Hartle
ch 7

5. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x\,dv^2 + 2\,dv\,dx.$$

(a) Calculate the light cone at a point (v, x).

(b) Draw a (v, x) spacetime diagram showing how the light cones change with x.

(c) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.

(*Comment*: The light cone structure of this model spacetime is in many ways analogous to that of black-hole spacetimes to be considered in Chapter 12, in particular in having a surface such as $x = 0$, out from which you cannot get.)

18. Consider the three-dimensional space with the line element

$$dS^2 = \frac{dr^2}{(1-2M/r)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.

(b) Calculate the spatial volume between the two spheres in part (a).

19. The surface of a sphere of radius R in four flat Euclidean dimensions is given by

$$X^2 + Y^2 + Z^2 + W^2 = R^2.$$

(a) Show that points on the sphere may be located by coordinates (χ, θ, ϕ), where

$$X = R\sin\chi\,\sin\theta\,\cos\phi, \qquad Z = R\sin\chi\,\cos\theta,$$
$$Y = R\sin\chi\,\sin\theta\,\sin\phi, \qquad W = R\cos\chi.$$

(b) Find the metric describing the geometry on the surface of the sphere in these coordinates.

20. *Make the cover* Consider the two-dimensional geometry with the line element

$$d\Sigma^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2.$$

Find a two-dimensional surface in three-dimensional flat space that has the *same* intrinsic geometry as this slice. Sketch a picture of your surface. (*Comment*: This is a slice of the Schwarzschild black-hole geometry to be discussed in Chapter 12. It is also the surface on the cover of this book.)

Carroll 3.4

$x = uv\cos\phi \qquad y = uv\sin\phi \qquad z = \frac{1}{2}(u^2 - v^2)$

$ds^2 = dx^2 + dy^2 + dz^2$

- Compute $g_{\mu\nu}$ in the (u, v, ϕ) coordinate system
- if $V^\mu = v\partial_u - u\partial_v$ compute the components of V_μ and $V_\mu V^\mu$
- if $U^\mu = \sin\phi\,\partial_u - \cos\phi\,\partial_v$ compute $V_\mu U^\mu$