Affine Connections - Covariant Derivatives

Parallel Transport - Geodesics

Derivative Operator

$$\nabla : \mathcal{T}^{(k,l)}M \to \mathcal{T}^{(k,l+1)}M \qquad \text{s.t.}$$

1. Linear : $\forall\ T, S \in \mathcal{T}^{(k,l)}M \quad , \alpha, \beta \in \mathbb{R}$

$$\nabla_\mu \left[\alpha T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l} + \beta S^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l} \right] = \alpha \nabla_\mu T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l} + \beta \nabla_\mu S^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}$$

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2. Leibniz : $\forall\ T \in \mathcal{T}^{(k,l)}\mathcal{M}\ ,\ S \in \mathcal{T}^{(k',l')}\mathcal{M}$

$$\nabla_\mu\left[T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}\ S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{l'}'}\right] = \left[\nabla_\mu T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}\right] S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{l'}'} + T^{\nu_1 \dots \nu_k}{}_{\mu_1 \dots \mu_l}\left[\nabla_\mu S^{\nu_1' \dots \nu_{k'}'}{}_{\mu_1' \dots \mu_{k'}'}\right]$$

Derivative Operator

3. Commutativity with contractions

$$\nabla_\mu \left[T^{\nu_1 \cdots \gamma \cdots \nu_k}{}_{\mu_1 \cdots \gamma \cdots \nu_l} \right] = \left(\nabla_\mu T \right)^{\nu_1 \cdots \gamma \cdots \nu_k}{}_{\mu_1 \cdots \gamma \cdots \nu_l}$$

↳ take contraction first then differentiate

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---

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4. On functions: same as df, or, since $df(v) = V(f)$

$$V^\mu \nabla_\mu f = V(f) \qquad \left(\text{in a coord. basis } = V^\mu \partial_\mu f \right)$$

---

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$$V^\mu \nabla_\mu f = V(f)$$

5. Torsion free (in GR): $\forall f \in \mathcal{F}(M)$

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$$

If not: $\nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f = - T^\lambda{}_{\mu\nu} \nabla_\lambda f$

$T^\lambda{}_{\mu\nu} \rightarrow$ Torsion tensor

∂_μ is a derivative operator (see Wald for this "point of view")

- pick a coordinate system (U, x), with $\{\partial_\mu\}$ and $\{dx^\mu\}$
- In U, take a tensor field $T^{\nu\ldots}{}_{\lambda\ldots}$ and define the tensor field $\partial_\mu T^{\nu\ldots}{}_{\lambda\ldots}$ whose components in U are $\frac{\partial}{\partial x^\mu} T^{\nu\ldots}{}_{\lambda\ldots}$
- ∂_μ satisfies all axioms

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But:
- $\partial_\mu T^{\nu\cdots}{}_{\lambda\cdots}$ defined only on U
- If (U', x'), with $\{\partial_{\mu'}\}$ $\{dx^{\mu'}\}$ a different coordinate system, then $\partial_{\mu'} T^{\nu}{}_{\lambda\cdots}$ a different tensor field on $U \cap U'$

Uniqueness of ∇

- There are many different derivative operators on M

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this depends only on value of X^ν at a point!

this could depend on values of X^ν in a neighborhood

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Uniqueness of ∇

- $(\nabla_\mu - \tilde{\nabla}_\mu) f = df - df = 0$ (1)
- $(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu = C^\nu{}_{\mu\rho} X^\rho$ (2)
- (1) + (2) $\Rightarrow$

$$(\nabla_\mu - \tilde{\nabla}_\mu) \omega_\nu = - C^\rho{}_{\mu\nu} \omega_\rho$$

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Indeed:

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$\hookrightarrow$ a function + (1)

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(Leibniz rule)

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$$\overset{(2)}{\Longrightarrow} \quad \left[(\nabla_\mu - \tilde{\nabla}_\mu)\omega_\nu\right] X^\nu + \omega_\nu\, C^\nu{}_{\mu\rho} X^\rho = 0$$

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Higher rank tensors: e.g.

$$(\nabla_\mu - \tilde{\nabla}_\mu)(F^\rho{}_\nu\,\omega_\rho X^\nu) = 0$$

↳ a function

$$\Rightarrow \left[(\nabla_\mu - \tilde{\nabla}_\mu)\omega_\nu + C^\rho{}_{\mu\nu}\,\omega_\rho \right] X^\nu = 0$$

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$$\Rightarrow \left[\nabla_\mu - \tilde{\nabla}_\mu\right] F^\rho{}_\nu\, \omega_\rho\, X^\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu\, \omega_\rho\, X^\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma\, \omega_\rho\, X^\nu = 0$$

$$(\nabla_\mu - \tilde{\nabla}_\mu)(F^\rho{}_\nu\, \omega_\rho\, X^\nu) = 0 \quad \Rightarrow$$

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$$\Rightarrow [\nabla_\mu - \tilde{\nabla}_\mu] F^\rho{}_\nu \, \omega_\rho \chi^\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu \, \omega_\rho \chi^\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma \, \omega_\rho \chi^\nu = 0 \quad \Rightarrow$$

$$\left\{ (\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma \right\} \omega_\rho \chi^\nu = 0 \quad \Rightarrow$$

$$(\nabla_\mu - \tilde{\nabla}_\mu)(F^\rho{}_\nu \, \omega_\rho X^\nu) = 0 \quad \Rightarrow$$

$$[(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu] \, \omega_\rho \chi^\nu + F^\rho{}_\nu [(\nabla_\mu - \tilde{\nabla}_\mu) \omega_\rho] X^\nu + F^\rho{}_\nu \, \omega_\rho [(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu] = 0$$

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$$(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu - C^\rho{}_{\mu\sigma} F^\sigma{}_\nu + C^\sigma{}_{\mu\nu} F^\rho{}_\sigma = 0 \Rightarrow$$

$$(\nabla_\mu - \tilde{\nabla}_\mu) F^\rho{}_\nu = C^\rho{}_{\mu\sigma} F^\sigma{}_\nu - C^\sigma{}_{\mu\nu} F^\rho{}_\sigma$$

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$$(\nabla_\mu - \tilde{\nabla}_\mu) X^\nu = C^\nu{}_{\mu\rho} X^\rho$$

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$$\nabla_\mu X^\nu = \tilde{\nabla}_\mu X^\nu + C^\nu{}_{\mu\rho} X^\rho$$

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$$\nabla_\mu T^{\nu_1 \cdots \nu_k}{}_{\rho_1 \cdots \rho_l} = \tilde{\nabla}_\mu T^{\nu_1 \cdots \nu_k}{}_{\rho_1 \cdots \rho_l} + C^{\nu_1}{}_{\mu\sigma} T^{\sigma \cdots \nu_k}{}_{\rho_1 \cdots \rho_l} + \cdots + C^{\nu_k}{}_{\mu\sigma} T^{\nu_1 \cdots \sigma}{}_{\rho_1 \cdots \rho_l}$$
$$- C^\sigma{}_{\mu\rho_1} T^{\nu_1 \cdots \nu_k}{}_{\sigma \cdots \rho_l} - \cdots - C^\sigma{}_{\mu\rho_l} T^{\nu_1 \cdots \nu_k}{}_{\rho_1 \cdots \sigma}$$

$$\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\rho} X^\rho$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho{}_{\mu\nu}\,\omega_\rho$$

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$$\nabla_\mu T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_\ell} = \partial_\mu T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\rho_\ell} + \Gamma^{\nu_1}{}_{\mu\sigma} T^{\sigma\cdots\nu_k}{}_{\rho_1\cdots\rho_\ell} + \cdots + \Gamma^{\nu_k}{}_{\mu\sigma} T^{\nu_1\cdots\sigma}{}_{\rho_1\cdots\rho_\ell}$$
$$- \Gamma^\sigma{}_{\mu\rho_1} T^{\nu_1\cdots\nu_k}{}_{\sigma\cdots\rho_\ell} - \cdots - \Gamma^\sigma{}_{\mu\rho_\ell} T^{\nu_1\cdots\nu_k}{}_{\rho_1\cdots\sigma}$$

If $\tilde{\nabla}_\mu = \partial_\mu$ then $C^\rho{}_{\mu\nu} \to \Gamma^\rho{}_{\mu\nu}$

$\Gamma^\mu{}_{\nu\rho}$ a $(1,2)$ tensor field giving $\nabla_\mu - \partial_\mu$ in given chart

If (U, x), (U', x'), $U \cap U' \neq \emptyset$ two coordinate systems, then

(U, x) has ∂_μ and $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\rho} X^\rho$

(U', x') has $\partial_{\mu'}$ and $\nabla_{\mu'} X^{\nu'} = \partial_{\mu'} X^{\nu'} + \Gamma^{\nu'}{}_{\mu'\rho'} X^{\rho'}$

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(U', x') has $\partial_{\mu'}$ and $\nabla_{\mu'} X^{\nu'} = \partial_{\mu'} X^{\nu'} + \Gamma^{\nu'}{}_{\mu'\rho'} X^{\rho'}$, then:

$$\underbrace{\nabla_{\mu'} X^{\nu'}}_{\text{the same tensor}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \underbrace{\nabla_\mu X^\nu}_{\text{the same tensor}}$$

component xfm law

the same tensor

If $\tilde{\nabla}_\mu = \partial_\mu$ then $C^\rho{}_{\mu\nu} \to \Gamma^\rho{}_{\mu\nu}$

$\Gamma^\mu{}_{\nu\rho}$ a $(1,2)$ tensor field giving $\nabla_\mu - \partial_\mu$ in given chart

If (U, x), (U', x'), $U \cap U' \neq \emptyset$ two coordinate systems, then

(U, x) has ∂_μ and $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\rho} X^\rho$

(U', x') has $\partial_{\mu'}$ and $\nabla_{\mu'} X^{\nu'} = \partial_{\mu'} X^{\nu'} + \Gamma^{\nu'}{}_{\mu'\rho'} X^{\rho'}$, then:

$$\nabla_{\mu'} X^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu X^\nu$$ component xfm law

$$\partial_{\mu'} X^{\nu'} \neq \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu X^\nu$$

different tensors, their components not related with tensor xfm law

If (U, x), (U', x'), $U \cap U' \neq \emptyset$ two coordinate systems, then

(U, x) has ∂_μ and $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu{}_{\mu\rho} X^\rho$

(U', x') has $\partial_{\mu'}$ and $\nabla_{\mu'} X^{\nu'} = \partial_{\mu'} X^{\nu'} + \Gamma^{\nu'}{}_{\mu'\rho'} X^{\rho'}$, then:

$$\nabla_{\mu'} X^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu X^\nu$$ component xfm law

$$\partial_{\mu'} X^{\nu'} \neq \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu X^\nu$$

$$\Gamma^{\mu'}{}_{\nu'\rho'} \neq \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \Gamma^\mu{}_{\nu\rho}$$

different tensors, correspond to $\nabla_{\mu'} - \partial_{\mu'}$ and $\nabla_\mu - \partial_\mu$

We can calculate the relation between $\Gamma^{\mu'}{}_{\nu'\rho'}$ and $\Gamma^{\mu}{}_{\nu\rho}$ from

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \nabla_{\mu} V^{\nu}$$

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$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} \right) V^{\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda}$$

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RHS: $\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\mu\lambda} V^{\lambda} \right)$

We can calculate the relation between $\Gamma^{\mu'}{}_{\nu'\rho'}$ and $\Gamma^{\mu}{}_{\nu\rho}$ from

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$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} \right) V^{\nu} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda}$$

RHS: $$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\mu\lambda} V^{\lambda} \right) =$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\lambda} V^{\lambda}$$

$$\text{LHS} = \text{RHS} \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} \right) V^{\lambda} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\lambda} V^{\lambda}$$

$$\text{LHS: } \nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}} \left[\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu} \right] + \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda'}$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \cancel{\frac{\partial x^{\nu'}}{\partial x^{\nu}}} \frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\cancel{\nu}\lambda}} \right) V^{\cancel{\nu}\lambda} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} V^{\lambda}$$

$$\text{RHS: } \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}{}_{\mu\lambda} V^{\lambda} \right) =$$

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$$LHS = RHS \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right)\cancel{V^{\lambda}} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'}\cancel{V^{\lambda}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}{}_{\mu\lambda}\cancel{V^{\lambda}}$$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right) + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}{}_{\mu\lambda}$$

$\hookrightarrow$ solve for this

$$\text{LHS} = \text{RHS} \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} \right) \cancel{V^{\lambda}} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} \cancel{V^{\lambda}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\lambda} \cancel{V^{\lambda}}$$

$$\frac{\partial x^{\mu'}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} \right) + \frac{\partial x^{\mu'}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} \Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}{}_{\mu\lambda} \frac{\partial x^{\mu'}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}}$$

$\delta_{\rho}{}^{\mu}$

$\delta^{\mu}{}_{\rho}$ $\quad \delta_{\nu}{}^{\sigma}$

$$LHS = RHS \Rightarrow \frac{\partial x^{\mu}}{\partial x^{\mu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right)\cancel{V^{\lambda}} + \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'}\cancel{V^{\lambda}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}{}_{\mu\lambda}\cancel{V^{\lambda}}$$

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$\delta_{\rho}{}^{\mu}$ $\qquad\qquad$ $\delta^{\mu}{}_{\rho}$ $\quad$ $\delta_{\nu}{}^{\sigma}$

$$\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\rho}\partial x^{\lambda}}\right) + \frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} = \Gamma^{\sigma}{}_{\rho\lambda}$$

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$\delta_{\rho}{}^{\mu}$ $\qquad$ $\delta^{\mu}{}_{\rho}$ $\qquad$ $\delta_{\nu}{}^{\sigma}$

$$\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\rho}\partial x^{\lambda}}\right) + \frac{\partial x^{\mu'}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} = \Gamma^{\sigma}{}_{\rho\lambda}$$

rename: $\sigma \to \nu$

$\rho \to \mu$

$$\Gamma^{\nu}{}_{\mu\lambda} = \frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\Gamma^{\nu'}{}_{\mu'\lambda'} + \frac{\partial x^{\nu}}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}}\right)$$

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$\delta_\rho{}^\mu$ $\delta^\mu{}_\rho$ $\delta_\nu{}^\sigma$

$$\frac{\partial x^\sigma}{\partial x^{\nu'}}\left(\frac{\partial^2 x^{\nu'}}{\partial x^\rho \partial x^\lambda}\right) + \frac{\partial x^{\mu'}}{\partial x^\rho}\frac{\partial x^\sigma}{\partial x^{\nu'}}\frac{\partial x^{\lambda'}}{\partial x^\lambda}\Gamma^{\nu'}{}_{\mu'\lambda'} = \Gamma^\sigma{}_{\rho\lambda}$$

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$$\Gamma^{\nu'}{}_{\mu'\lambda'} = \frac{\partial x^{\nu'}}{\partial x^\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\Gamma^\nu{}_{\mu\lambda} + \frac{\partial x^{\nu'}}{\partial x^\nu}\left(\frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\lambda'}}\right)$$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0$$

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$$\nabla_\mu \nabla_\nu f = \nabla_\nu \tilde{\nabla}_\mu f$$

$$\hookrightarrow \nabla_\mu f = \tilde{\nabla}_\mu f = \partial_\mu f$$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0 \quad , \quad \text{then}$$

$$\nabla_\mu \nabla_\nu f = \nabla_\nu (\tilde{\nabla}_\mu f) = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu f) - C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f$$

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If $\nabla, \tilde{\nabla}$ are torsion free, then

$$\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$$

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f$$

Torsion free ∇

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$$\nabla_\mu \nabla_\nu f = \nabla_\nu (\tilde{\nabla}_\mu f) = \tilde{\nabla}_\mu (\tilde{\nabla}_\nu f) - C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f \qquad (1)$$

$$\nabla_\nu \nabla_\mu f = \qquad \tilde{\nabla}_\nu (\tilde{\nabla}_\mu f) - C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f \qquad (2)$$

If $\nabla, \tilde{\nabla}$ are torsion free, then

$$\left.\begin{aligned} \nabla_\mu \nabla_\nu f &= \nabla_\nu \nabla_\mu f \\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f &= \tilde{\nabla}_\nu \tilde{\nabla}_\mu f \end{aligned}\right\} \underset{(2)}{\overset{(1)}{\Longrightarrow}} \quad C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f = C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f$$

Torsion free ∇

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = 0 \quad , \quad \text{then}$$

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$$\left.\begin{array}{l} \nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f \\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f = \tilde{\nabla}_\nu \tilde{\nabla}_\mu f \end{array}\right\} \underset{(2)}{\overset{(1)}{\Longrightarrow}} C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f = C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f \quad , \quad \forall f \Rightarrow$$

$$C^\rho{}_{\mu\nu} = C^\rho{}_{\nu\mu} \Leftrightarrow \begin{cases} C^\rho{}_{[\mu\nu]} = 0 \\ C^\rho{}_{(\mu\nu)} = C^\rho{}_{\mu\nu} \end{cases}$$

Torsion free ∇

∂_μ is torsion free, since $\partial_\mu \partial_\nu f = \partial_\nu \partial_\mu f$, so if ∇_μ is torsion free

$$\Rightarrow \Gamma^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu}$$

$$\left.\begin{aligned} \nabla_\mu \nabla_\nu f &= \nabla_\nu \nabla_\mu f \quad (1) \\ \tilde{\nabla}_\mu \tilde{\nabla}_\nu f &= \tilde{\nabla}_\nu \tilde{\nabla}_\mu f \quad (2) \end{aligned}\right\} \Longrightarrow C^\rho{}_{\mu\nu} \tilde{\nabla}_\rho f = C^\rho{}_{\nu\mu} \tilde{\nabla}_\rho f, \quad \forall f \Rightarrow$$

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Exercise: If $\Gamma^\rho{}_{[\mu\nu]} \neq 0$, then $T^\rho{}_{\mu\nu} = 2\,\Gamma^\rho{}_{[\mu\nu]}$ is a tensor

s.t. $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = - T^\rho{}_{\mu\nu}\, \partial_\rho f$ $\qquad$ $T^\rho{}_{\mu\nu}$: torsion tensor

Metric Compatibility of ∇_μ

∇_μ is metric compatible if $\nabla_\mu g_{\nu\rho} = 0$

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Theorem: $\exists$ unique ∇_μ that is metric compatible and torsion-free

Metric Compatibility of ∇_μ

∇_μ is metric compatible if $\nabla_\mu g_{\nu\rho} = 0$

- Metric compatible connections preserve the inner product of paralelly-transported vectors

<u>Theorem</u>: $\exists$ unique ∇_μ that is metric compatible and torsion-free

<u>Proof</u>: Let $\tilde{\nabla}_\mu$ be any torsion-free derivative operator. Then

$$\nabla_\mu g_{\nu\rho} = 0 \Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} - C^\lambda{}_{\mu\nu} g_{\lambda\rho} - C^\lambda{}_{\mu\rho} g_{\nu\lambda} = 0 \,, \; C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$$

Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = C^\lambda{}_{\mu\nu} g_{\lambda\rho} + C^\lambda{}_{\mu\rho} g_{\nu\lambda}$$

<u>Proof</u>: Let $\tilde{\nabla}_\mu$ be any torsion-free derivative operator. Then

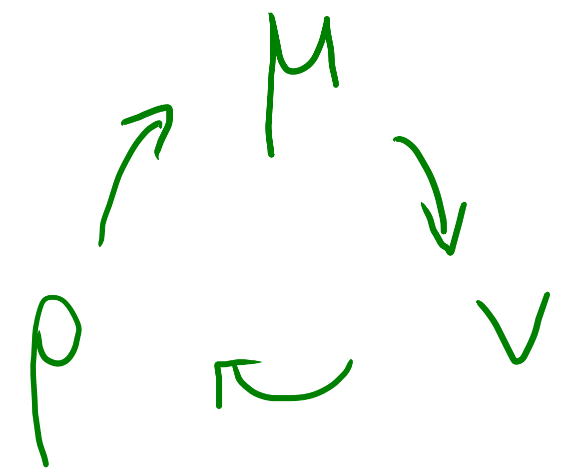
$$\nabla_\mu g_{\nu\rho} = 0 \Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} - C^\lambda{}_{\mu\nu} g_{\lambda\rho} - C^\lambda{}_{\mu\rho} g_{\nu\lambda} = 0 \ , \ C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$$

Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = C^{\lambda}{}_{\mu\nu}\, g_{\lambda\rho} + C^{\lambda}{}_{\mu\rho}\, g_{\nu\lambda}$$

$$\tilde{\nabla}_\rho g_{\mu\nu} = C^{\lambda}{}_{\rho\mu}\, g_{\lambda\nu} + C^{\lambda}{}_{\rho\nu}\, g_{\mu\lambda}$$

$$\tilde{\nabla}_\nu g_{\rho\mu} = C^{\lambda}{}_{\nu\rho}\, g_{\lambda\mu} + C^{\lambda}{}_{\nu\mu}\, g_{\rho\lambda}$$



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Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = C^\lambda{}_{\cancel{\mu\nu}} g_{\lambda\rho} + C^\lambda{}_{\cancel{\mu\rho}} g_{\nu\lambda} \qquad (-)$$

$$\tilde{\nabla}_\rho g_{\mu\nu} = C^\lambda{}_{\cancel{\rho\mu}} g_{\lambda\nu} + C^\lambda{}_{\rho\nu} g_{\mu\lambda} \qquad (+)$$

$$\tilde{\nabla}_\nu g_{\rho\mu} = C^\lambda{}_{\nu\rho} g_{\lambda\mu} + C^\lambda{}_{\cancel{\nu\mu}} g_{\rho\lambda} \qquad (+)$$

use torsion free condition $C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$

<u>Proof</u>: Let $\tilde{\nabla}_\mu$ be any torsion-free derivative operator. Then

$$\nabla_\mu g_{\nu\rho} = 0 \Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} - C^\lambda{}_{\mu\nu} g_{\lambda\rho} - C^\lambda{}_{\mu\rho} g_{\nu\lambda} = 0, \quad C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$$

Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = \cancel{C^\lambda{}_{\mu\nu}} g_{\lambda\rho} + \cancel{C^\lambda{}_{\mu\rho}} g_{\nu\lambda} \qquad (-)$$

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$$\tilde{\nabla}_\nu g_{\rho\mu} = C^\lambda{}_{\nu\rho} g_{\lambda\mu} + \cancel{C^\lambda{}_{\nu\mu}} g_{\rho\lambda} \qquad (+)$$

use torsion + free condition $C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$

$$-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu} = 2\, C^\lambda{}_{\rho\nu} g_{\mu\lambda}$$

Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = \cancel{C^\lambda{}_{\mu\nu}}\, g_{\lambda\rho} + \cancel{C^\lambda{}_{\mu\rho}}\, g_{\nu\lambda} \qquad (-)$$

$$\tilde{\nabla}_\rho g_{\mu\nu} = \cancel{C^\lambda{}_{\rho\mu}}\, g_{\lambda\nu} + C^\lambda{}_{\rho\nu}\, g_{\mu\lambda} \qquad (+)$$

$$\tilde{\nabla}_\nu g_{\rho\mu} = C^\lambda{}_{\nu\rho}\, g_{\lambda\mu} + \cancel{C^\lambda{}_{\nu\mu}}\, g_{\rho\lambda} \qquad (+)$$

use torsion + free condition

$$C^\lambda{}_{\mu\nu} = C^\lambda{}_{\nu\mu}$$

$$\left(-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu}\right) g^{\mu\sigma} = 2\, C^\lambda{}_{\rho\nu}\, \underbrace{g_{\mu\lambda}\, g^{\mu\sigma}}_{\delta_\lambda{}^\sigma}$$

Metric Compatibility of ∇_μ

$$\Rightarrow \tilde{\nabla}_\mu g_{\nu\rho} = \cancel{C^\lambda{}_{\mu\nu}\, g_{\lambda\rho}} + \cancel{C^\lambda{}_{\mu\rho}\, g_{\nu\lambda}} \qquad (-)$$

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$$C^\sigma{}_{\rho\nu} = \frac{1}{2} g^{\mu\sigma}\left(\tilde{\nabla}_\rho g_{\nu\mu} + \tilde{\nabla}_\nu g_{\rho\mu} - \tilde{\nabla}_\mu g_{\rho\nu}\right)$$

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$\hookrightarrow$ $(-)$ sign here!

Metric Compatibility of ∇_μ

In a coordinate system $\tilde{\nabla}_\mu \to \partial_\mu$, $C^\mu{}_{\nu\rho} \to \Gamma^\mu{}_{\nu\rho}$, so

$$\Gamma^\sigma{}_{\rho\nu} = \frac{1}{2} g^{\sigma\mu} \left(\partial_\rho g_{\nu\mu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\rho\nu} \right)$$

$$\left(-\tilde{\nabla}_\mu g_{\nu\rho} + \tilde{\nabla}_\rho g_{\mu\nu} + \tilde{\nabla}_\nu g_{\rho\mu} \right) g^{\mu\sigma} = 2 C^\lambda{}_{\rho\nu} \underbrace{g_{\mu\lambda} g^{\mu\sigma}}_{\delta_\lambda{}^\sigma}$$

$$C^\sigma{}_{\rho\nu} = \frac{1}{2} g^{\mu\sigma} \left(\tilde{\nabla}_\rho g_{\nu\mu} + \tilde{\nabla}_\nu g_{\rho\mu} - \tilde{\nabla}_\mu g_{\rho\nu} \right)$$

(–) sign here!

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∇_μ is the unique $\left\{ \begin{array}{c} \text{Christoffel} \\ \text{or} \\ \text{Levi-Civita} \end{array} \right\}$ connection associated w/ g

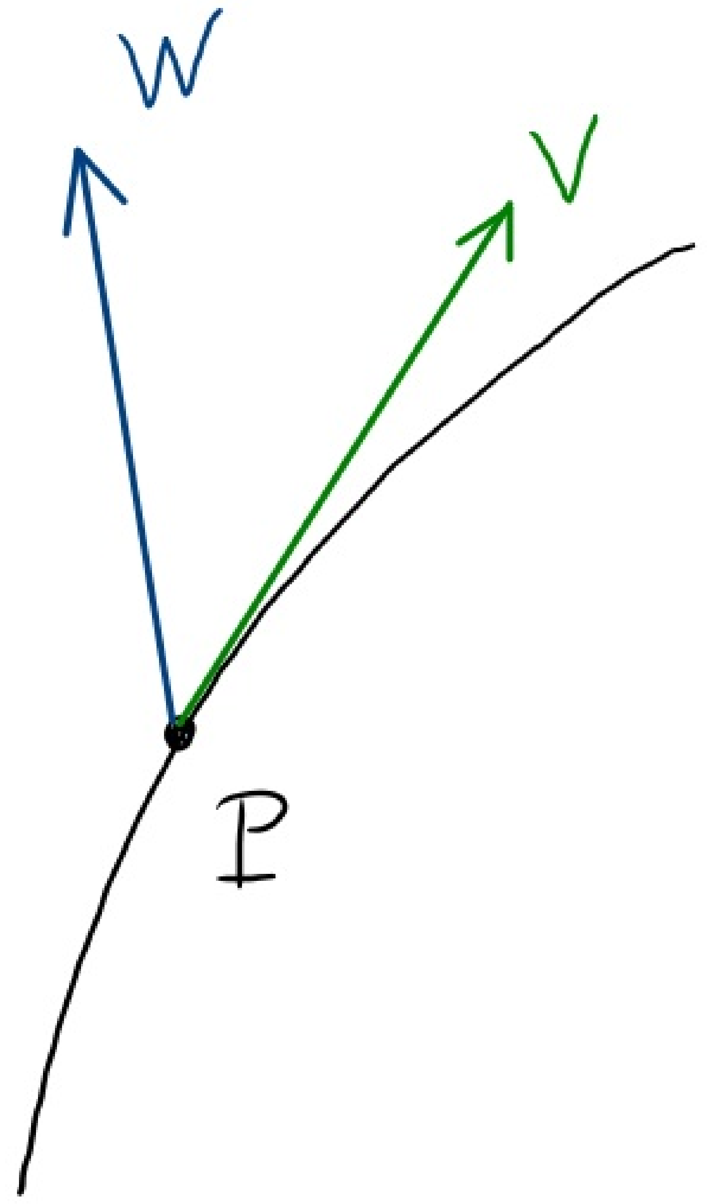
$\Gamma^\mu{}_{\nu\rho}$ are its Christoffel symbols (*)

(*) $\Gamma^\mu{}_{\nu\rho}$ is a tensor the way we view it. Traditionally $\Gamma^\mu{}_{\nu\rho}$ are the set of "symbols" transforming like $\Gamma^{\mu'}{}_{\nu'\rho'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\rho}{\partial x^{\rho'}} \Gamma^\mu{}_{\nu\rho} + \frac{\partial x^{\nu'}}{\partial x^\nu} \left(\frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^{\lambda'}} \right)$

Directional Covariant Derivative

If $\gamma(t)$ is a curve, and V^μ a vector field tangent to it, then

$$D_V W^\mu \equiv V^\nu \nabla_\nu W^\mu$$



Directional Covariant Derivative

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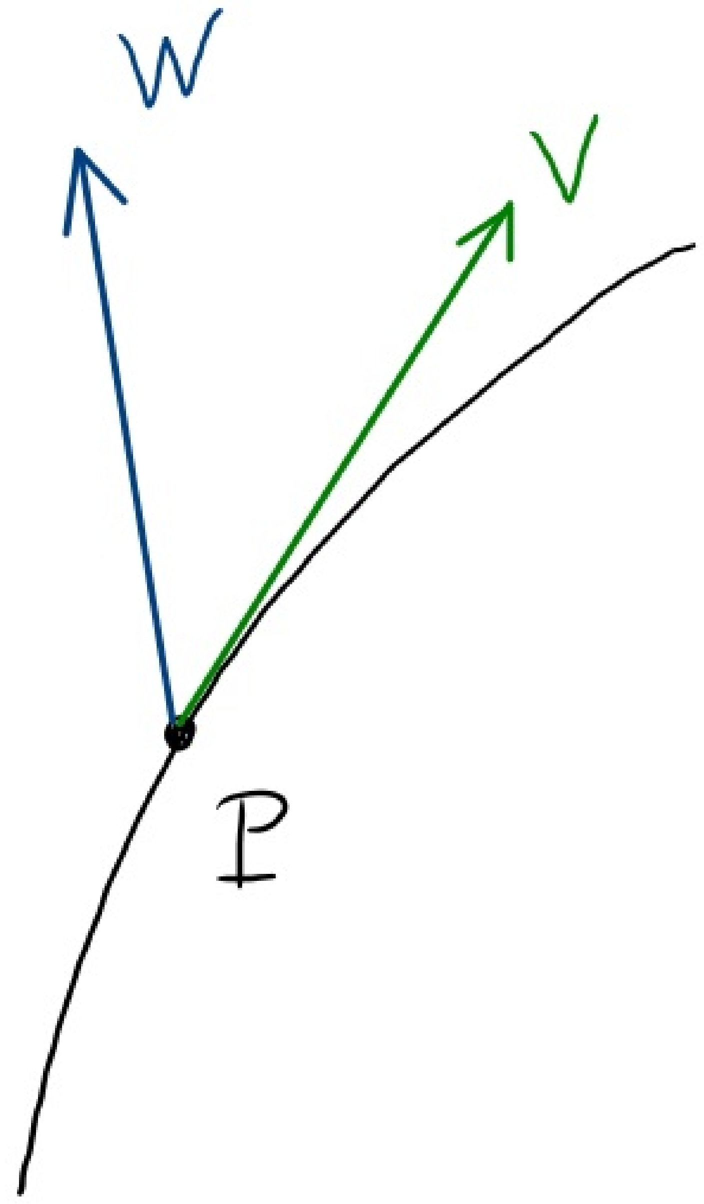
We may also write: $D_V W^\mu = \dfrac{DW^\mu}{dt}$



Properties

(1) $D_\nu(\alpha W^\mu + \beta U^\mu) = \alpha \, D_\nu W^\mu + \beta \, D_\nu U^\mu \qquad \alpha, \beta \in \mathbb{R}$

Properties

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(2) $D_V f = V(f) = \dfrac{df}{dt} = V^\mu \partial_\mu f$

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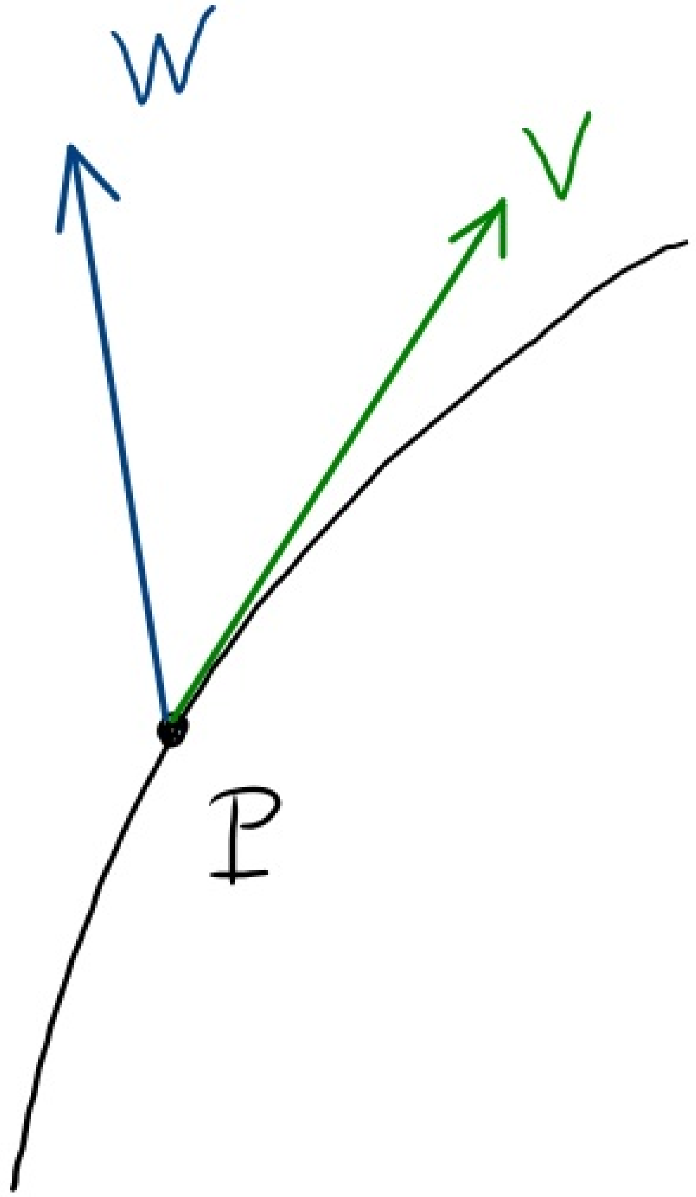
(6) $D_V (\omega_\mu W^\mu) = [D_V \omega_\mu] W^\mu + \omega_\mu [D_V W^\mu]$

(7) $(D_V D_W - D_W D_V) f = [V, W]^\mu \partial_\mu f = (V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu)\, \partial_\mu f$

$$D_V W^\mu = V^\nu \nabla_\nu W^\mu = V^\nu \partial_\nu W^\mu + V^\nu \Gamma^\mu{}_{\nu\rho} W^\rho$$



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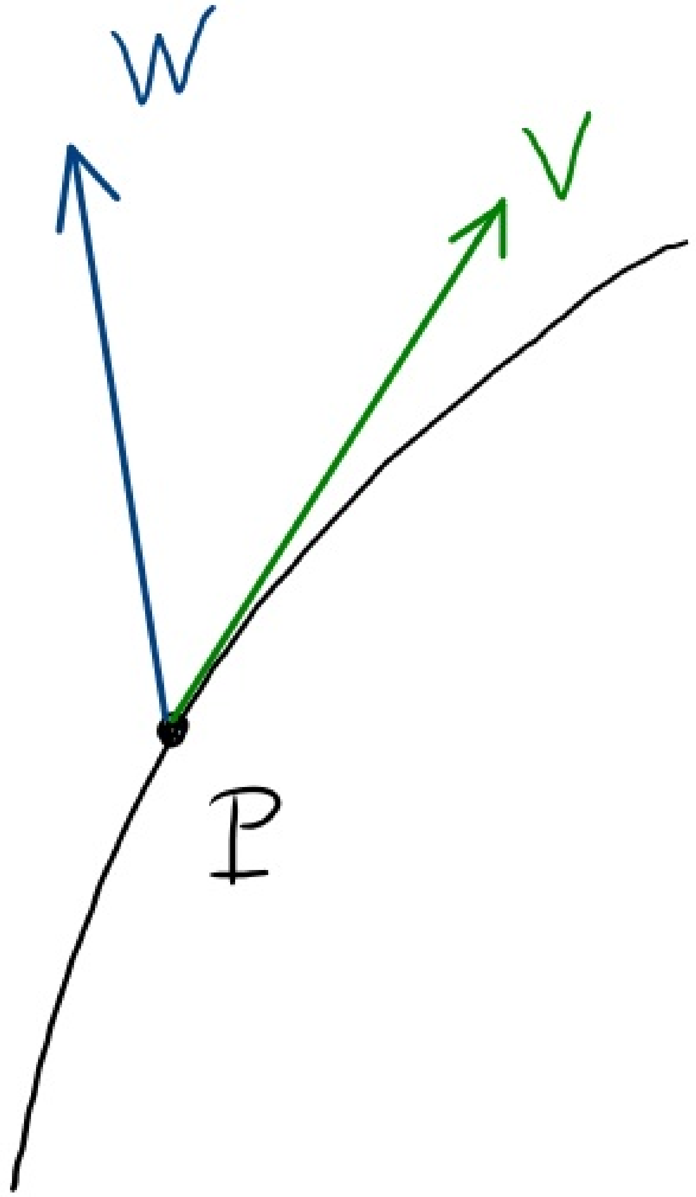
If $\{x^\mu\}$ are coordinates, $V^\mu = \dfrac{dx^\mu}{dt}$



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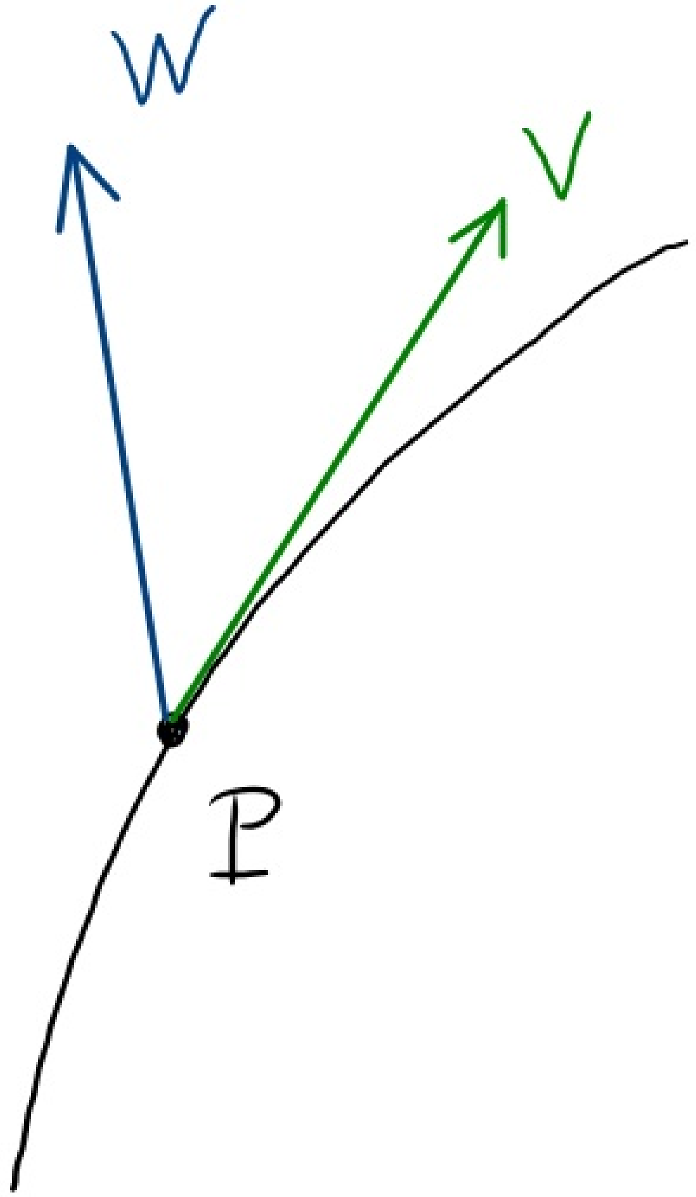

If $\{x^\mu\}$ are coordinates, $V^\mu = \dfrac{dx^\mu}{dt}$, and

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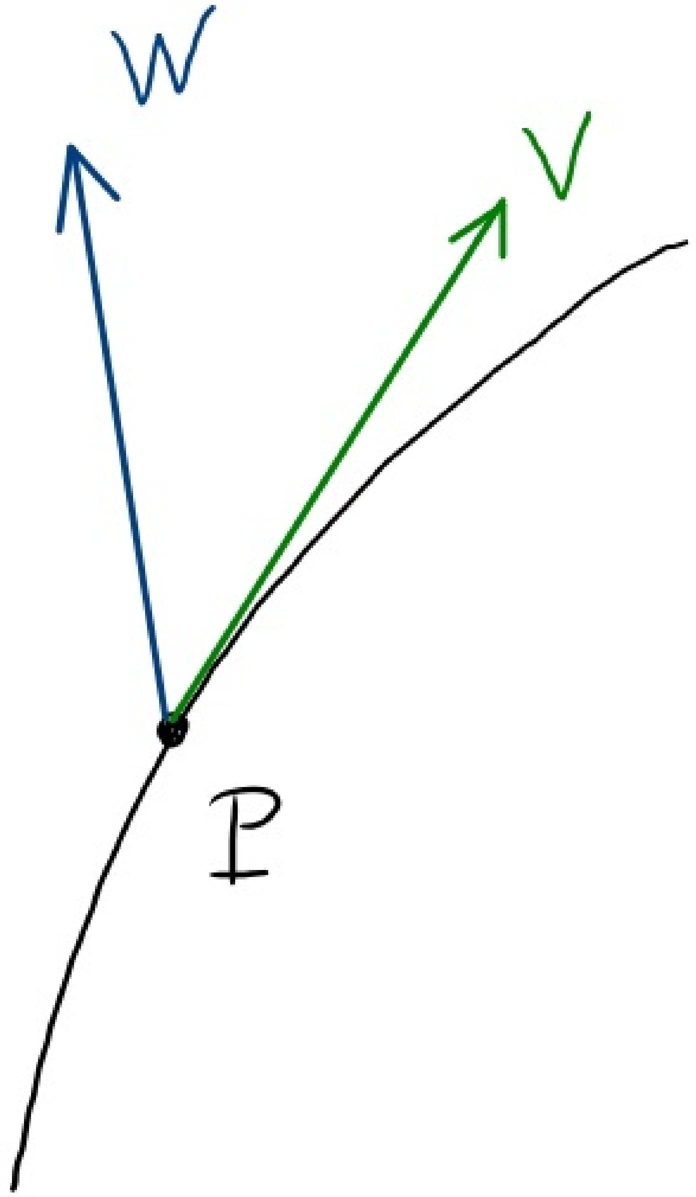


$$D_V W^\mu = \frac{dx^\nu}{dt} \partial_\nu W^\mu + \frac{dx^\nu}{dt} \Gamma^\mu{}_{\nu\rho} W^\rho$$

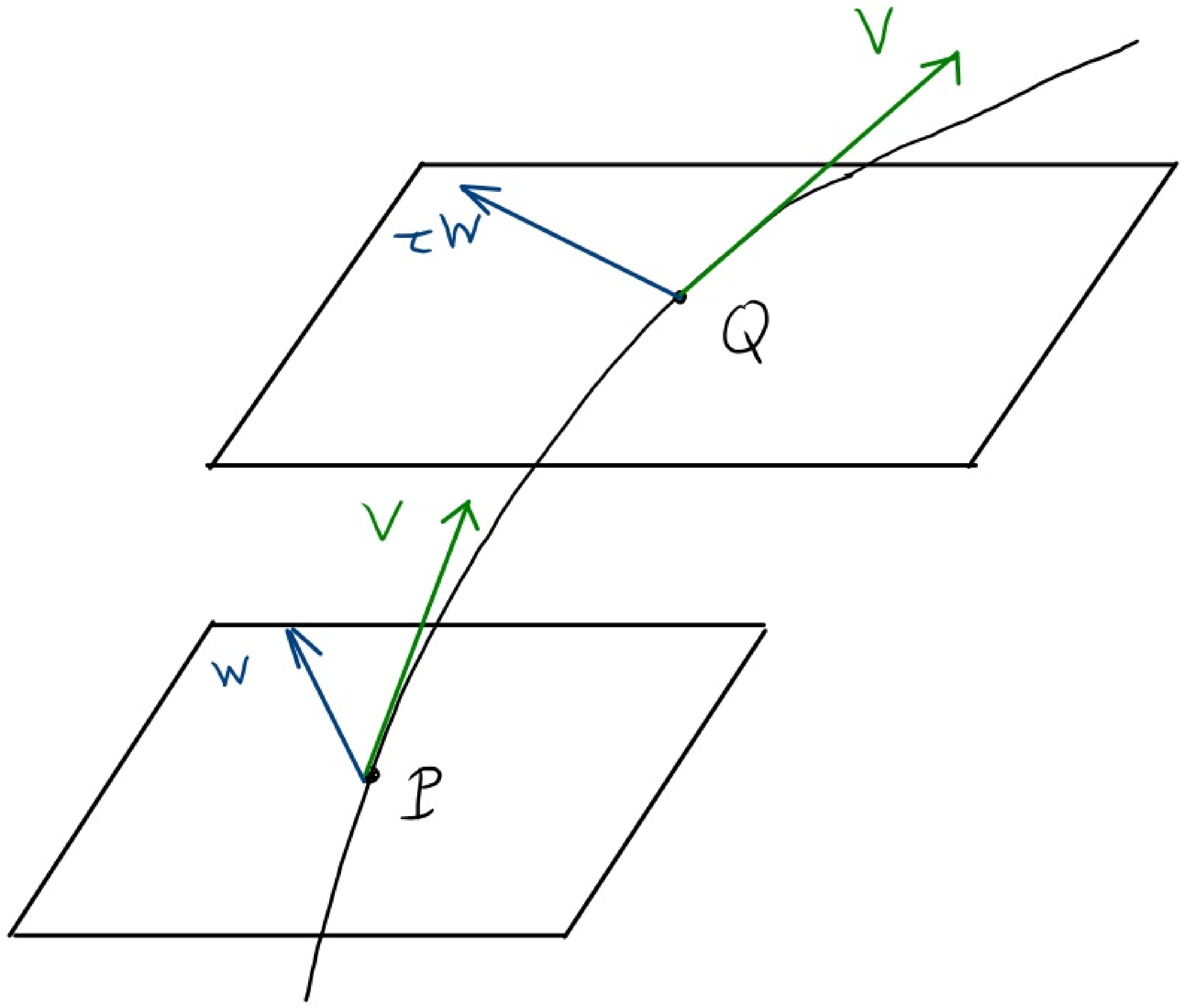
$$= \frac{dW^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dt} W^\rho$$

$\hookrightarrow$ depends only on values of W^μ on curve

Parallel Transport of Vector

W^μ is parallel transported along $\gamma(t)$ if:

$$D_V W^\mu = 0 \qquad \forall\ P \in \gamma(t)$$



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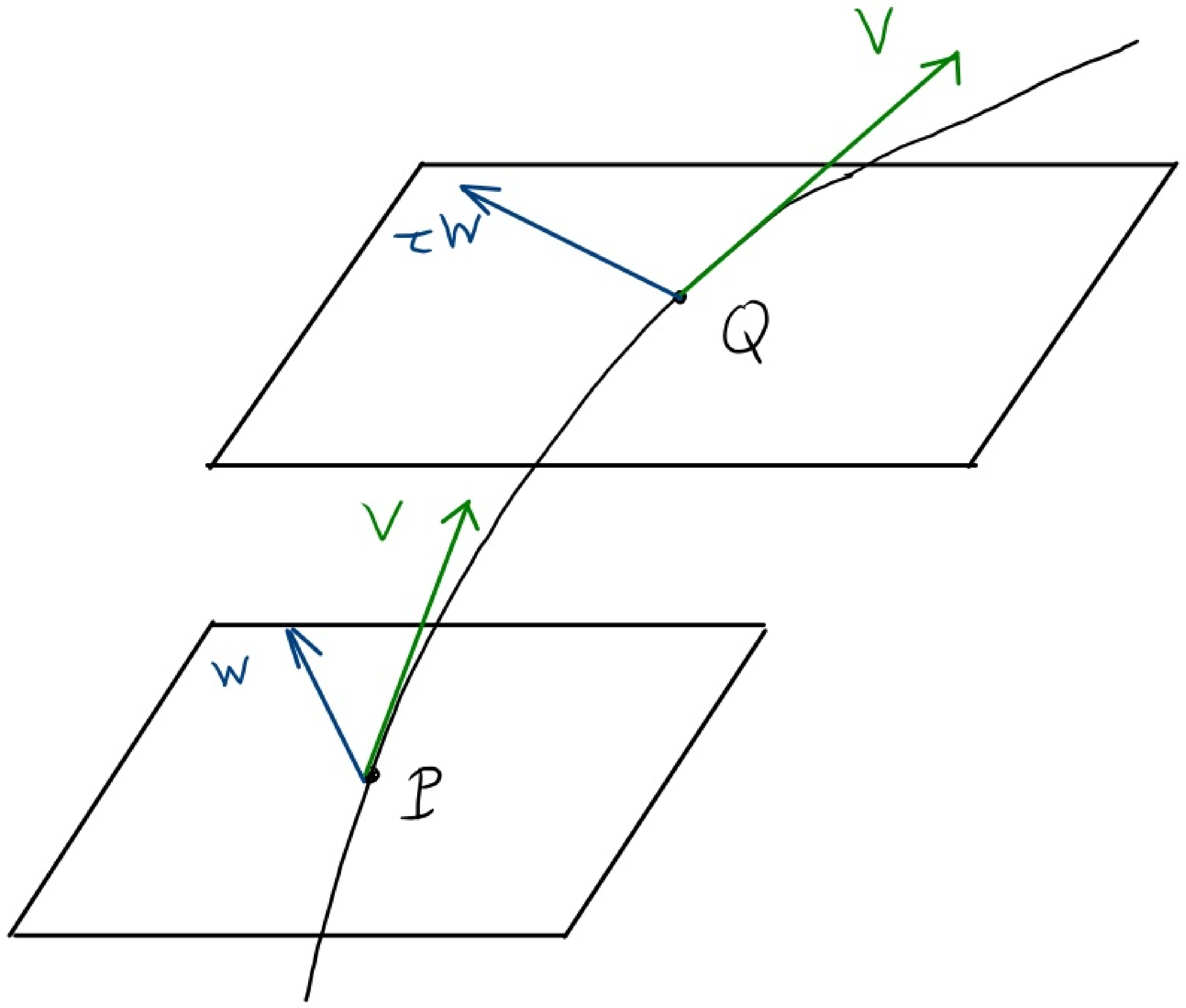

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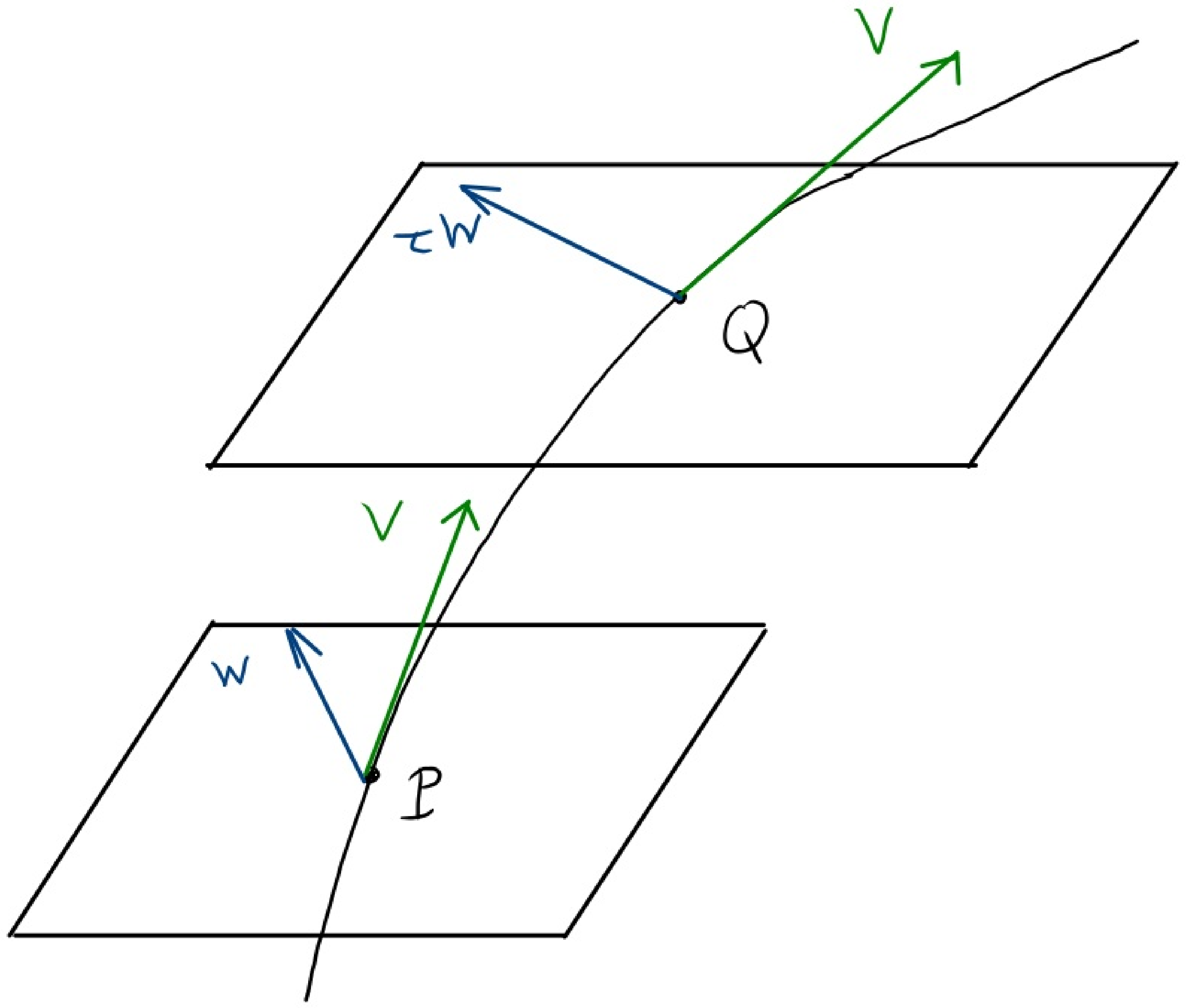

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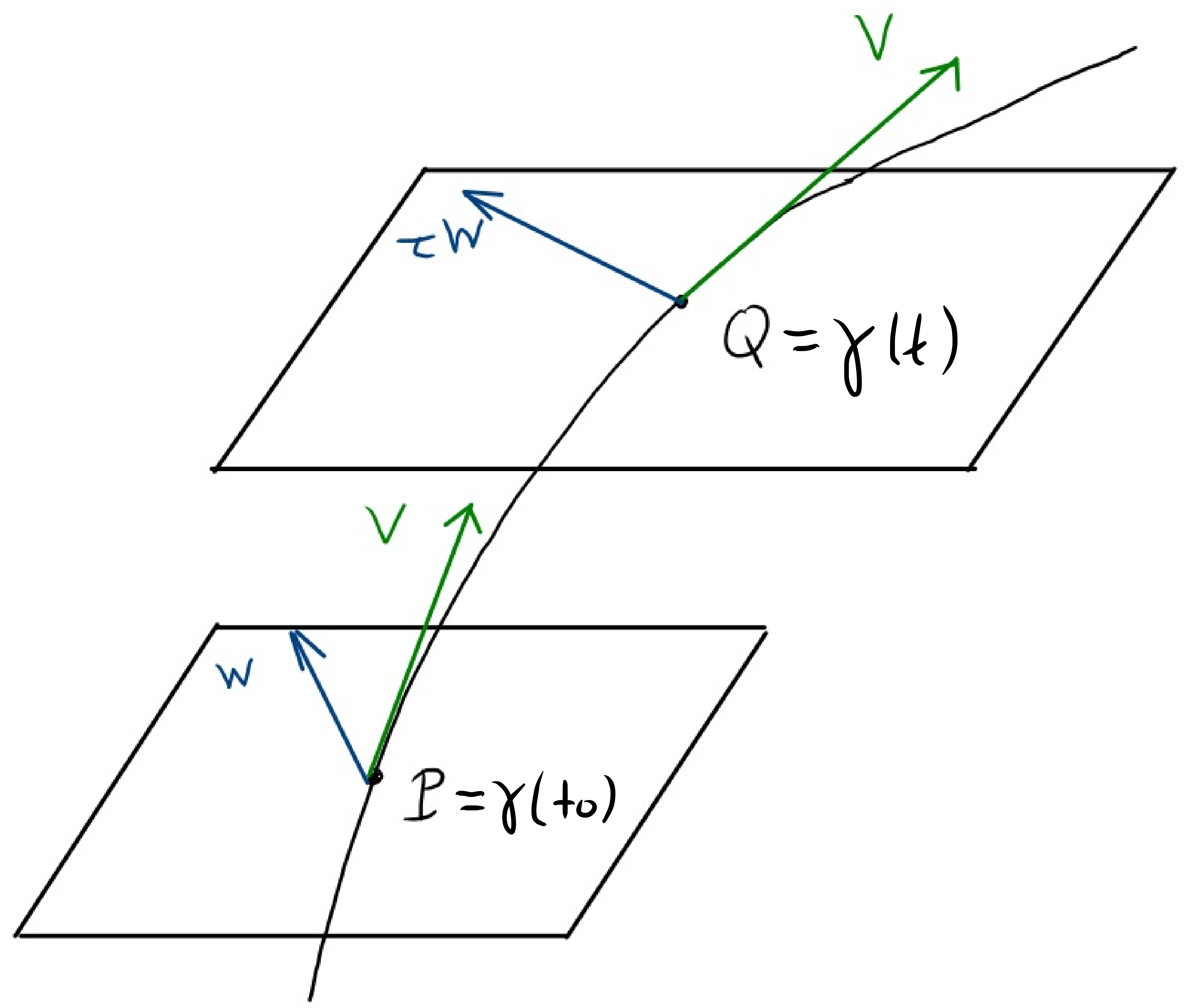
- If $W^\mu(\underline{P})$ is given $\Rightarrow$ unique solution along $\gamma(t)$

Parallel Transport of Vector



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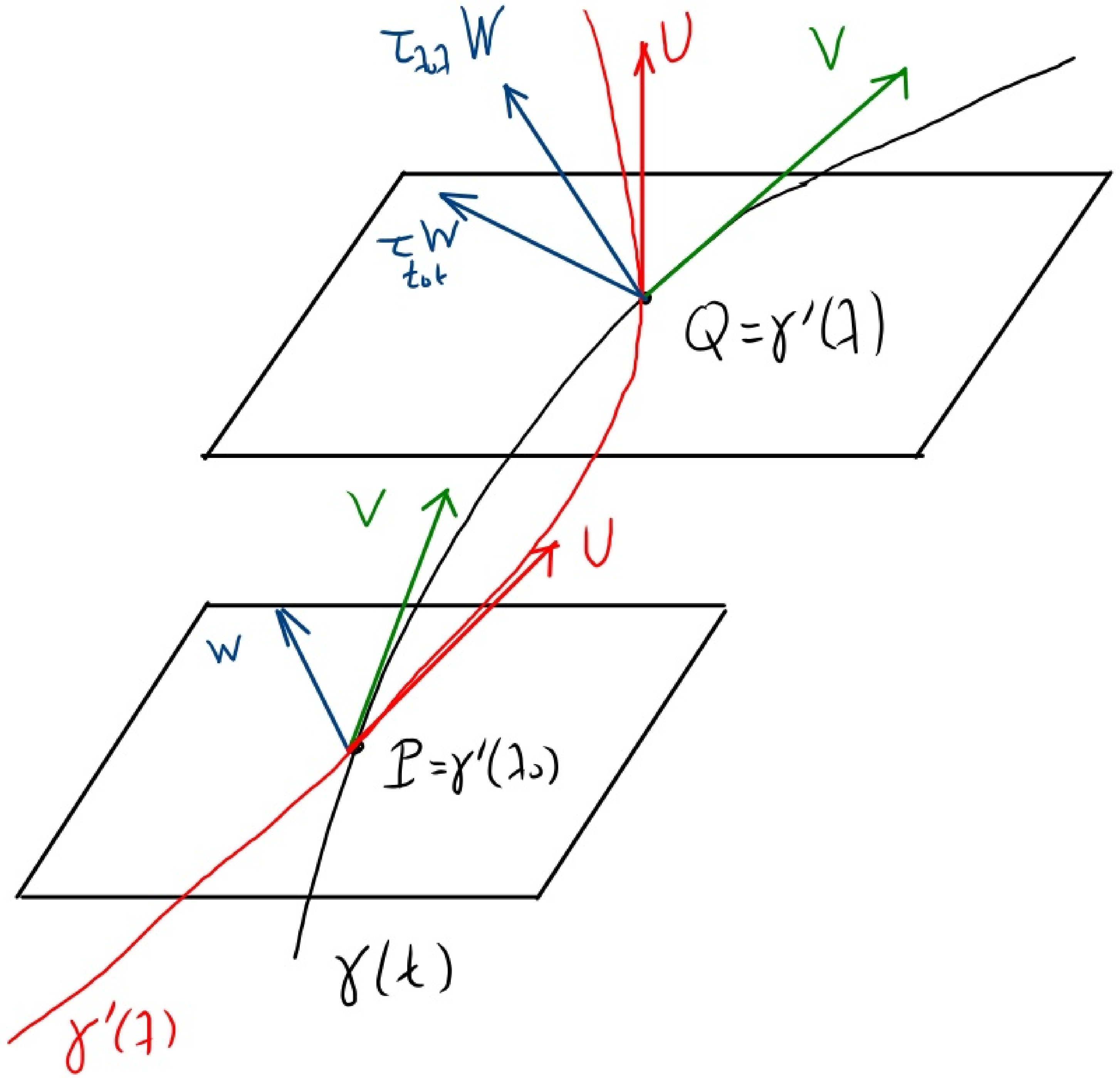
- If $W^\mu(P)$ is given $\Rightarrow$ unique solution along $\gamma(t)$

$\Rightarrow$ a 1-1 map between $T_P M$ and $T_Q M$, $P=\gamma(t_0)$, $Q=\gamma(t)$

$$W^\mu(t_0) \mapsto \tau_{tt_0} W^\mu(t_0)$$

Parallel Transport of Vector

- Parallel transport is path dependent



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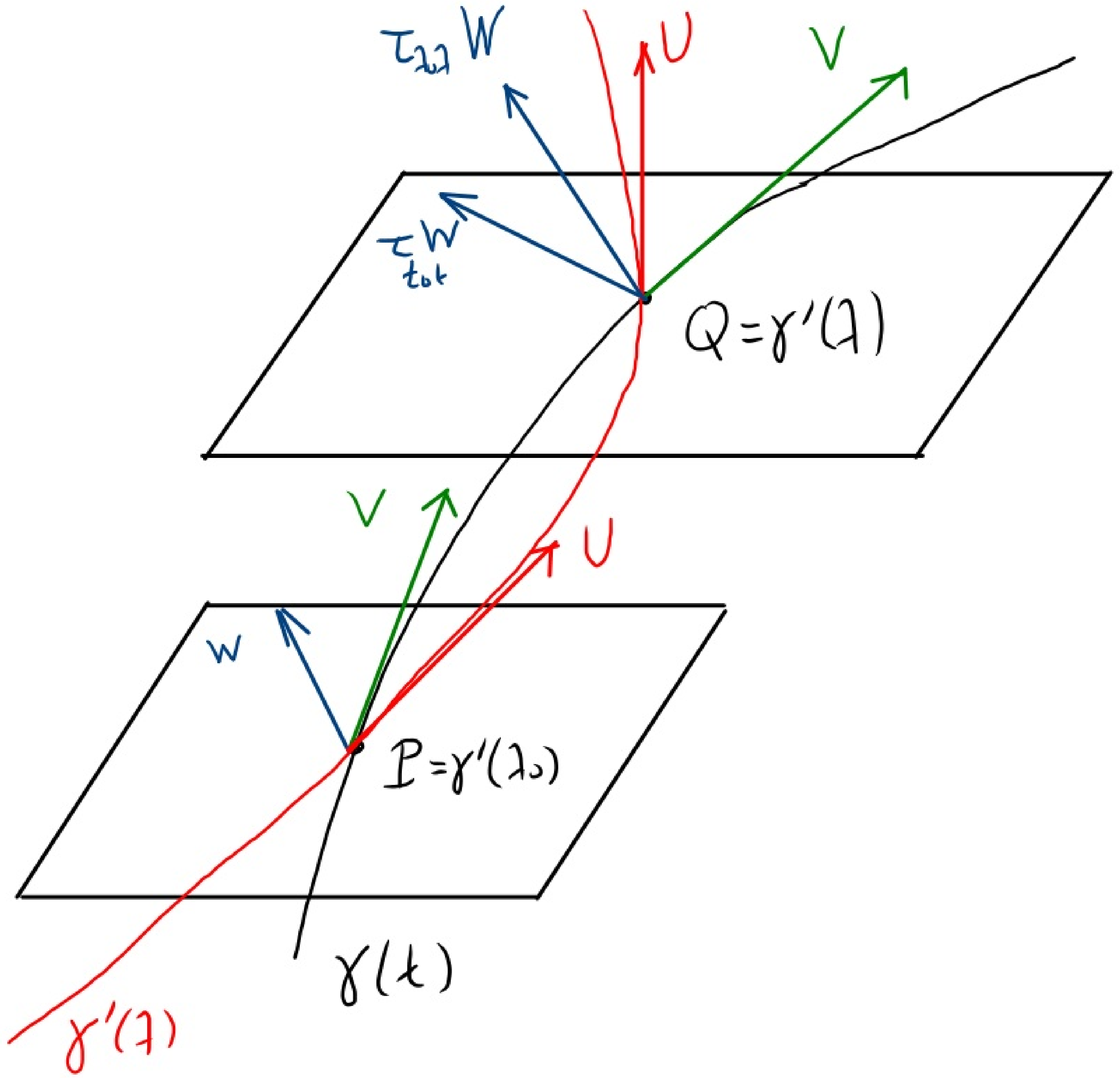
Parallel Transport of Vector

- Parallel transport is path dependent
- Parallel transport is connection dependent
  - ⇒ if we change $g_{\mu\nu}$
  - ⇒ metric compatible ∇_μ will change
  - ⇒ parallel transported vector will change



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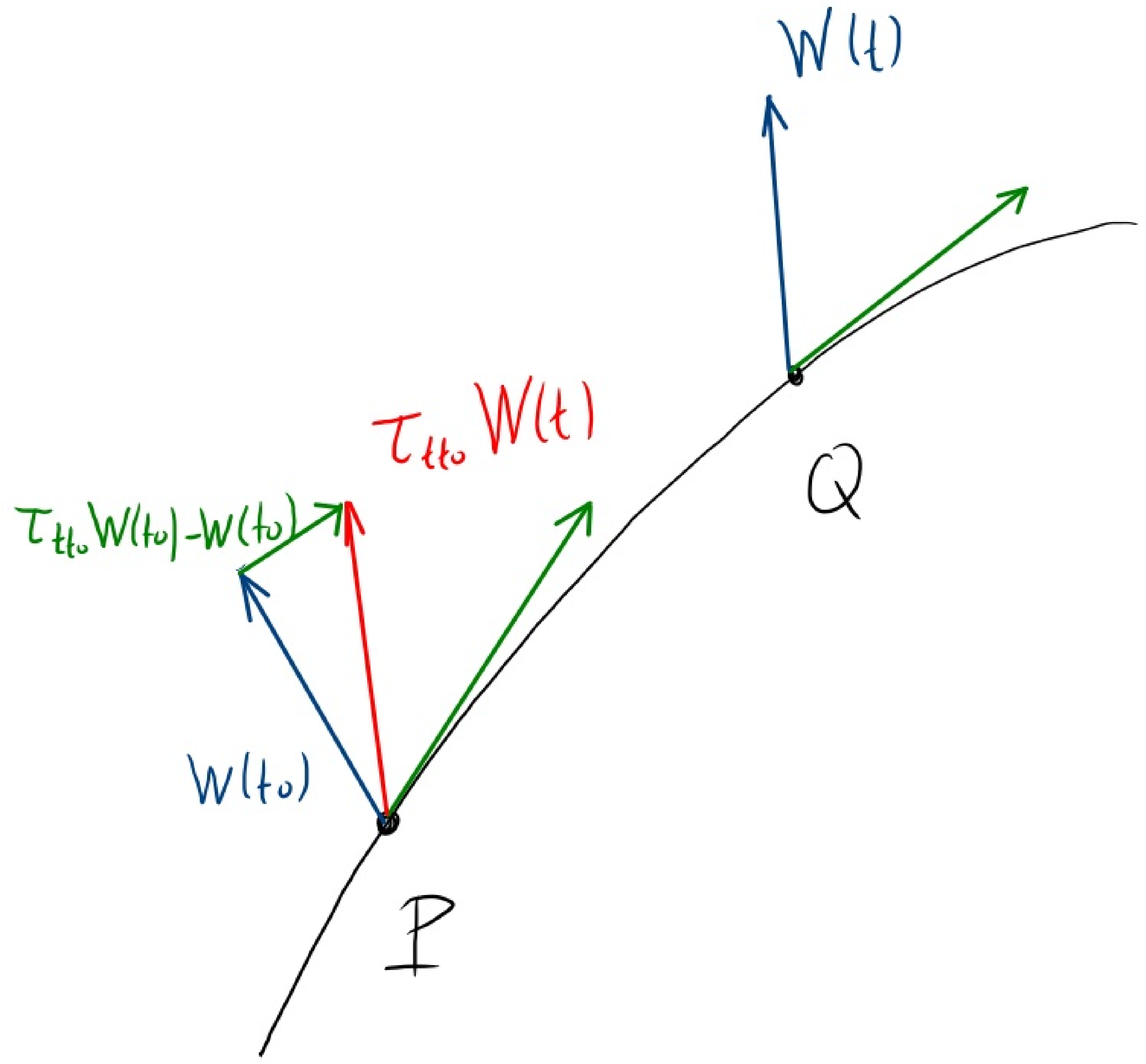
One can show that

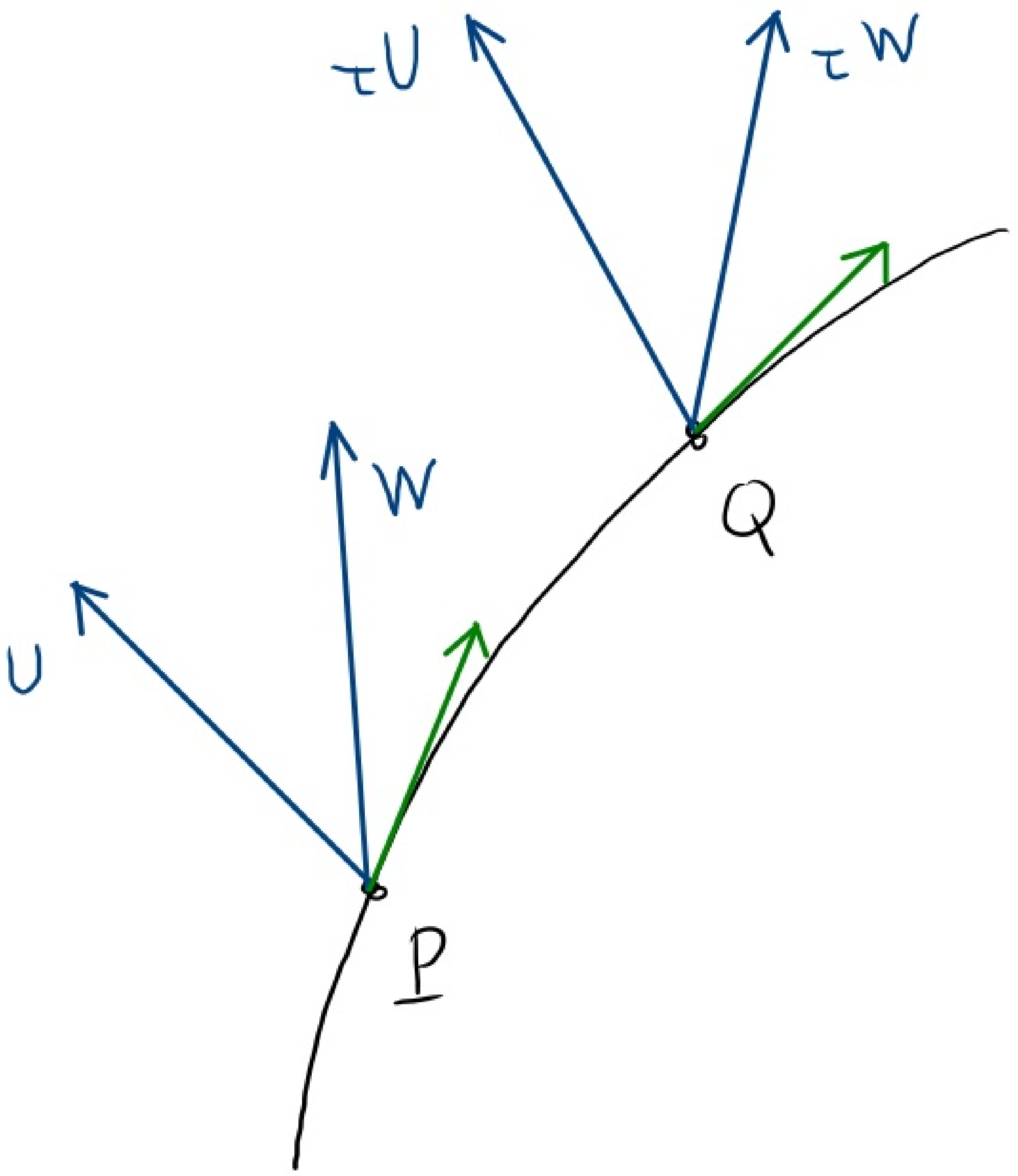
$$D_v W^\mu = \lim_{t \to t_0} \frac{\tau_{t t_0} W^\mu(t) - W^\mu(t_0)}{t - t_0}$$



If ∇_μ is metric compatible, then

$$\frac{d}{dt} U \cdot W = D_v \, U \cdot W$$

↳ a function, so $\frac{d}{dt} = D_v$



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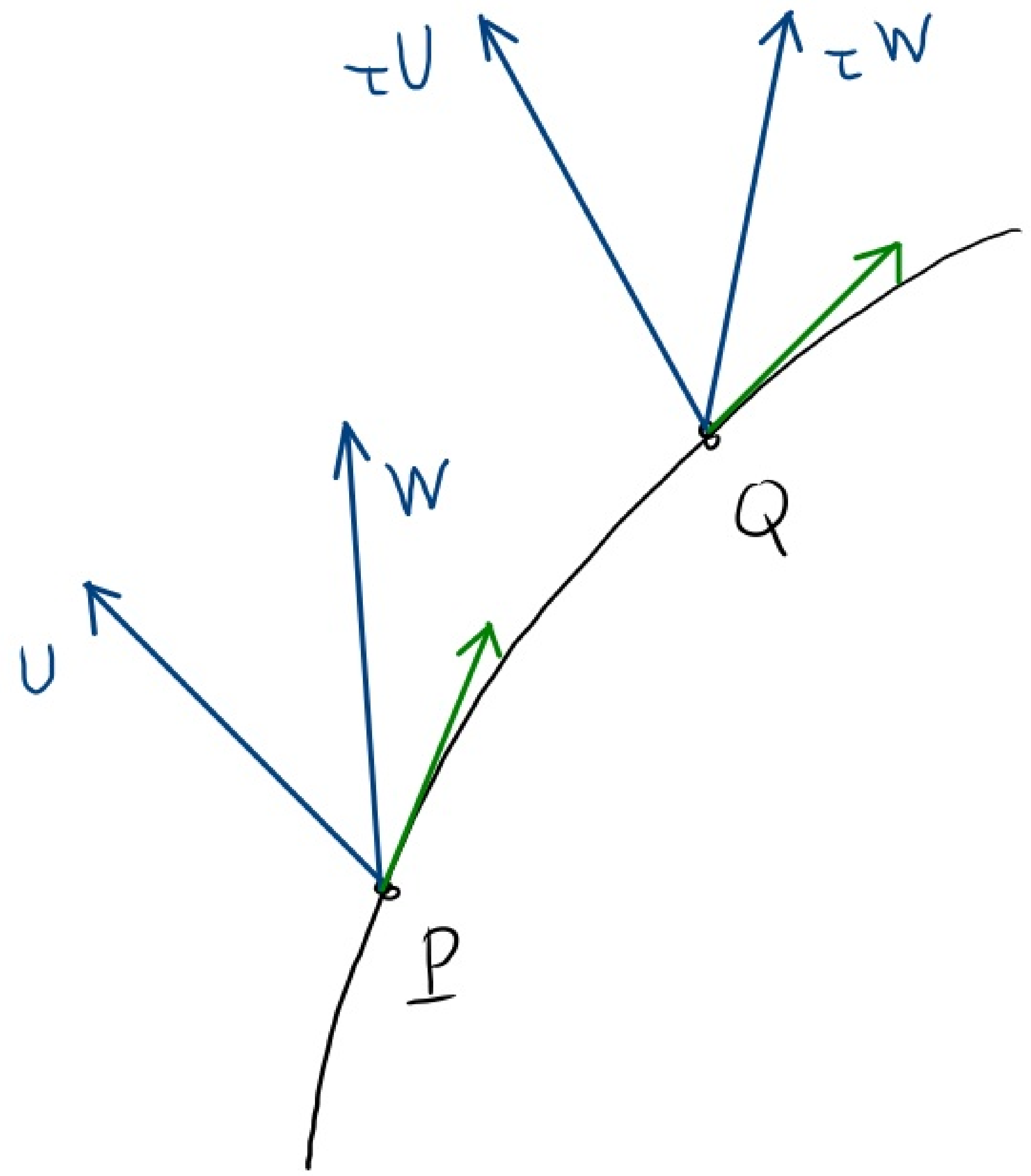

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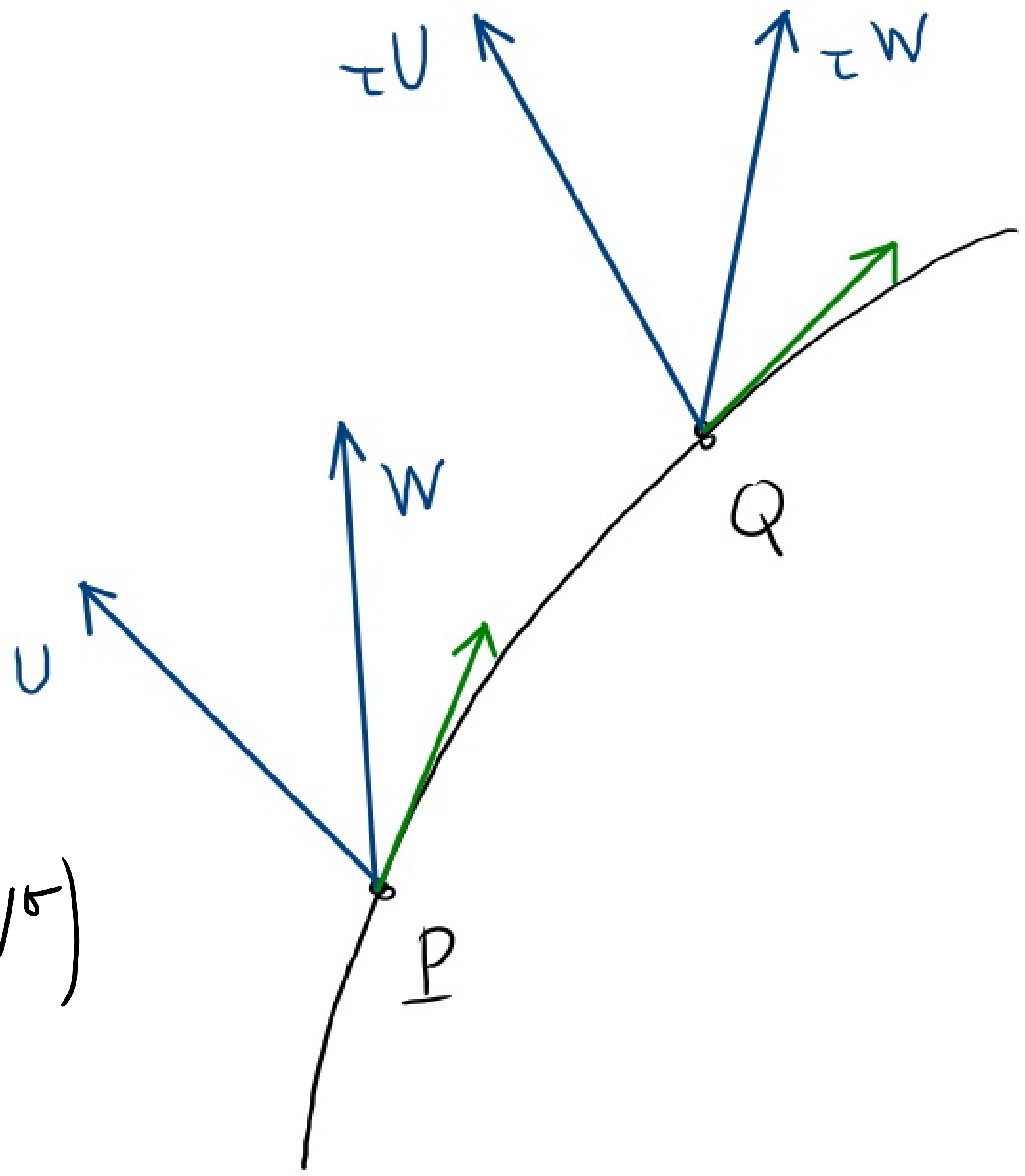
$$= V^\mu (\nabla_\mu g_{\nu\sigma}) U^\nu W^\sigma + V^\mu g_{\nu\sigma} (\nabla_\mu U^\nu) W^\sigma + V^\mu g_{\nu\sigma} U^\nu (\nabla_\mu W^\sigma)$$



If ∇_μ is metric compatible, then

$$\frac{d}{dt} U \cdot W = D_V \, U \cdot W$$

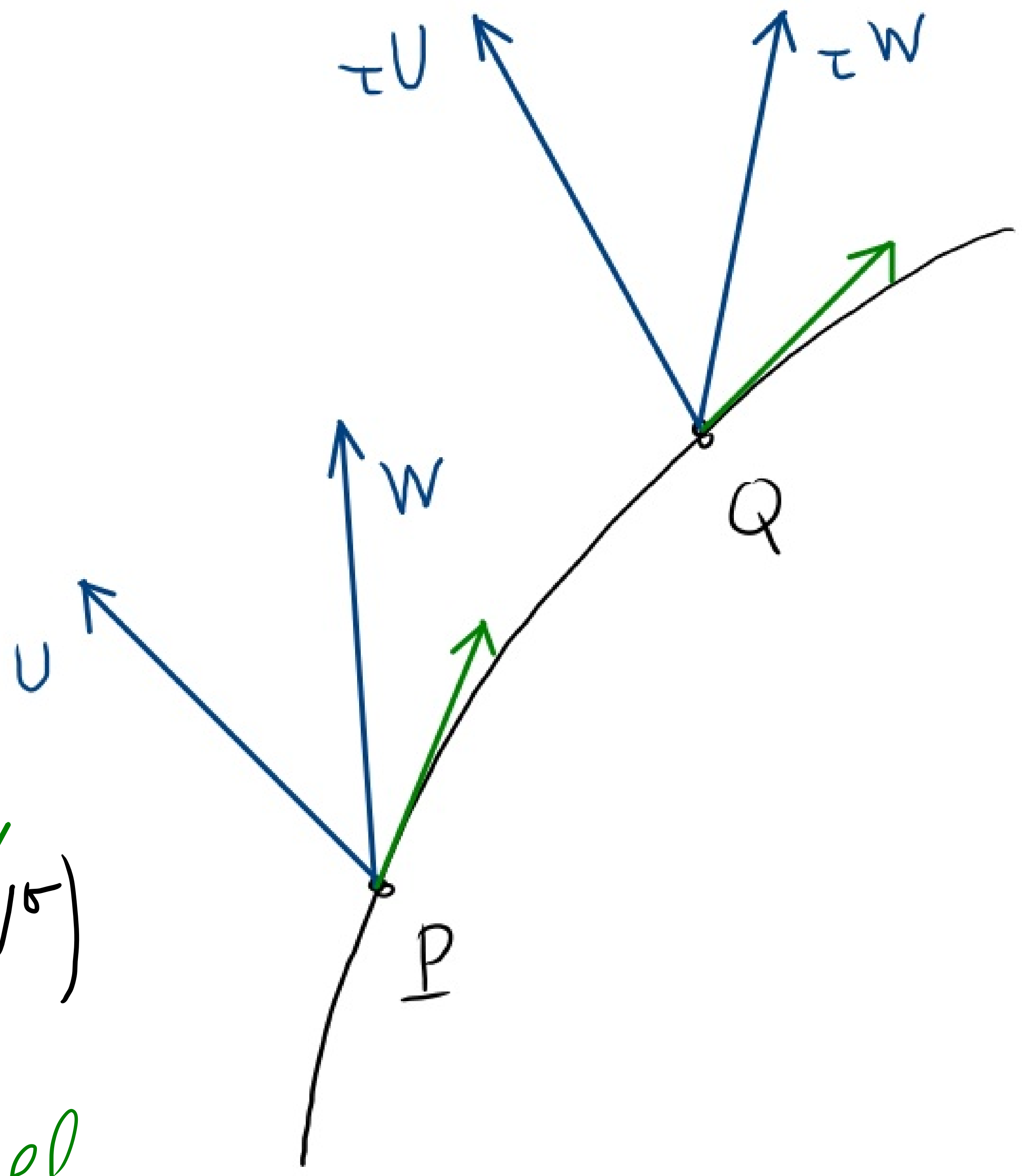
$$= V^\mu \nabla_\mu (g_{\nu\sigma} U^\nu W^\sigma)$$

$$= V^\mu \cancel{(\nabla_\mu g_{\nu\sigma})} U^\nu W^\sigma + V^\mu g_{\nu\sigma} \cancel{(\nabla_\mu U^\nu)} W^\sigma + V^\mu g_{\nu\sigma} U^\nu \cancel{(\nabla_\mu W^\sigma)}$$

metric compatibility   parallel transported   parallel transported

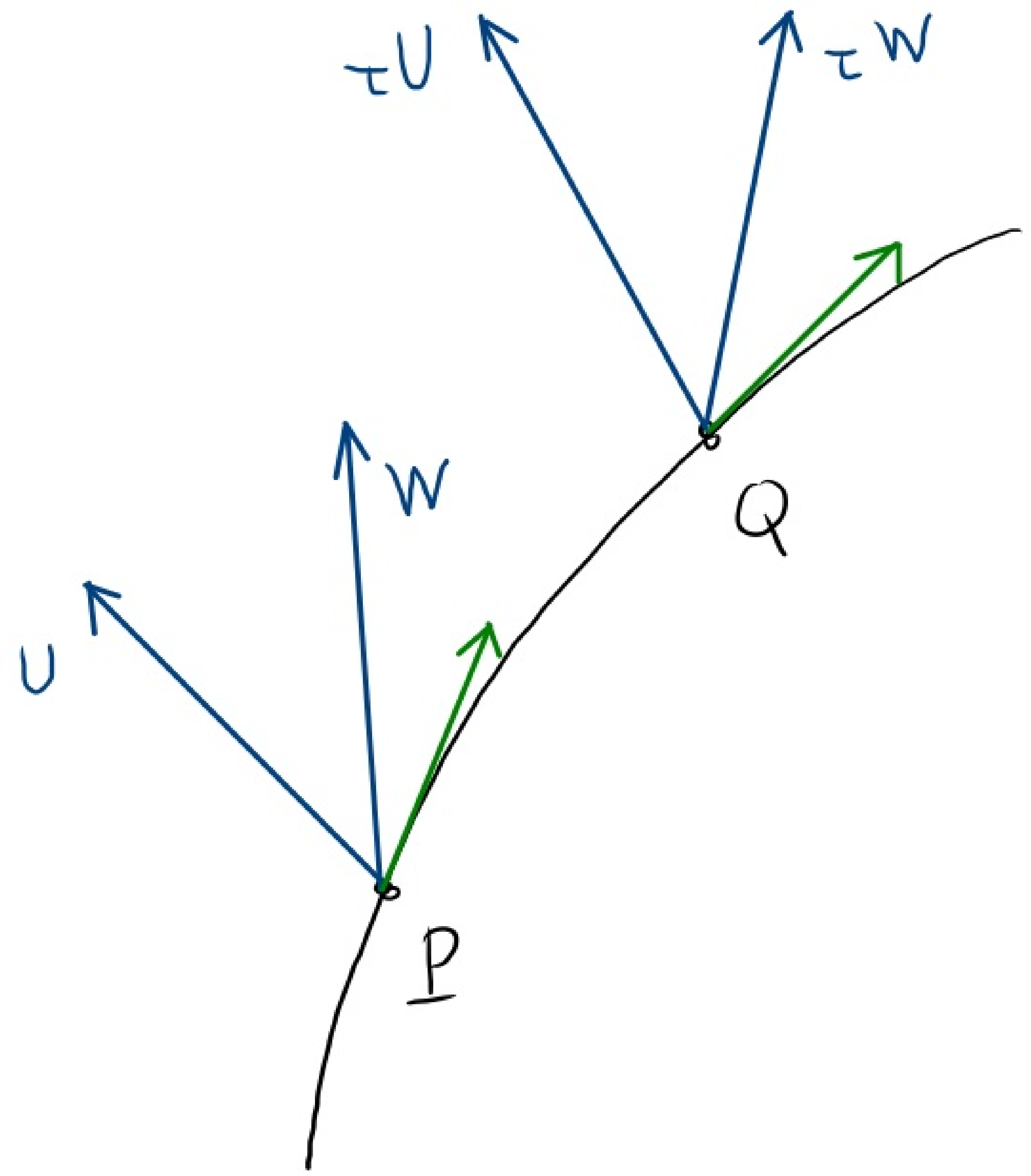
$$= 0$$

$\Rightarrow$ $U \cdot W$ is constant along the curve



$\Rightarrow$ angles are preserved

$\Rightarrow$ norms are preserved



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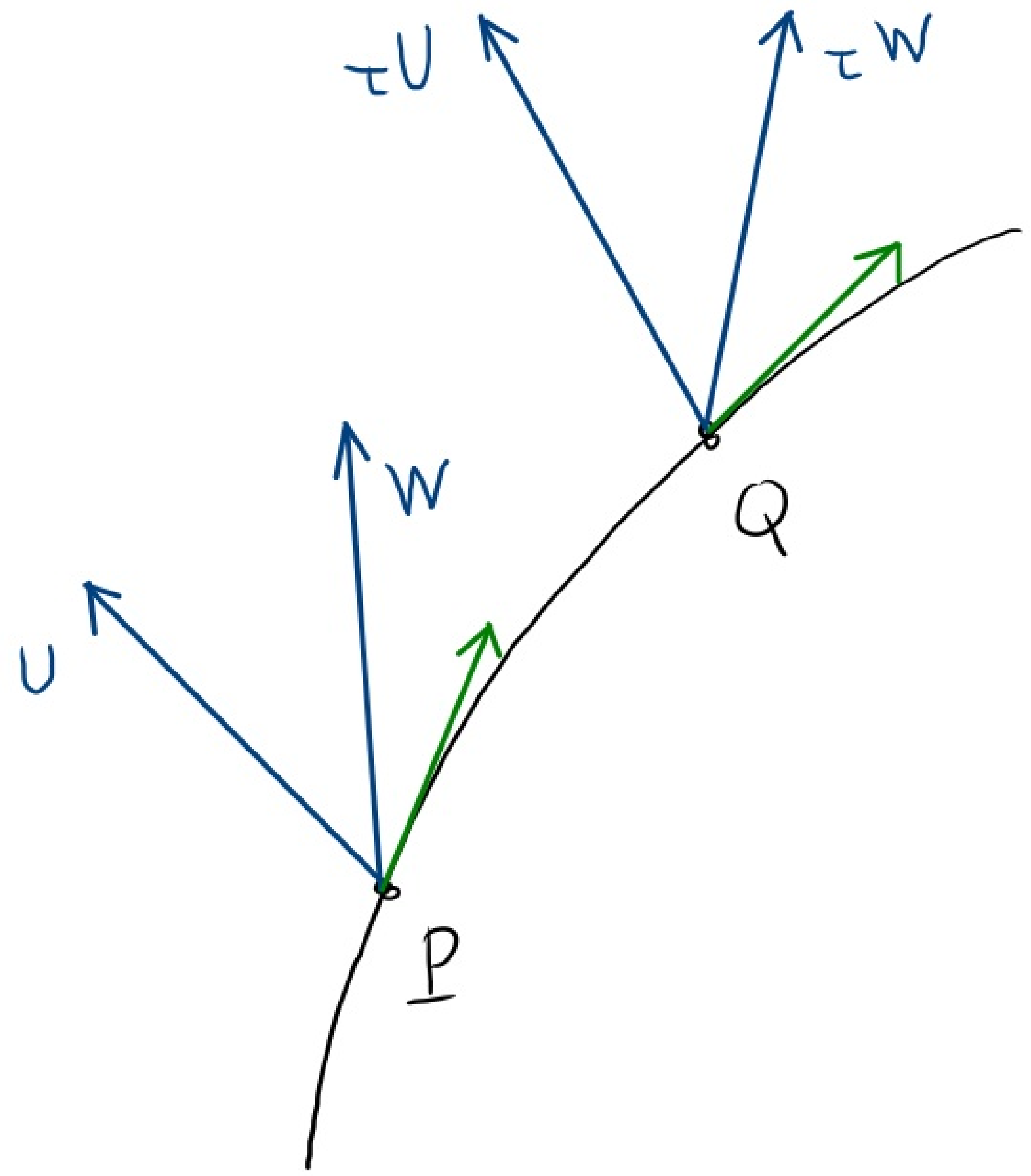


These are the properties of parallel transport that we can keep

We can't get rid of path-dependence

(unless we have a flat connection)

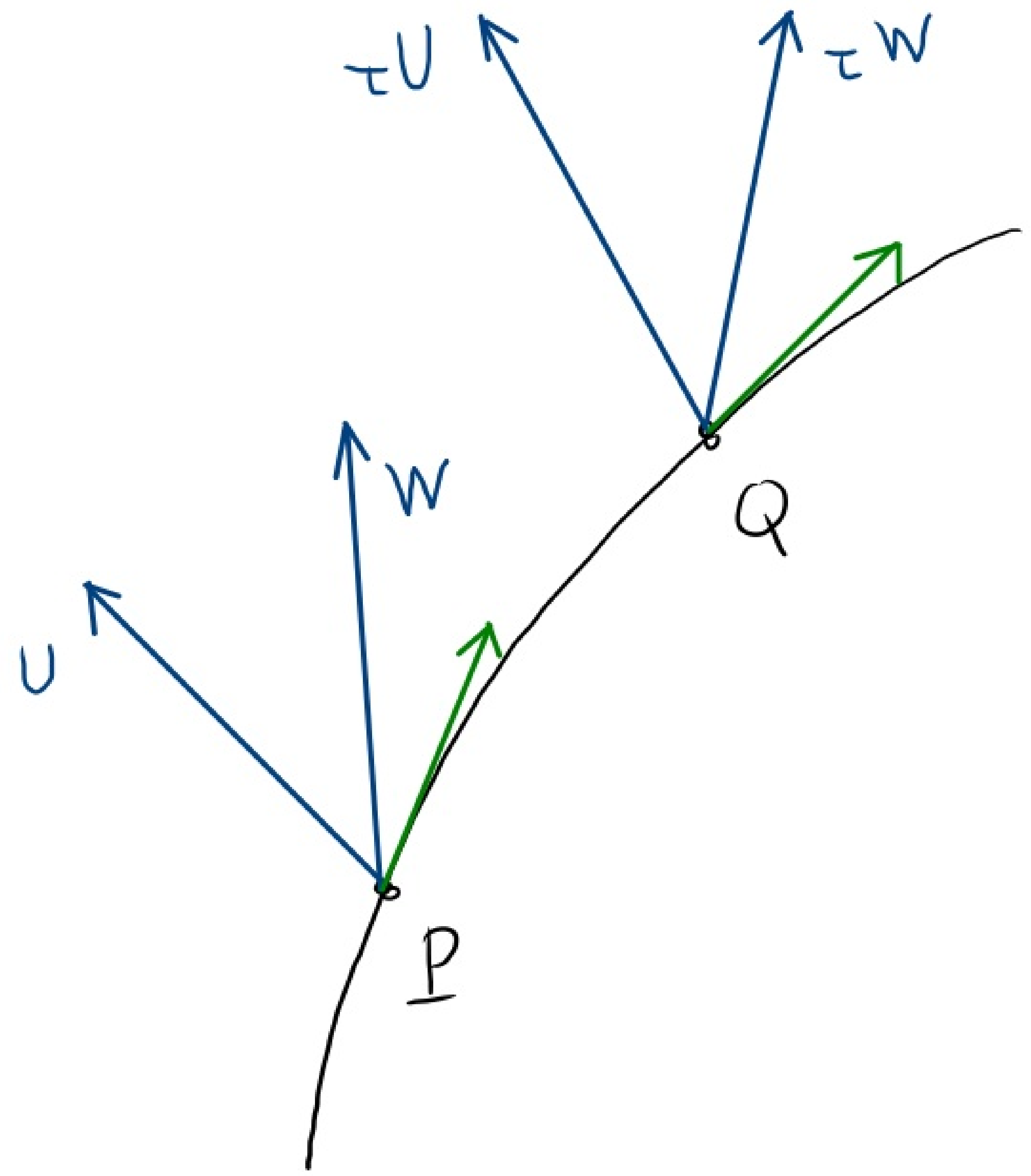
$\Rightarrow$ angles are preserved

$\Rightarrow$ norms are preserved

If T is any (k,l) tensor:

$D_V T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = V^\mu \nabla_\mu T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$, and if

$D_V T = 0 \Rightarrow T$ parallel-transported along $\gamma(t)$



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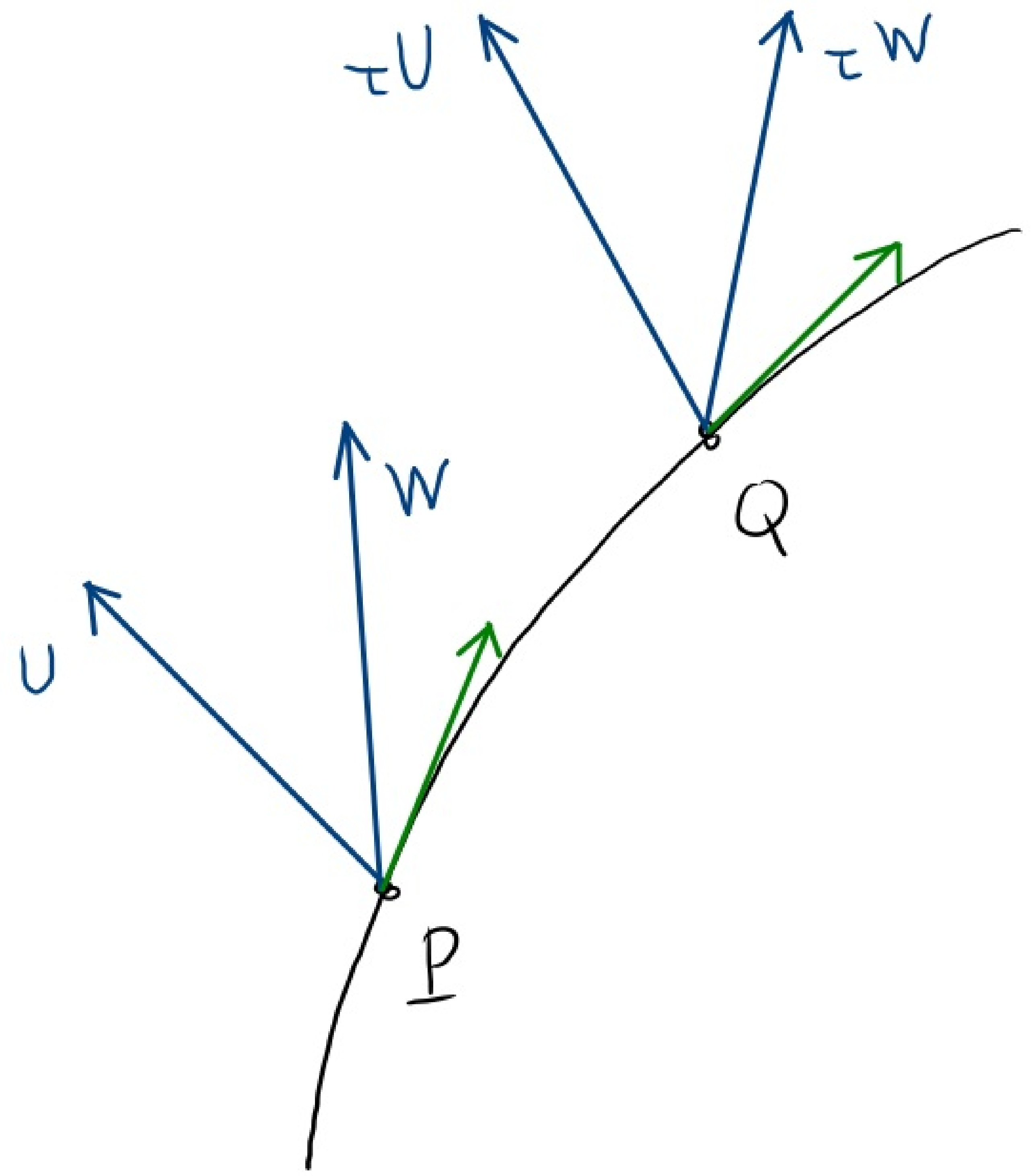

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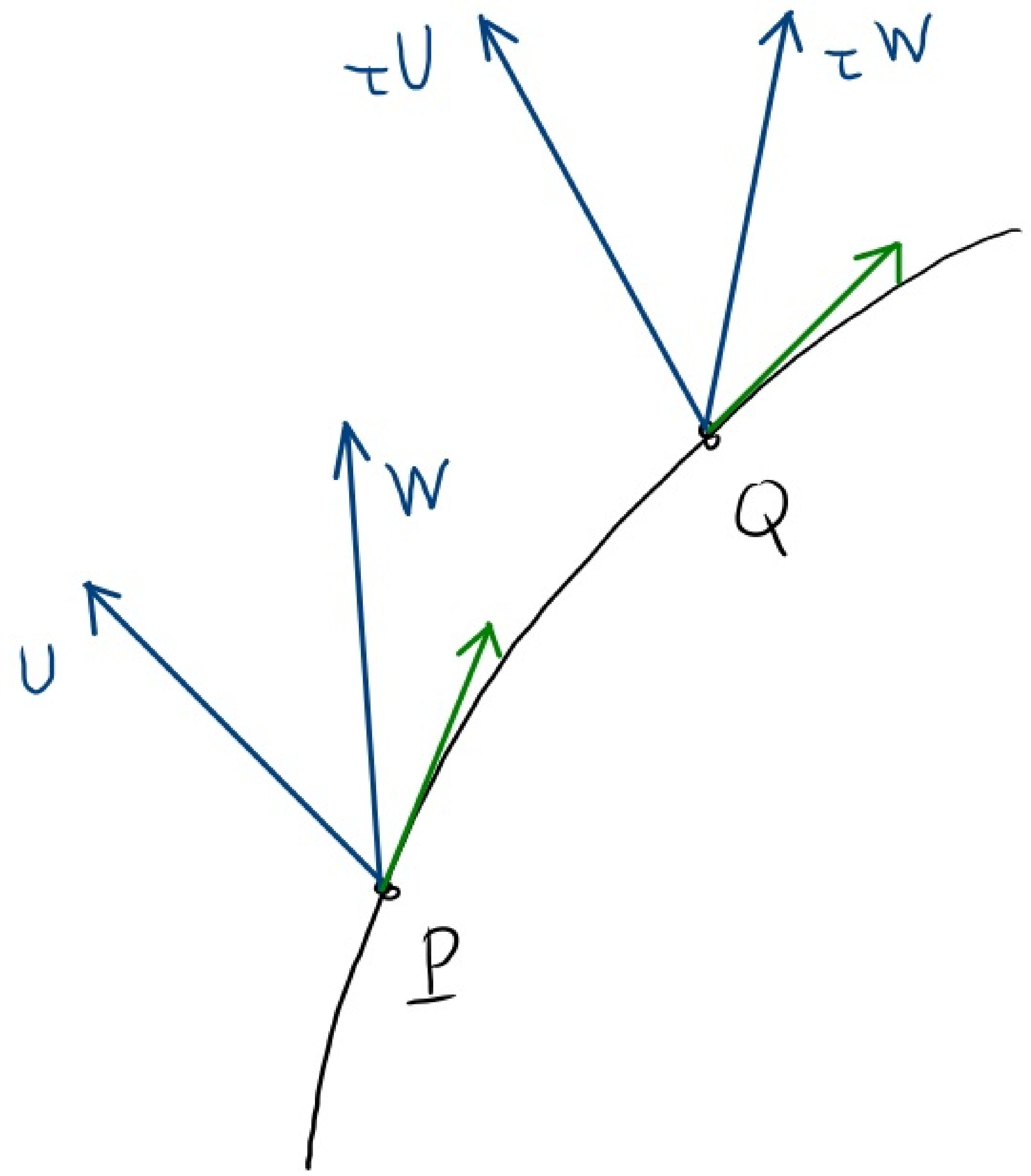
$D_V T = 0 \Rightarrow T$ parallel-transported along $\gamma(t)$

- $D_V T = \left(\begin{array}{l}\text{rate of change of } T \text{ compared to} \\ \text{what it would have been if parallel-transported}\end{array}\right)$

$\Rightarrow$ angles are preserved

$\Rightarrow$ norms are preserved



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- $D_V T = \left(\begin{array}{l}\text{rate of change of } T \text{ compared to} \\ \text{what it would have been if parallel-transported}\end{array}\right)$
- Contractions of p-t tensors are preserved: $D_V(S\,T \dots) = 0$