Lecture 4: Problem solutions (from Hartle's book, ch 7)

5. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x\,dv^2 + 2\,dv\,dx.$$

(a) Calculate the light cone at a point (v, x).

(b) Draw a (v, x) spacetime diagram showing how the light cones change with x.

(c) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.

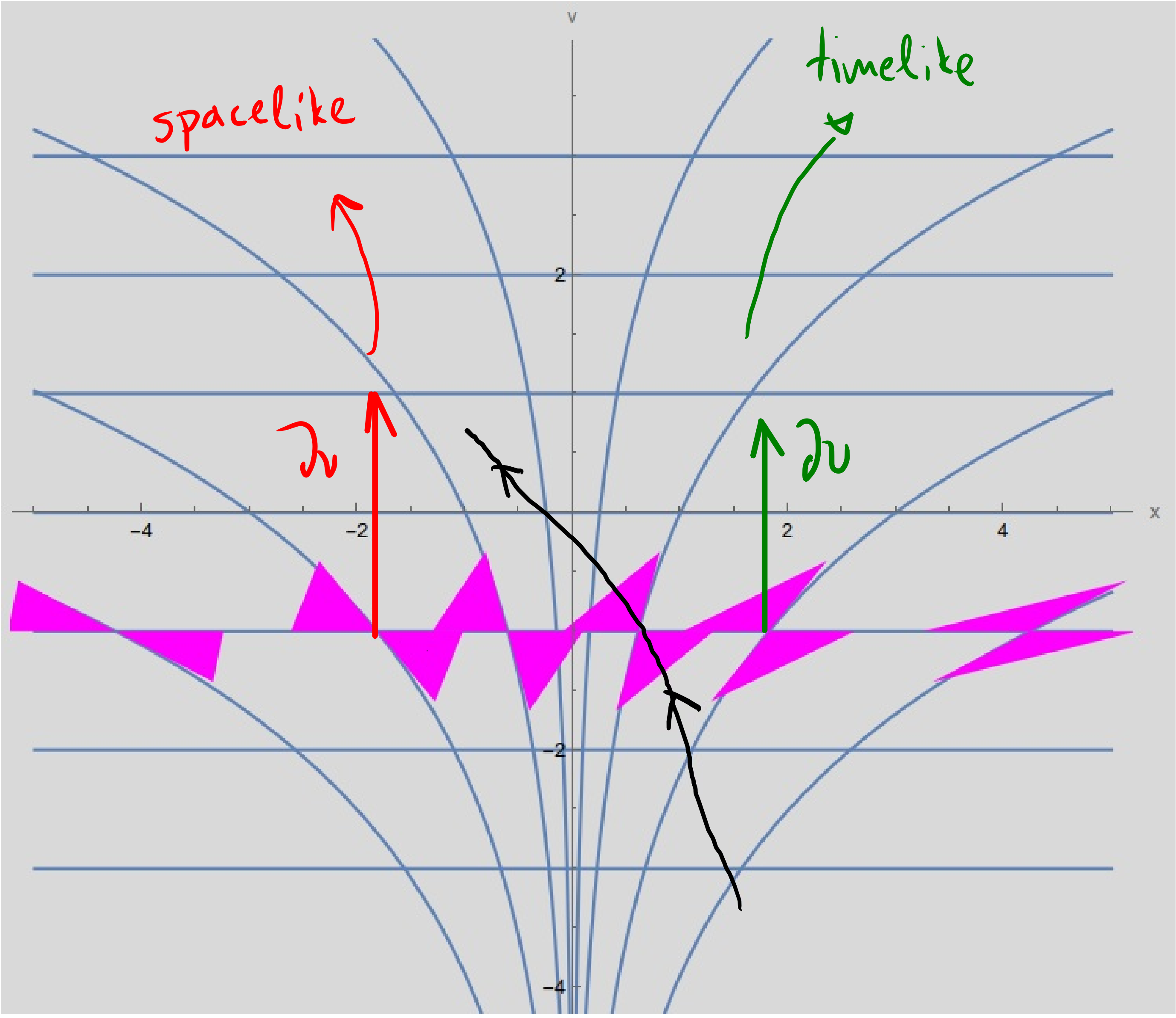
(*Comment*: The light cone structure of this model spacetime is in many ways analogous to that of black-hole spacetimes to be considered in Chapter 12, in particular in having a surface such as $x = 0$, out from which you cannot get.)

$$ds^2 = 0 \Rightarrow (-x\,dv + 2\,dx)\,dv = 0 \Rightarrow \left\{\begin{array}{l} dv = 0 \\ -x\,dv + 2\,dx = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} v = v_0 \\ \text{or} \\ \frac{1}{2}dv = \frac{dx}{x} \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{c} v = v_0 \\ \text{or} \\ \frac{1}{2}\int dv = \int \frac{dx}{x} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} v = v_0 \\ \frac{1}{2}v = \ln|x| + \ln|x_0| \end{array}\right\} \Rightarrow \begin{array}{c} v = v_0 \\ v = \ln\left(\frac{x}{x_0}\right)^2 \end{array}$$

with slopes $\frac{dv}{dx} = 0$

$$\frac{dv}{dx} = \frac{2}{x}$$



Light cones are in the directions for which $ds^2 < 0 \Rightarrow$ $dv > 0$ and $-x\,dv + 2dx < 0$ (1) or $dv < 0$ $-x\,dv + 2dx > 0$ (2)

For $x<0$, (1) $\Rightarrow 2\,dx < x\,dv < 0$, so light cones are tilted to the left, and we can't cross from $x<0$ to $x>0$.

We notice that ∂_v has norm $\partial_v \cdot \partial_v = g_{vv} = -x \begin{cases} <0 \text{ for } x>0 & \text{timelike} \\ >0 \text{ for } x<0 & \text{spacelike} \end{cases}$

So for $x>0$, ∂_v is <u>in</u> the light cone

$x<0$ " " outside "

18. Consider the three-dimensional space with the line element

$$dS^2 = \frac{dr^2}{(1-2M/r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.

(b) Calculate the spatial volume between the two spheres in part (a).

(a) As we move along the radial distance $d\theta = d\phi = 0$

$$S = \int_{2M}^{3M} \frac{dr}{\left(1-\frac{2M}{r}\right)^{1/2}} = r\left(1-\frac{2M}{r}\right)^{1/2} + 2M \tan^{-1}\left[\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\right] \Bigg|_{2M}^{3M}$$

$$= M\left[\sqrt{3} + 2\tan^{-1}\sqrt{3}\right] \approx 3.049\, M$$

(b) The determinant of the metric is

$$g = \frac{1}{1-\frac{2M}{r}} \cdot r^2 \cdot r^2 \sin^2\theta \Rightarrow \sqrt{g} = \left(1-\frac{2M}{r}\right)^{-1/2} r^2 \sin\theta$$

$$V = \int \sqrt{g}\, dr\, d\theta\, d\phi = \int_{2M}^{3M} dr \left(1-\frac{2M}{r}\right)^{-1/2} r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi \int_{2M}^{3M} dr\, r^2 \left(1-\frac{2M}{r}\right)^{-1/2}$$

$$V = 4\pi \left. \frac{1}{6}\left(1-\frac{2M}{r}\right)^{-1/2}\left(2r^3 + Mr^2 + 5M^2 r - 30M^3\right) + 5M^3 \tan^{-1}\left[\left(1-\frac{2M}{r}\right)^{-1/2}\right] \right|_{2M}^{3M}$$

$$= 2\pi M^3 \left[16\sqrt{3} + \ln\left(362 + 209\sqrt{3}\right)\right] \approx 215.50\, M^3$$

19. The surface of a sphere of radius R in four flat Euclidean dimensions is given by

$$X^2 + Y^2 + Z^2 + W^2 = R^2.$$

(a) Show that points on the sphere may be located by coordinates (χ, θ, ϕ), where

$$X = R \sin\chi \sin\theta \cos\phi, \qquad Z = R \sin\chi \cos\theta,$$
$$Y = R \sin\chi \sin\theta \sin\phi, \qquad W = R \cos\chi.$$

(b) Find the metric describing the geometry on the surface of the sphere in these coordinates.

(a) $X^2 + Y^2 = R \sin^2\chi \sin^2\theta$

$X^2 + Y^2 + Z^2 = R \sin^2\chi \left(\sin^2\theta + \cos^2\theta\right) = R^2 \sin^2\chi$

$X^2 + Y^2 + Z^2 + W^2 = R^2\left(\sin^2\chi + \cos^2\chi\right) = R^2$

(b) $dX = \dfrac{\partial X}{\partial \chi} d\chi + \dfrac{\partial X}{\partial \theta} d\theta + \dfrac{\partial X}{\partial \phi} d\phi = R\cos\chi \sin\theta \cos\phi \, d\chi + R \sin\chi \cos\theta \cos\phi \, d\theta - R \sin\chi \sin\theta \sin\phi \, d\phi$

$dY = R\cos\chi \sin\theta \sin\phi \, d\chi + R \sin\chi \cos\theta \sin\phi \, d\theta + R \sin\chi \sin\theta \cos\phi \, d\phi$

$dZ = R\cos\chi \cos\theta \, d\chi - R \sin\chi \sin\theta \, d\theta$

$dW = -R \sin\chi \, d\chi$

$$ds^2 = dX^2 + dY^2 + dZ^2 + dW^2$$

$$dW^2 = R^2 \sin^2\chi \, d\chi^2$$

$$dZ^2 = R^2 \left[\cos^2\chi \cos^2\theta \, d\chi^2 + \sin^2\chi \sin^2\theta \, d\theta^2 - 2 \cos\chi \sin\cancel{\chi} \cos\theta \sin\theta \, d\chi \, d\theta \right]$$

$$dY^2 = R^2 \{ \cos^2\chi \sin^2\theta \sin^2\phi \, d\chi^2 + \sin^2\chi \cos^2\theta \sin^2\phi \, d\theta^2 + \sin^2\chi \sin^2\theta \cos^2\phi \, d\phi^2$$

$$+ 2 \cos\chi \sin\chi \cancel{\sin\theta} \cos\theta \sin^2\phi \, d\chi \, d\theta$$

$$+ 2 \cos\chi \sin\chi \cancel{\sin^2\theta} \sin\phi \cos\phi \, d\chi \, d\phi$$

$$+ 2 \sin^2\chi \sin\theta \cancel{\cos\theta} \sin\phi \cos\phi \, d\theta \, d\phi \,]$$

$$dX^2 = R^2 [\cos^2\chi \sin^2\theta \cos^2\phi \, d\chi^2 + \sin^2\chi \cos^2\theta \cos^2\phi \, d\theta^2 + \sin^2\chi \sin^2\theta \sin^2\phi \, d\phi^2$$

$$+ 2 \sin\chi \cos\chi \cancel{\sin\theta} \cos\theta \cos^2\phi \, d\chi \, d\theta$$

$$- 2 \sin\chi \cos\chi \cancel{\sin^2\theta} \sin\phi \cos\phi \, d\chi \, d\phi$$

$$- 2 \sin^2\chi \sin\theta \cancel{\cos\theta} \sin\phi \cos\phi \, d\theta \, d\phi \,]$$

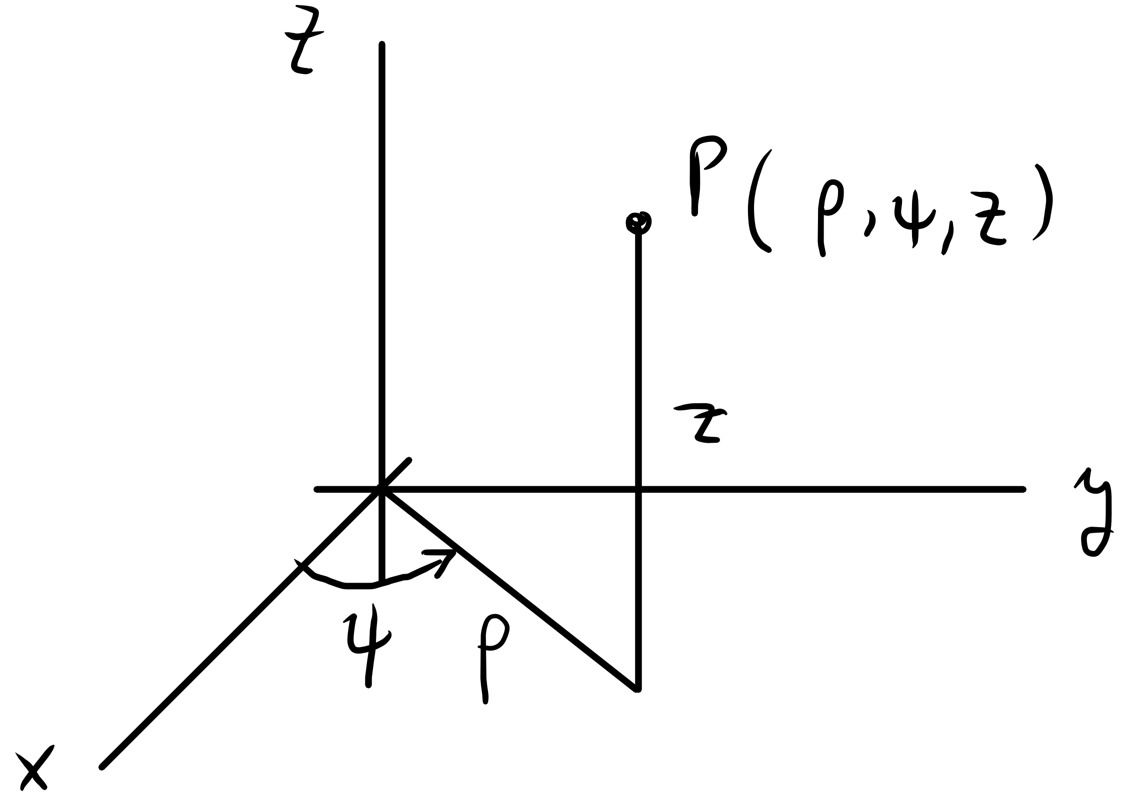
$$ds^2 = R^2\left[\, d\chi^2 + \sin^2\chi \, d\theta^2 + \sin^2\chi \sin^2\theta \, d\phi^2 \right]$$

20. *Make the cover* Consider the two-dimensional geometry with the line element

$$d\Sigma^2 = \frac{dr^2}{(1-2M/r)} + r^2 d\phi^2.$$

Find a two-dimensional surface in three-dimensional flat space that has the *same* intrinsic geometry as this slice. Sketch a picture of your surface. (*Comment*: This is a slice of the Schwarzschild black-hole geometry to be discussed in Chapter 12. It is also the surface on the cover of this book.)

Embed in 3d flat space with cylindrical coordinates (ρ, ψ, z)



$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= d\rho^2 + \rho^2 d\psi^2 + dz^2$$

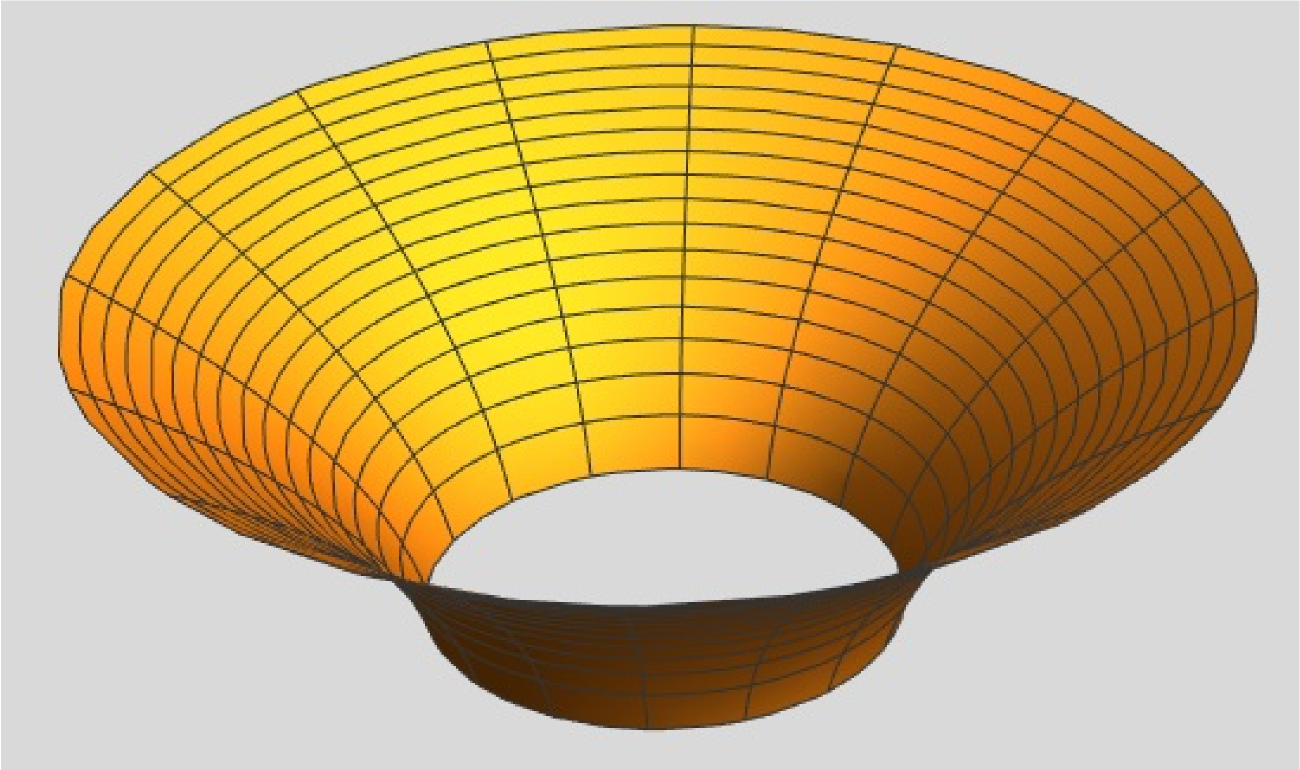
We will make an axisymmetric embedding $z = z(\rho)$, so

$$d\Sigma^2 = d\rho^2 + \rho^2 d\psi^2 + \left(\frac{\partial z}{\partial \rho}\right)^2 d\rho^2 = \left[1 + \left(\frac{\partial z}{\partial \rho}\right)^2\right] d\rho^2 + \rho^2 d\psi^2$$

So we should have $\psi = \phi$, $\rho = r$ and $1 + \left(\frac{\partial z}{\partial \rho}\right)^2 = \left(1 - \frac{2M}{\rho}\right)^{-1} \Rightarrow \left(\frac{\partial z}{\partial \rho}\right)^2 = -1 + \frac{1}{1-\frac{2M}{\rho}} \Rightarrow$

$$\left(\frac{\partial z}{\partial \rho}\right)^2 = \frac{2M/\rho}{1-\frac{2M}{\rho}} = \frac{2M}{(\rho-2M)} \Rightarrow \frac{\partial z}{\partial \rho} = \sqrt{2M}\ (\rho-2M)^{-1/2}$$

$$\Rightarrow z = \int d\rho \sqrt{2M}\ (\rho-2M)^{-1/2} = 2\sqrt{2M}\ (\rho-2M)^{1/2}$$



Carroll 3.4 $\quad x = uv\cos\phi \quad y = uv\sin\phi \quad z = \frac{1}{2}(u^2 - v^2)$

$$ds^2 = dx^2 + dy^2 + dz^2$$

- Compute $g_{\mu\nu}$ in the (u, v, ϕ) coordinate system
- if $V^\mu = v\partial_u - u\partial_v$ compute the components of V_μ and $V_\mu V^\mu$
- if $U^\mu = \sin\phi\,\partial_u - \cos\phi\,\partial_v$ compute $V_\mu U^\mu$

$$dx = v\cos\phi\, du + u\cos\phi\, dv - uv\sin\phi\, d\phi$$

$$dy = v\sin\phi\, du + u\sin\phi\, dv + uv\cos\phi\, d\phi$$

$$dz = u\,du - v\,dv$$

$$dx^2 = v^2\cos^2\phi\, du^2 + u^2\cos^2\phi\, dv^2 + u^2v^2\sin^2\phi\, d\phi^2$$
$$+ 2vu\cos^2\phi\, du\,dv \cancel{- 2uv^2\cos\phi\sin\phi\, du\,d\phi} \cancel{- 2u^2v\cos\phi\sin\phi\, dv\,d\phi}$$

$$dy^2 = v^2\sin^2\phi\, du^2 + u^2\sin^2\phi\, dv^2 + u^2v^2\cos^2\phi\, d\phi^2$$
$$+ 2vu\sin^2\phi\, du\,dv \cancel{+ 2uv^2\sin\phi\cos\phi\, du\,d\phi} \cancel{+ 2u^2v\sin\phi\cos\phi\, dv\,d\phi}$$

$$dz^2 = u^2 du^2 + v^2 dv^2 - \cancel{2uv}\, du\, dv$$

$$dx^2 + dy^2 = v^2 du^2 + u^2 dv^2 + u^2 v^2 d\phi^2 + \cancel{2uv}\, du\, dv$$

$$dx^2 + dy^2 + dz^2 = (u^2+v^2)(du^2 + dv^2) + u^2 v^2 d\phi^2$$

$$(g_{\mu\nu}) = \begin{pmatrix} u^2+v^2 & & \\ & u^2+v^2 & \\ & & u^2v^2 \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} \frac{1}{u^2+v^2} & & \\ & \frac{1}{u^2+v^2} & \\ & & \frac{1}{u^2v^2} \end{pmatrix}$$

$$V^\mu = [v, -u, 0]$$

$$V_u = g_{uu} V^u = (u^2+v^2)\, v$$

$$V_v = g_{vv} V^v = -(u^2+v^2)\, u$$

$$V^\mu V_\mu = g_{\mu\nu} V^\mu V^\nu = (u^2+v^2)\, v^2 + (u^2+v^2)\, u^2 = (u^2+v^2)^2$$

$$V_\phi = g_{\phi\phi} V^\phi = 0$$

$$U^\mu = [\sin\phi, -\cos\phi, 0]$$

$$V_\mu U^\mu = g_{\mu\nu} V^\mu U^\nu = g_{uu} V^u U^u + g_{vv} V^v U^v$$

$$= (u^2+v^2)\, v \sin\phi + (u^2+v^2)(-u)(-\cos\phi)$$

$$= (u^2+v^2)\left[v\sin\phi + u\cos\phi\right]$$