q}}, Q] \,, \qquad p \neq 0 mod\, q \tag{7.2.41}$$

where (3.4.52) $Q = \partial^q + u_{q-2}\partial^{q-2} + \ldots$. Note that $Q^{\frac{1}{q}} = \partial + \sum_{i\geq 1} f_i \partial^i$ so that $P_p = Q^{\frac{p}{q}} = \partial^p + \sum_{-\infty<i<p} \alpha_i \partial^i$.

A similar equation for the mKdV hierarchy follows from the analysis of section 6.2:

$$\frac{\partial}{\partial t_{2k+1}} \mathcal{Q}_- = [\mathcal{P}_k, \mathcal{Q}_-] \,, \tag{7.2.42}$$

with $\mathcal{Q}_-$ and $\mathcal{P}_k$ given by (6.2.39) and (6.2.50) respectively. It is also possible to put these equations in first order form in terms of a Lax pair. This is discussed in detail in [72]

The KdV equation (7.2.41) is the q-reduced KP hierarchy. The KP system can be derived from a system of linear differential equations for $w(t, z)$:

$$\begin{aligned} L(t,\partial)\, w(t,z) &= z\, w(t,z) \\ \frac{\partial}{\partial t_n} w(t,z) &= B_n(t,\partial)\, w(t,z) \,, \end{aligned} \tag{7.2.43}$$

where $L = \partial + \sum_{k\geq 1} L_k \partial^{-l}$, $B_n(t,\partial) = \partial^n + \sum_{0\leq k\leq n} B_{nk}\partial^k$ is a differential operator, $t = (t_1, t_2, \ldots)$ and $\partial = \partial/\partial t_1$. The compatibility conditions for (7.2.43)

$$\begin{aligned} \frac{\partial L}{\partial t_n} &= [B_n, L] \\ \frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} &= [B_n, B_m] \,, \end{aligned}$$

are satisfied by $B_n = L^n_+$ so that the first of the above conditions becomes

$$\frac{\partial}{\partial t_n} L = [L^n_+, L] \,. \tag{7.2.44}$$

This gives the KP hierarchy as a deformation equation of the ΨDO operator L.

The general KdV hierarchy (7.2.41) follows in the case where L^q is a differential operator. Then the solutions are independent of the times t_{kq} and $Q \equiv L^q$. In particular the ordinary KdV hierarchy follows when $Q = L^2 = \partial^2 - u$ and the solutions depend on the odd times only.

We define the monic, normalized ΨDO $S = 1 + \sum_{k\geq 1} s_k \partial^k \in \mathcal{G}$ by

$$LS = S\partial \Rightarrow L^n = S\partial S^{-1} \,. \tag{7.2.45}$$

Then

$$\begin{aligned}\frac{\partial}{\partial t_n} S &= - L^n_- S \\ &= - (S\partial^n S^{-1})_- S \,.\end{aligned} \tag{7.2.46}$$

The function $w(t,z)$ is the Baker wavefunction. Is is easy to see that

$$\begin{aligned} w(t,z) =& S \mathrm{e}^{\sum_k t_k z^k} \\ =&(1 + \sum_i s_k(i) z^{-i}) \mathrm{e}^{\sum_k t_k z^k} \\ =&\widehat{w}(t,z) \mathrm{e}^{\sum_k t_k z^k} \end{aligned} \tag{7.2.47}$$

is a solution to (7.2.43). The adjoint wavefunction $w^*(t,z)$ is defined by

$$\begin{aligned} w^*(t,z) =&(S^\dagger)^{-1} \mathrm{e}^{\sum_k t_k z^k} \\ =&(1 + \sum_i s^*_k(t) z^{-i}) \mathrm{e}^{\sum_k t_k z^k} \\ =&\widehat{w}^*(t,z) \mathrm{e}^{\sum_k t_k z^k} \end{aligned} \tag{7.2.48}$$

where $S^\dagger = (1 + \sum_i (-1)^i \partial^{-i} s_k(t))$ is the formal adjoint of S.

The wavefunctions $w(t,z)$ and $w^*(t,z)$ satisfy the "bilinear identity"

$$\oint_c w(t,z) w^*(t',z) = 0 \tag{7.2.49}$$

for every t and t'. The contour is taken around ∞. The converse is also true. If the functions $w(t,z)$ and $w^*(t,z)$ are of the form (7.2.47) and (7.2.48) and satisfy the bilinear identity, then $w(t,z)$ is the wavefunction of the KP hierarchy and $w^*(t,z)$ its adjoint and L is defined by (7.2.45).

The τ-functions of the KP hierarchy are defined by

$$\begin{aligned} w(t,z) &= \frac{\tau(t_1-\frac{1}{z}, t_2-\frac{1}{2z^2}, t_3-\frac{1}{3z^3}, \ldots)}{\tau(t)} e^{\sum_k t_k z^k} \\ w^*(t,z) &= \frac{\tau(t_1+\frac{1}{z}, t_2+\frac{1}{2z^2}, t_3+\frac{1}{3z^3}, \ldots)}{\tau(t)} e^{-\sum_k t_k z^k}\,. \end{aligned} \tag{7.2.50}$$

The τ-function is associated with the existence of a closed one form

$$\omega(t,dt) = \sum_{i\geq 1} dt_i Res_{z=\infty}\, z^i (\sum_{j\geq 1} z^{-1-j}\frac{\partial}{\partial t_j} - \frac{\partial}{\partial z}) \ln \widehat{w}(t,z)$$

where $d\omega = 0$. The closeness of ω is equivalent to the bilinear identity (7.2.49) (for a proof see [66]). Then at least locally

$$\omega(t,dt) = d\ln\tau(t)\,,$$

and

$$\frac{\partial}{\partial t_n}\ln\tau = Res_{z=\infty}\, z^n (\sum_{j\geq 1} z^{-1-j}\frac{\partial}{\partial t_j} - \frac{\partial}{\partial z}dz) \ln \widehat{w}(t,z)\,,$$

which is (7.2.50).

It is easy to check that for the second reduced KP or KdV system

$$u = -2\partial^2 \ln\tau\,. \tag{7.2.51}$$

To see this, simply expand (7.2.50) and from (7.2.47) get $s_1 = -\partial\ln\tau$. Then (7.2.46) gives $\frac{\partial}{\partial t_n}s_1 = -L_1^n$, where $L^n = \partial^n + \sum_{k<n} L_k^n \partial^{-k}$. Since $Q = \partial^2 - u = L^2 = L_+^2 = \partial^2 + L_1$ we obtain (7.2.51).

The bilinear identity (7.2.49) is just the Hirota equations for the τ-functions defined in (7.2.50).

$$\begin{aligned}
0 &= \int \tau(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, t_3 - \frac{1}{3z^3}, \ldots)\, \tau(t'_1 + \frac{1}{z}, t'_2 + \frac{1}{2z^2}, t'_3 + \frac{1}{3z^3}, \ldots) \mathrm{e}^{\sum_k (t_k - t'_k) z^k} dz \\
&= \int \tau(x_1 - y_1 - \frac{1}{z}, x_1 - y_1 - \frac{1}{2z^2}, x_1 - y_1 - \frac{1}{3z^3}, \ldots) \\
&\qquad \times \tau(x_1 + y_1 + \frac{1}{z}, x_1 + y_1 + \frac{1}{2z^2}, x_1 + y_1 + \frac{1}{3z^3}, \ldots) \mathrm{e}^{-2\sum_k y_k z^k} dz \\
&= \int \mathrm{e}^{2\sum_k y_k z^k} \mathrm{e}^{-\sum_{n\geq 1} \frac{z^{-n}}{n} (\frac{\partial}{\partial y_n})} \tau(t+y, z)\, \tau(t-y, z) z^m dz \,,
\end{aligned}$$

which is (7.1.28). This establishes the connection between the τ-function formalism described in the previous section and the definition I gave originally to the KdV hierarchy in chapter 3. A similar connection can be established for the mKdV hierarchy as well. This is to be expected since the τ-functions of the mKdV hierarchy are two separate KdV τ-functions related by the Miura transform. For a more complete discussion on the mKdV hierarchies and their generalizations see [70–71,72].

7.3. The String Equation and the Sato Grassmannian

As I discussed in chapter 6, the string equation of UMM in the operator formalism is written in the form

$$[\mathcal{P}, \mathcal{Q}_-] = 1 \,, \tag{7.3.52}$$

where $\mathcal{P}$ and $\mathcal{Q}_-$ are 2×2 matrices of differential operators of the form (6.2.39) and (6.2.50). The solutions to this equation were shown to flow between multicritical points according to the mKdV hierarchy. To this hierarchy correspond two τ-functions τ_1 and τ_2 which obey the bilinear Hirota equations. Using the Plücker embedding discussed in section 7.1 we can associate to each τ-function a point in the big cell of the Sato Grassmannian $Gr^{(0)}$ satisfying

$$\frac{\partial}{\partial t_{2k+1}} V_i = z^{2k+1} V_i \,, \qquad z^2 V_i \subset V_i \,, \quad i = 1, 2 \,. \tag{7.3.53}$$

The two τ-function are related by the Miura transform (7.1.32) and we will see that this imposes an extra condition on the spaces V_i, namely that

$$z\,V_1 \subset V_2 \qquad \text{and} \qquad z\,V_2 \subset V_1\,. \tag{7.3.54}$$

Therefore the solutions to the mKdV hierarchy are given by a pair of points V_1 and V_2 in $Gr^{(0)}$ that satisfy (7.3.53) and (7.3.54). The string equation imposes an extra condition on them that picks out a unique pair of points from this subspace of $Gr^{(0)} \times Gr^{(0)}$. It is possible to solve these conditions for the spaces V_1 and V_2 satisfying these conditions and obtain all the solutions to the string equation.

Given an operator $\mathcal{Q}_-$ we can associate to it unique points V_1 and V_2 in $Gr^{(0)}$. This is based on the fact that the spaces V_i can be written in terms of a unique ΨDO operator $S_i \in \mathcal{G}$ as $V_i = S_i\,H_+$. I remind to the reader that the S_i are of the form

$$S_i = 1 + \sum_{k\geq 1} s_k^{(i)} \partial^{-k}\,. \tag{7.3.55}$$

We will see that the S_i correspond to the Baker wavefunctions of the KdV hierarchy (7.2.47) discussed in the previous section.

Indeed, consider S_1 and $S_2 \in \mathcal{G}$ such that

$$\widehat{S}\mathcal{Q}_-\widehat{S}^{-1} = \tilde{\mathcal{Q}}_- \tag{7.3.56}$$

where

$$\widehat{S} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \; \tilde{\mathcal{Q}}_- = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}. \tag{7.3.57}$$

Then

$$\begin{aligned} S_1(\partial + v)S_2^{-1} &= \partial\,, \\ S_2(\partial - v)S_1^{-1} &= \partial\,, \end{aligned} \tag{7.3.58}$$

which imply that

$$\begin{aligned} S_1(\partial^2 - u_1)S_1^{-1} &= \partial^2 \quad u_1 = v^2 + v'\,, \\ S_2(\partial^2 - u_2)S_2^{-1} &= \partial^2 \quad u_2 = v^2 - v'\,. \end{aligned} \tag{7.3.59}$$

The existence of $S_1 \in \mathcal{G}$ follows from the general fact [73] that for every monic normalized pseudodifferential operator $\mathcal{L}$ of order n there exists an S such that $S\mathcal{L}S^{-1} = \partial^n$.

Given S_1, one can determine S_2 from

$$S_1(\partial + v) = \partial S_2 .$$

By taking formal adjoints of (7.3.58) and (7.3.59), it is easy to show that S_1 and S_2 can be made simultaneously unitary. Indeed, from (7.3.59) we obtain

$$\begin{aligned} (S_1^{-1})^\dagger(\partial^2 - \tilde{u})S_1^\dagger &= \partial^2 \Rightarrow \\ (S_1 S_1^\dagger)^{-1}\partial^2(S_1 S_1^\dagger) &= \partial^2 \Rightarrow \\ S_1 S_1^\dagger &= f(\partial^2)\,, \end{aligned} \tag{7.3.60}$$

where f is arbitrary. Similarly $S_2 S_2^\dagger = g(\partial^2)$. But since (7.3.56) implies

$$(\hat{S}\hat{S}^\dagger)^{-1}\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}(\hat{S}\hat{S}^\dagger) = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \tag{7.3.61}$$

then

$$(\hat{S}\hat{S}^\dagger) = \begin{pmatrix} f(\partial^2) & 0 \\ 0 & g(\partial^2) \end{pmatrix}$$

gives

$$\begin{aligned} \partial g &= f\partial\,, \\ \partial f &= g\partial\,, \end{aligned}$$

or, $f = g$. Therefore S_1 and S_2 can be simultaneously chosen to be unitary, i.e $S_1 S_1^\dagger = 1$ and $S_2 S_2^\dagger = 1$.

Since $V \in Gr^{(0)}$ is given uniquely by $V = SH_+$, the operator $\mathcal{Q}_-$ determines two spaces $V_1 = S_1 H_+$ and $V_2 = S_2 H_+$. Conversely given spaces V_1 and V_2 determine $\mathcal{Q}_-$ uniquely. The operator $\mathcal{Q}_-$, however, is a differential operator and V_1, V_2 cannot be arbitrary. Indeed, since every differential operator leaves H_+ invariant, we obtain

$$\begin{aligned} (\partial + v)\, H_+ \subset H_+ &\Leftrightarrow S_1^{-1}\partial S_2\, H_+ \subset H_+ \\ &\Leftrightarrow \partial\, V_2 \subset V_1 \\ &\Leftrightarrow z\, V_2 \subset V_1 \end{aligned} \tag{7.3.62}$$

Similarly, $z V_2 \subset V_1$.

The string equation imposes further conditions on V_1 and V_2. After transformation with the operator $\widehat{S}$ equation (7.3.52) becomes

$$[\widetilde{\mathcal{P}}_{(k)}, \widetilde{\mathcal{Q}}_{-}] = 1 \tag{7.3.63}$$

where $\widetilde{\mathcal{P}}_{(k)} = \widehat{S}\mathcal{P}_{(k)}\widehat{S}^{-1}$. The solution to (7.3.63) is

$$\widetilde{\mathcal{P}}_{(k)} = \begin{pmatrix} 0 & -x + \tilde{f}_k(\partial) \\ -x + \tilde{f}_k(\partial) & 0 \end{pmatrix} \tag{7.3.64}$$

which gives $\mathbf{P}_{(k)} = S_1^{-1}(-x + \tilde{f}_k(\partial))S_2$ and $\mathbf{P}^{\dagger}_{(k)} = S_2^{-1}(-x + \tilde{f}_k(\partial))S_1$. Consistency requires therefore that $-x + \tilde{f}_k(\partial)$ must be self adjoint $\tilde{f}_k(\partial) = f_k(\partial^2)$. For the k^{th} multicritical point $\mathbf{P}_{(k)}$ is a differential operator of order $2k$. Therefore $f_k(\partial^2) = \partial^{2k} + \ldots$. By using the freedom to redefine S_i by a monic, zeroth-order, pseudodifferential operator $R = 1 + \sum_{i\geq 1} r_i\partial^{-i}$ with constant coefficients r_i, it is easy to show that all negative powers in $f_k(\partial^2)$ may be eliminated. The proof shows that all powers below ∂^{-1} can be eliminated by R, and a ∂^{-1} term is forbidden by self-adjointness. Therefore

$$f_k(\partial^2) = \partial^{2k} + \sum_{1\leq i\leq k} f_{k,i}\partial^{2(k-i)} \tag{7.3.65}$$

By Fourier transforming, the action of $\widetilde{\mathcal{P}}$ on H is represented by

$$\widetilde{\mathcal{P}}_{(k)} = \begin{pmatrix} 0 & A_k \\ A_k & 0 \end{pmatrix}, \text{where} \quad A_k = \frac{d}{dz} + \sum_{i=0}^{k} \alpha_i z^{2i} \quad \text{and } \alpha_i \equiv f_{k,i} = \text{const.} \tag{7.3.66}$$

Given the constants α_i, we can calculate the operator $\mathbf{P}_{(k)}$. Since $S_2(\partial - v)(\partial + v)S_2^{-1} = \partial^2$ implies $S_2[(\partial - v)(\partial + v)]^{i-\frac{1}{2}}S_2^{-1} = \partial^{2i-1}$ then using $S_1(\partial + v)S_2^{-1} = \partial$ we obtain

$$S_1(\partial + v)[(\partial - v)(\partial + v)]^{i-\frac{1}{2}}S_2^{-1} = \partial^{2i}. \tag{7.3.67}$$

Transforming back to H_+ we obtain

$$\begin{aligned}\mathbf{P}_{(k)} &= S_1^{-1}(-x+\sum_{i=0}^{k}\alpha_i\partial^{2i})S_2 \\ &= S_1^{-1}(-x+\alpha_0)S_2+\sum_{i=1}^{k}\alpha_i S_1^{-1}\partial^{2i}S_2 \\ &= S_1^{-1}(-x+\alpha_0)S_2+\sum_{i=1}^{k}\alpha_i(\partial+v)[(\partial-v)(\partial+v)]^{i-\frac{1}{2}}\end{aligned} \tag{7.3.68}$$

Comparing with (6.2.55) and since $S_1^{-1}xS_2 = x+\sum_{i\geq 1} q_i(x)\partial^{-i}$, we conclude that at the k$^{\text{th}}$ multicritical point, $\alpha_k = 1$ and $\alpha_i = 0$ for $i < k$. Moreover, by perturbing away from the multicritical points we see that

$$\alpha_i(t) = -(2i+1)t_{2i+1}\,. \tag{7.3.69}$$

The requirement that $\mathcal{P}$ be a differential operator is equivalent to the conditions $A_k V_1 \subset V_2$ and $A_k V_2 \subset V_1$. The space of solutions to the string equation is the space of operators $\mathcal{Q}_-$ such that there exists $\mathcal{P}_{(k)}$ with $[\mathcal{P}_{(k)}, \mathcal{Q}_-] = 1$. We conclude that this space is isomorphic to the set of elements $V_1, V_2 \subset Gr^{(0)}$ that satisfy the conditions [24]:

$$\begin{aligned} z\,V_1 \subset V_2 \quad z\,V_2 \subset V_1 \\ A_k\,V_1 \subset V_2 \quad A_k\,V_2 \subset V_1\end{aligned} \tag{7.3.70}$$

for some $A_k = \frac{d}{dz}+\sum_{i=0}^{k}\alpha_i z^{2i}$.

It is now easy to show that the string equation is compatible with the mKdV flows. Since the mKdV flows are given by the equation

$$\frac{\partial}{\partial t_{2k+1}}\,V_i = z^{2k+1}\,V_i \quad (i=1,2)\,, \tag{7.3.71}$$

$V_i(t) = \exp\{\sum_k t_{2k+1}z^{2k+1}\}V_i \equiv \gamma(t,z)V_i$ and (7.3.70) imply

$$\begin{aligned} z\,\gamma(z,t)V_1 \subset \gamma(t,z)V_2 \Rightarrow z\,V_1(t) \subset V_2(t) \\ A_k(t)\,\gamma(z,t)V_1 \subset \gamma(t,z)V_2 \Rightarrow A_k(t)\,V_1(t) \subset V_2(t)\,,\end{aligned} \tag{7.3.72}$$

where

$$A_k(t) \equiv \gamma A_k \gamma^{-1} = A_k - \sum_k (2k+1) t_{2k+1} z^{2k} \tag{7.3.73}$$

and analogous equations with V_1 and V_2 interchanged. This is clearly consistent with (7.3.69).

In order to understand better the connection of (7.3.71) with the mKdV hierarchy, note that (7.3.71) implies that

$$\begin{aligned}
(\frac{\partial}{\partial t_{2k+1}} - z^{2k+1})\, V_i =&0 \subset V_i \Rightarrow \\
(S_i^{-1} \frac{\partial S}{\partial t_{2k+1}} - S_i^{-1} \partial^{2k+1} S_i) H_+ \subset& H_+ \Rightarrow \\
(S_i^{-1} \frac{\partial S}{\partial t_{2k+1}} - S_i^{-1} \partial^{2k+1} S_i)_- =&0 \Rightarrow ,
\end{aligned}$$

which is the equation (7.2.46) for the Baker function of the mKdV hierarchy

$$\frac{\partial S}{\partial t_{2k+1}} = (S_i^{-1} \partial^{2k+1})_- S_i \,.$$

The conditions (7.3.54), (7.3.62) are the Miura transform (7.3.59). In order to prove this, note that (7.3.62) imply that the coefficients $s_1^{(i)}$ are related to u_i by

$$u_i = -2 \partial s_1^{(i)} \,. \tag{7.3.74}$$

Then compute

$$S_1^{-1} \partial S_2 = \partial + (s_1^{(2)} s_1^{(1)}) + (s_1'^{(2)} + s_2^{(2)} + s_1^{(1)} s_1^{(2)} + (s_1^{(i)})^2 - s_1^{(2)}) \partial^{-1} \ldots \tag{7.3.75}$$

and set $v = s_1^{(2)} - s_1^{(1)}$ or $v' = \frac{1}{2}(u_1 - u_2)$. We only need to show that $v^2 = \frac{1}{2}(u_1 + u_2)$. But

$$\begin{aligned}
z\, V_2 \subset V_1 \Rightarrow & S_1^{-1} \partial S_2 \; H_+ \subset H_+ \\
\Rightarrow & (S_1^{-1} \partial S_2)_- = 0
\end{aligned}$$

The ∂^{-1} coefficient of the equation $(S_1^{-1} \partial S_2)_- + (S_2^{-1} \partial S_1)_- = 0$ gives $v^2 = \frac{1}{2}(u_1 + u_2)$.

The string equation can be expressed as a set of constraints on the τ-functions τ_1 and τ_2 that obey the Virasoro algebra. The idea is to transform the bosonic Virasoro generators into fermionic operators using the boson-fermion equivalence (7.1.24). Then using the correspondence between states in the GL(∞)-orbits of the vacuum and $Gr^{(0)}$, annihilation of the τ-function by the Virasoro constraints L_n is shown to be equivalent to the invariance of $V_i \in Gr^{(0)}$ under the action of operators $z^{2n+1}A$.

Consider the fermionic operators obeying the centerless Virasoro algebra

$$\mathrm{L}_n = \frac{1}{2}\sum_{p=-\infty}^{2n-1} J_p J_{2n-p} + \frac{1}{16}\delta_{n,0} \quad n \geq 0 \tag{7.3.76}$$

acting on the τ-functions associated with the states $|g>_i$

$$\tau_i(t) = < i-1| \exp\{\sum_{p\geq 1} t_p J_p\}|g>_i \quad i = 1,2\,. \tag{7.3.77}$$

Then shift the times $t_{2i+1} \to t_{2i+1} + \frac{\alpha_i}{2i+1}$ for $i \leq k$, where the α_i are defined in (7.3.66). Then

$$\begin{aligned} \tau_i(t) \to \tau_i'(t) &= < i-1| \exp\{\sum_{p\geq 1}(t_p + t_p^{(0)})J_p\}|g>_i\,, \\ \mathrm{L}_n \to \mathrm{L}_n' &= \mathrm{e}^{\sum_{p=0}^{k} \frac{\alpha_p}{2p+1} J_{2p+1}} \mathrm{L}_n \mathrm{e}^{-\sum_{p=0}^{k} \frac{\alpha_p}{2p+1} J_{2p+1}} \\ &= \mathrm{L}_n + \sum_{p=0}^{k} \alpha_p J_{2(n+p)+1}\,. \end{aligned} \tag{7.3.78}$$

In [25,26] it was shown that the fermion operators L_n' correspond via (7.1.10) to the operators

$$\frac{1}{2}z^{2n+1}A = \frac{1}{2}z^{2n+1}\left(\frac{d}{dz} + \sum_{p=0}^{k} \alpha_i z^{2i}\right)\,. \tag{7.3.79}$$

Then, because of (7.1.11), invariance of $V_{1,2}$ under $z^{2n+1}A$ (see (7.3.70)) implies that the τ-functions τ_i are annihilated by the L_n's for $n \geq 1$ and

$$\mathrm{L}_0 \tau_i = \mu \tau_i\,. \tag{7.3.80}$$

Note that the Virasoro constraints arise as a result of the consistency of the mKdV flows with the UMM. The above derivation also shows that if L_n annihilates the τ-function, all L_{n+m} will annihilate it too.

The constant μ is an arbitrary parameter. Such a parameter does not appear for $L_n (n \geq 1)$ by closure of the Virasoro algebra. As pointed out in [19] it is the same for the two τ-functions and it cannot be determined by the closure of the algebra since, contrary to the HMM, L_{-1} is absent. If one includes boundary conditions then there exists a one parameter family of solutions to the string equation with the correct scaling behaviour at infinity [20,62]. It has been suggested in [19] that the parameter of such a particular solution is related to μ. The Virasoro constraints are then those of a highest weight state of conformal dimension μ. Although L_{-1} is absent one should bear in mind the additional constraints arising from the interrelation of τ_1 and τ_2 determined by equation (7.3.70). The authors of [57] give in open closed string theory interpretation of UMM. In this picture the L_{-1} constraint is not absent. In order to appear one has to introduce a new "time" that corresponds to the boundary cosmological constant of the open space time. This effectively annihilates L_{-1}.

7.4. Algebraic Description of the Moduli Space

In this section I attempt to give a complete description of the moduli space of the string equation (7.3.52). As already mentioned, the space of solutions to (7.3.52) is isomorphic to the set of points V_1 and V_2 of $Gr^{(0)}$ that satisfy the conditions (7.3.70) [24]. Therefore I start by describing the spaces V_1 and V_2.

First choose vectors $\phi_1(z), \phi_2(z) \in V_1$, such that

$$\phi_1(z) = 1 + \text{lower order terms}\,, \quad \phi_2(z) = z + \text{lower order terms}$$

Then the condition $z^2 V_1 \subset V_1$ and $\pi_+(V_1) \cong H_+$ shows that we can choose a basis for V_1

$$\phi_1, \phi_2, z^2\phi_1, z^2\phi_2, \ldots$$

Since $z V_1 \subset V_2$ and $\pi_+(V_2) \cong H_+$ we can choose a basis for V_2 to be

$$\psi, z\phi_1, z\phi_2, z^3\phi_1, z^3\phi_2, \ldots$$

where $\psi(z) = 1 +$ lower order terms. Using $z V_2 \subset V_1$ we have $z\psi = \alpha\phi_1 + \beta\phi_2$. Choose ϕ_1, ϕ_2 such that $z\psi = \phi_2$. Then we obtain the following basis for V_1, V_2 $(\phi \equiv \phi_1)$:

$$\begin{aligned} V_1 &: \quad \phi, z\psi, z^2\phi, z^3\psi, \ldots \\ V_2 &: \quad \psi, z\phi, z^2\psi, z^3\phi, \ldots \end{aligned} \tag{7.4.81}$$

Then it is clear that ϕ, ψ specify the spaces V_1, V_2. Using the conditions $AV_1 \subset V_2$ and $AV_2 \subset V_1$ we obtain

$$\begin{aligned} (\frac{d}{dz} + f_k(z^2))\phi &= P_{00}(z)\phi + P_{01}(z)\psi \\ (\frac{d}{dz} + f_k(z^2))\psi &= P_{10}(z)\phi + P_{11}(z)\psi\,. \end{aligned} \tag{7.4.82}$$

The polynomials $P_{00}(z)$ and $P_{11}(z)$ are odd whereas $P_{01}(z), P_{10}(z)$ are even. Comparing both sides of (7.4.82) we find that because $\deg(f_k) = 2k$, $\deg(P_{01}(z)) = \deg(P_{10}(z)) = 2k$ and $\deg(P_{11}(z))$, $\deg(P_{00}(z)) < 2k$ and that the coefficients of the leading terms of $P_{01}(z)$ and $P_{10}(z)$ are equal to α_k.

Equations (7.4.82) can be rewritten in the form

$$D\chi = B_{2k}(z)\chi \tag{7.4.83}$$

where $\chi = \binom{\phi}{\psi}$,

$$D = \begin{pmatrix} \frac{d}{dz} & 0 \\ 0 & \frac{d}{dz} \end{pmatrix}, \quad B_{2k}(z) = \begin{pmatrix} P_{00}(z) - f_k(z^2) & P_{01}(z) \\ P_{10}(z) & P_{11}(z) - f_k(z^2) \end{pmatrix}. \tag{7.4.84}$$

The requirement that ϕ, ψ be solutions of the form $1 +$ (lower order terms), rather than exponential, puts further constraints on the matrix $B_{2k}(z)$. It requires that the eigenvalues $\lambda(z)$ of B must vanish up to $\mathcal{O}(z^{-2})$, i.e $\lambda(z) = \sum_{i\geq 1} \lambda_i z^{-i-1}$. Indeed then $\chi \sim \exp \int^z \lambda(z')dz' \sim \exp\{-\frac{\lambda_1}{z}\} \sim 1 - \lambda_1 z^{-1} + \ldots$, as desired. But then $\det B_{2k}(z)$ is of $\mathcal{O}(z^{-4})$ and

$$f_{2k}(z^2) = \frac{1}{2}(P_{00}(z) + P_{11}(z)) \pm \sqrt{\frac{1}{4}(P_{00}(z) + P_{11}(z))^2 - \Delta + \mathcal{O}(z^{-4})} \quad (7.4.85)$$

where $\Delta(z) = P_{00}(z)P_{11}(z) - P_{01}(z)P_{10}(z)$. Since $f(z^2)$ is an even function of z, the odd parity of $P_{00}(z)$ and $P_{11}(z)$ determine that $P_{00}(z) + P_{11}(z) = 0$.

Conversely, given a 2×2 matrix $\left(P_{ij}(z)\right)$ with $P_{01}(z), P_{10}(z)$ even polynomials of degree $2k$ and $P_{00}(z), P_{11}(z)$ odd polynomials of degree $< 2k$ such that $P_{00}(z)+P_{11}(z) = 0$, we will show that we obtain exactly two solutions to the string equation (7.3.63). The eigenvalues $\lambda^{(1,2)}(z)$ of $\left(P_{ij}(z)\right)$ are given by

$$\lambda^{(1,2)}(z) = \pm\sqrt{-\Delta(z)} \quad (7.4.86)$$

and $\lambda^{(i)}(z) = \sum_{j=-\infty}^{k} \lambda_j^{(i)} z^{2j} \quad (i = 0, 1)$. Then the matrix B_{2k} of (7.4.84) with

$$f_k^{(i)}(z^2) = \sum_{m=-\infty}^{k} \alpha_m^{(i)} z^{2m} \quad \alpha_m^{(i)} - \lambda_m^{(i)} = \begin{cases} 0 & m \geq 0 \\ \neq 0 & \text{at least for } 0 \gg m \end{cases} \quad (7.4.87)$$

has determinant at most of $\mathcal{O}(z^{-4})$. Then the system (7.4.82) has solutions $\phi(z)$ and $\psi(z)$ of the form $\phi(z)$, $\psi(z)$ = const. + lower order terms. We can set the constant to one by requiring that the leading terms of the polynomials $P_{01}(z)$ and $P_{10}(z)$ are equal. Since we know from the discussion at the end of section 3 that the $m < 0$ terms of the operator A can be gauged away, we see that each eigenvalue $\lambda^{(i)}(z)$ specifies a unique solution to the string equation (7.3.63).

Hence the space of solutions to the string equation (7.3.52) is the two fold covering of the space of matrices $\left(P_{ij}(z)\right)$ with polynomial entries in z such that $P_{01}(z)$ and $P_{10}(z)$ are even polynomials having equal degree and leading terms and $P_{00}(z)$ and $P_{11}(z)$ are odd polynomials satisfying the conditions $P_{00}(z)+P_{11}(z) = 0$ and $\deg P_{00}(z) < \deg P_{01}(z)$.

The space of solutions to the string equation was also studied by Moore [74] where the moduli space was described in terms of the Stokes parameters of the corresponding isomonodromic deformation problem [50]. This description arises when the string equation is written as flatness conditions on meromorphic gauge fields and then it may be viewed as consistency conditions for isomonodromic deformation of an equation with an irregular singularity. The Stokes parameters of the isomonodromy problem determine the flat connection up to meromorphic gauge transformations. Therefore they are the moduli of these fields under the orbits of the gauge transformations and the space of solutions to the string equation can be described in terms of them. Guha and Mañas [75] discuss the connection between the algebraic description given here and that of [74].

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BIOGRAPHICAL SKETCH

Born on the 21st of July 1964, the author of this thesis was brought up in Athens, Greece. He graduated from high school in 1982 and studied at the physics department of Athens University, Athens, Greece from 1982 until 1987. In August 1987, he joined the Physics Department of Syracuse University as a prospective graduate student of the relativity group. He was impressed by the progress made in string theory, however, and he started doing related work with his advisor Mark Bowick in the spring of 1988. Since then, he has worked on several areas of theoretical physics, such as matrix models and integrable hierarchies, $2+1$ Quantum Gravity, hot fundamental strings, computational physics as well as experimental high energy physics with the Syracuse CLEO group. After finishing his PhD in August 1993, he plans to move to Gainesville, Florida to work with the high energy group of the Institute for Fundamental Theory at the University of Florida.