Geodesics

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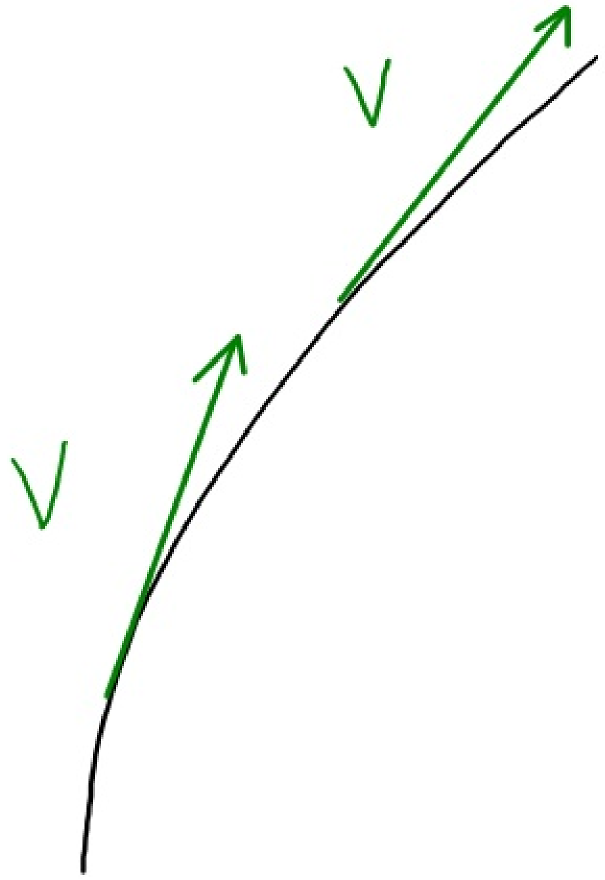
- Straightest curves – longest proper times (locally)
- Curvature makes parallel geodesics deviate
  – relative acceleration, "gravity"

* a curve is a geodesic if it parallel transports its tangent vector:

$$D_V V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0$$



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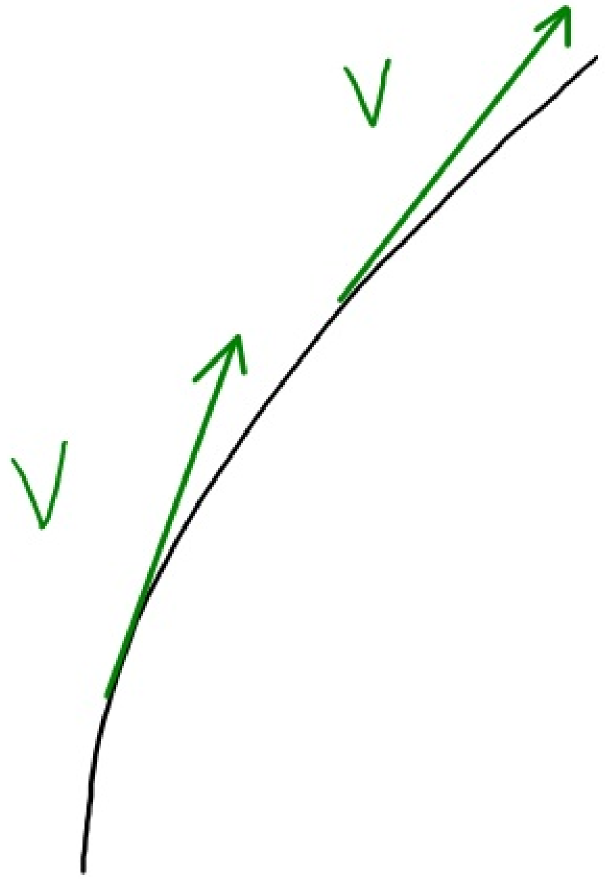

$$D_V V^\mu = 0 \Leftrightarrow V^\nu \nabla_\nu V^\mu = 0 \quad (1)$$

- a weaker condition is $D_V V^\mu = f V^\mu$, but with a reparametrization of the curve it can be recast to (1)

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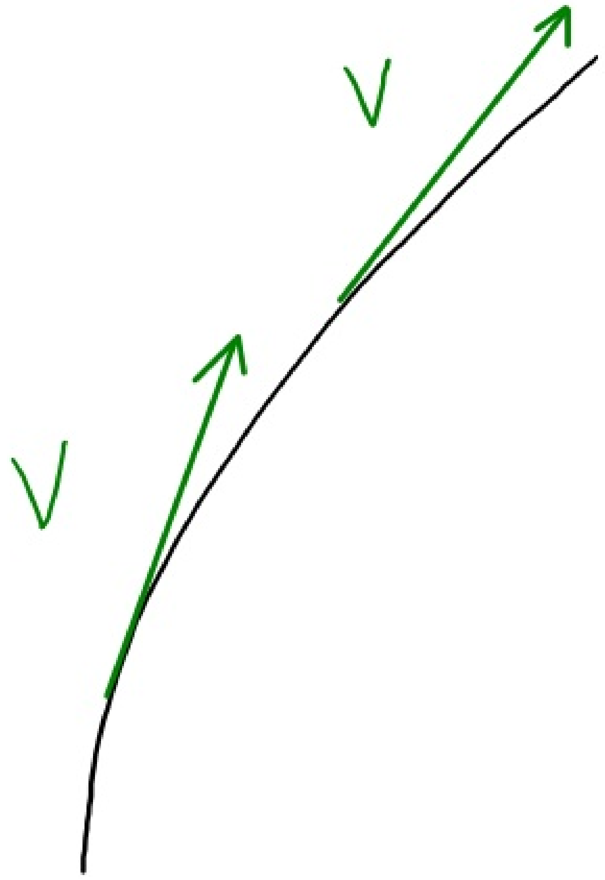

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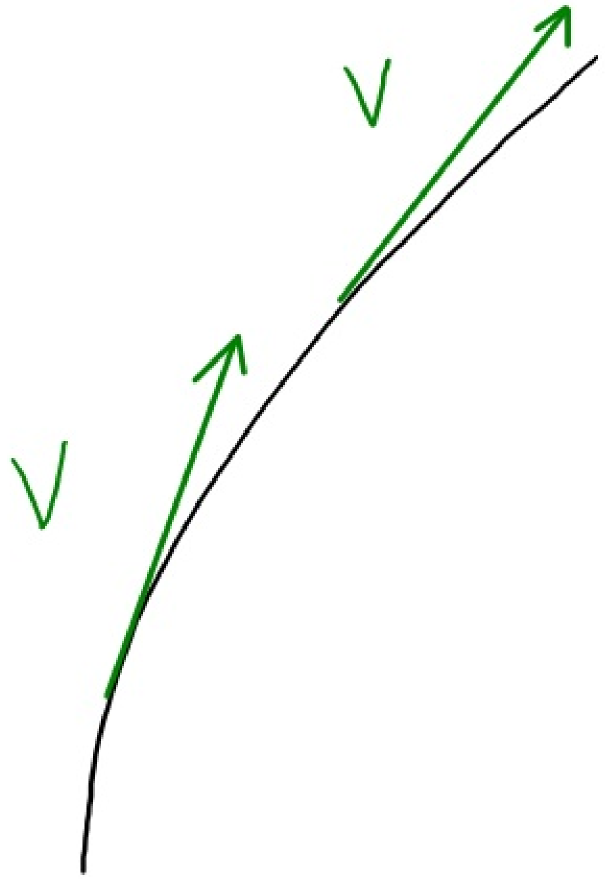

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  $\tau' = \alpha \tau + \beta$, $\alpha, \beta \in \mathbb{R}$ is also an affine parameter

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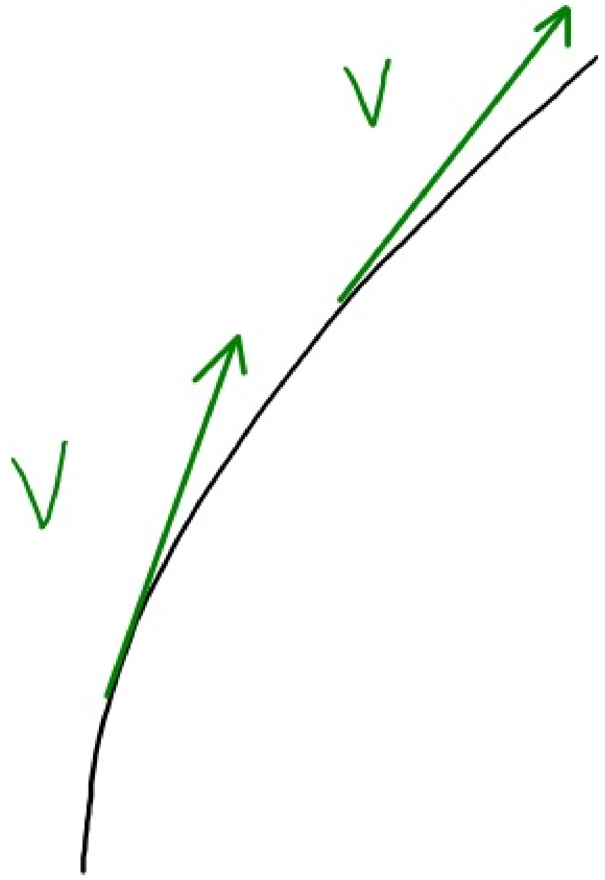

$$D_V V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0$$

$$\Rightarrow V^\nu \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\rho} V^\nu V^\rho = 0$$

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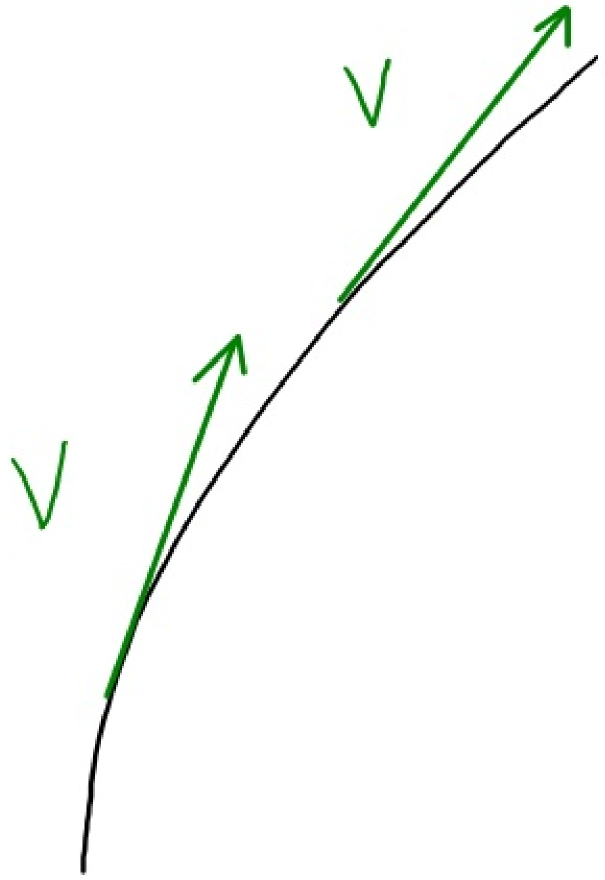

$$D_v V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0$$

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If $\{x^\mu\}$ are coordinates, $V^\mu = \dfrac{dx^\mu}{d\tau}$, and

$$\frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

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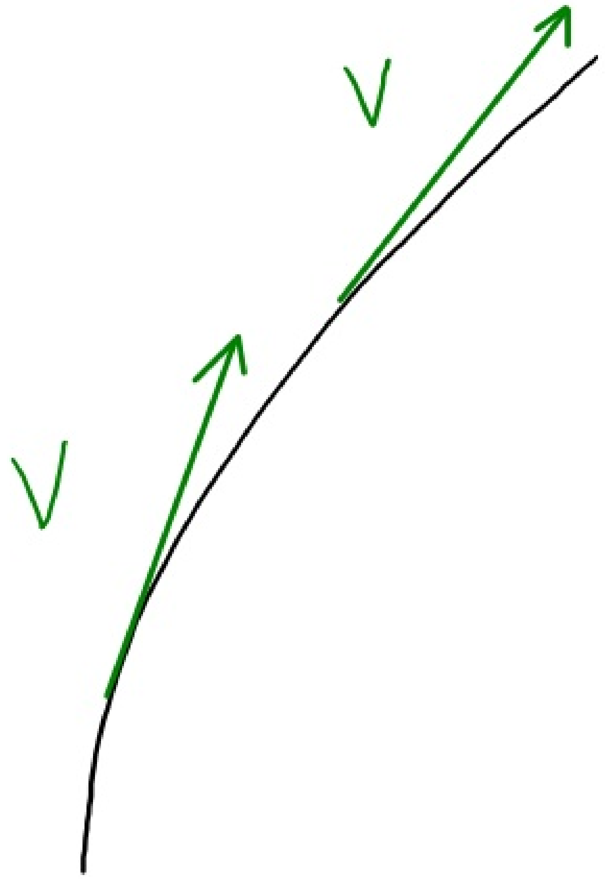

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$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

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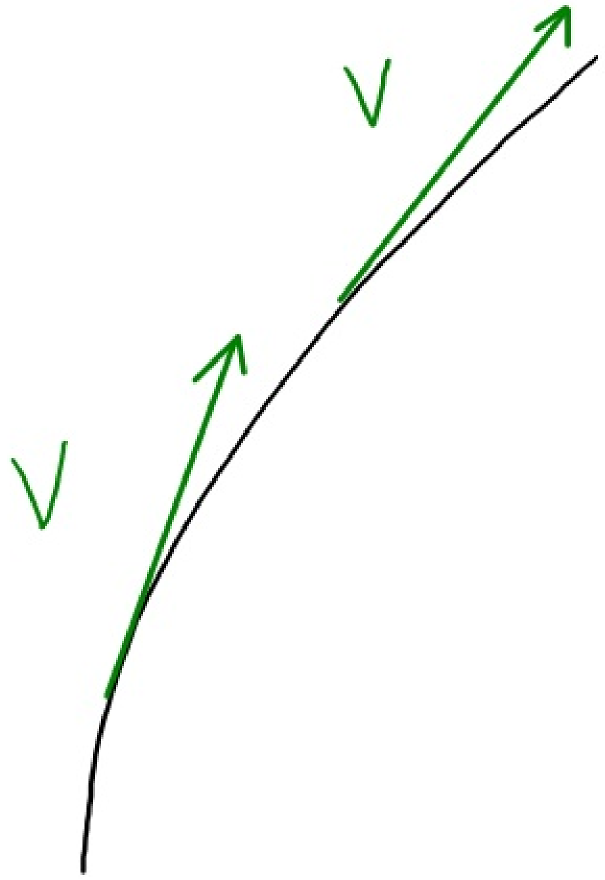

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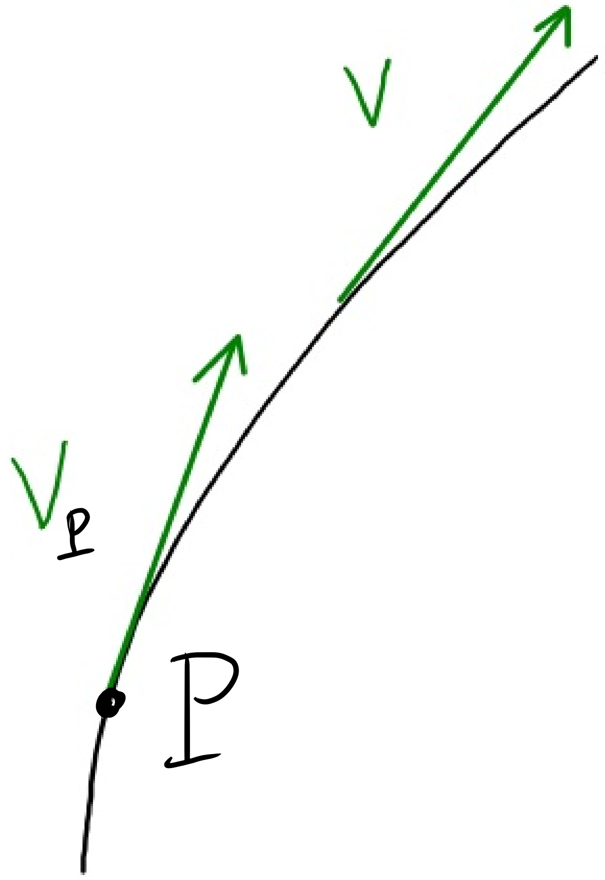
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu}{d\tau}(0)\right) \rightarrow \text{unique solution}$$

$\Rightarrow$ There is a unique geodesic through P with tangent vector V^μ_P



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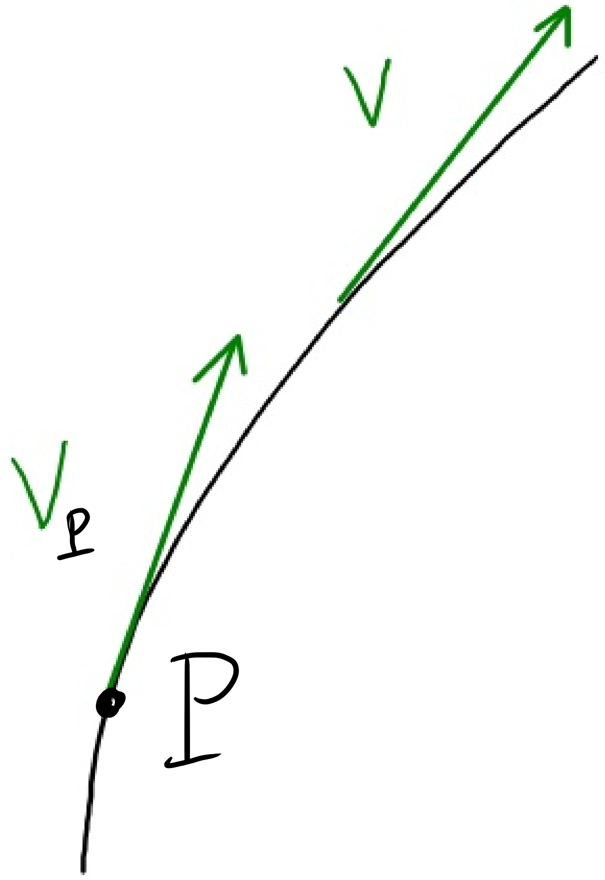
$\Rightarrow$ There is a unique geodesic through P with tangent vector V_P^μ

$\Rightarrow$ Geodesics depend only on symmetric part of $\Gamma^\mu{}_{\nu\rho}$

Adding $\Gamma^\mu{}_{[\nu\rho]} \neq 0$ would not affect geodesics (torsion)



symmetric term under $\nu \leftrightarrow \rho$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \underbrace{\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}} = 0 \Rightarrow \left(x^\mu(0), \frac{dx^\mu(0)}{d\tau}\right) \to \text{unique solution}$$

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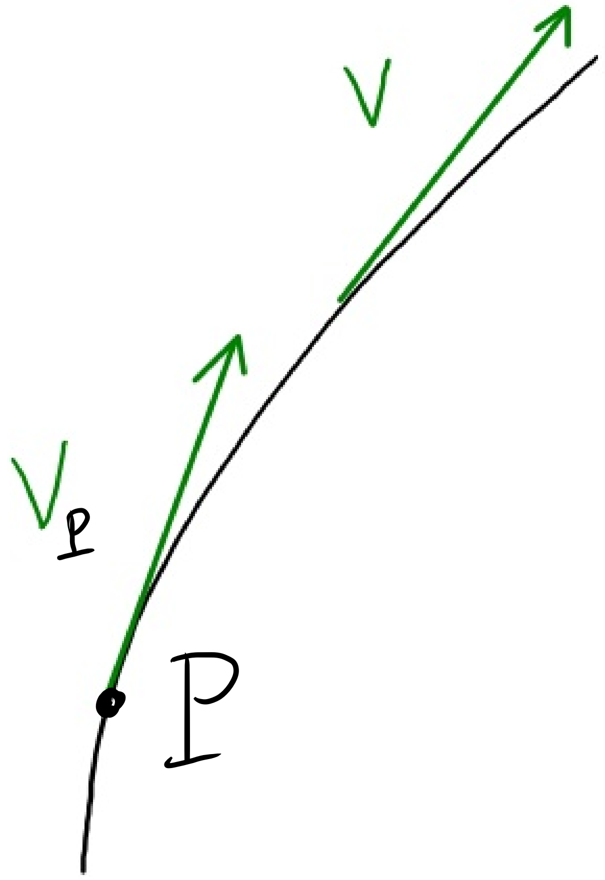

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$\Rightarrow$ $\Gamma^\mu{}_{\nu\rho} = 0$ everywhere $\rightsquigarrow$ straight lines

$\hookrightarrow$ Flat spacetime has straight lines as geodesics!

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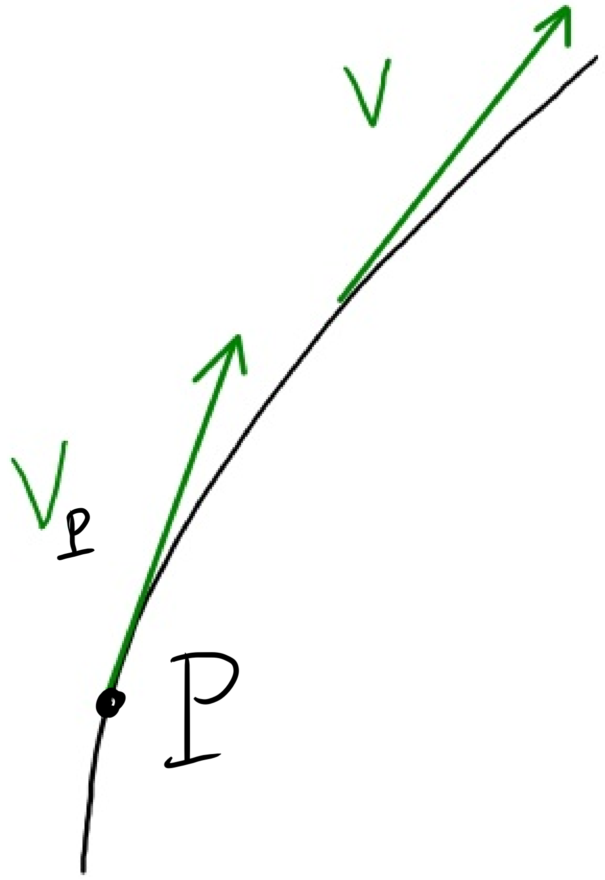

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- Free particles in GR move along geodesics

↳ Equivalence principle: Free particles appear to move on straight lines in inertial frames

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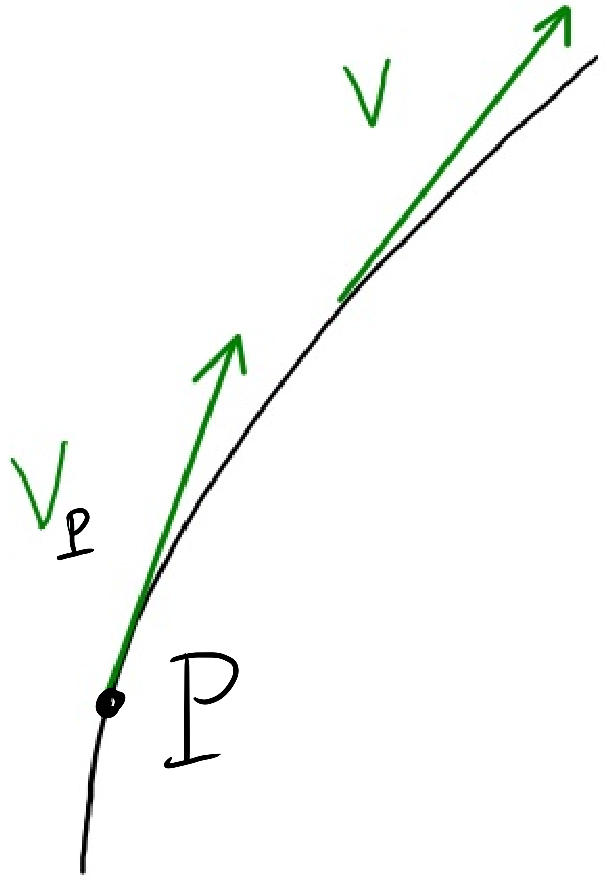

- Free particles in GR move along geodesics

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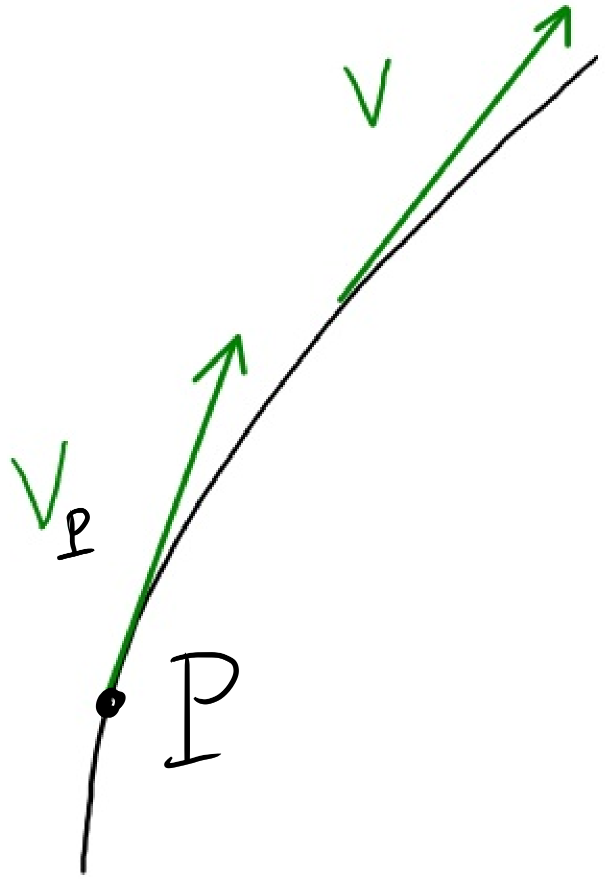

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- Free particles in GR move along geodesics
- Character of geodesics (timelike/null/spacelike) does not change

  Inner product $g_{\mu\nu} V^\mu V^\nu$ is constant: V^μ parallel transported

- Free massive particles: $U^\mu = \frac{dx^\mu}{d\tau}$ 4-velocity

$p^\mu = m U^\mu$

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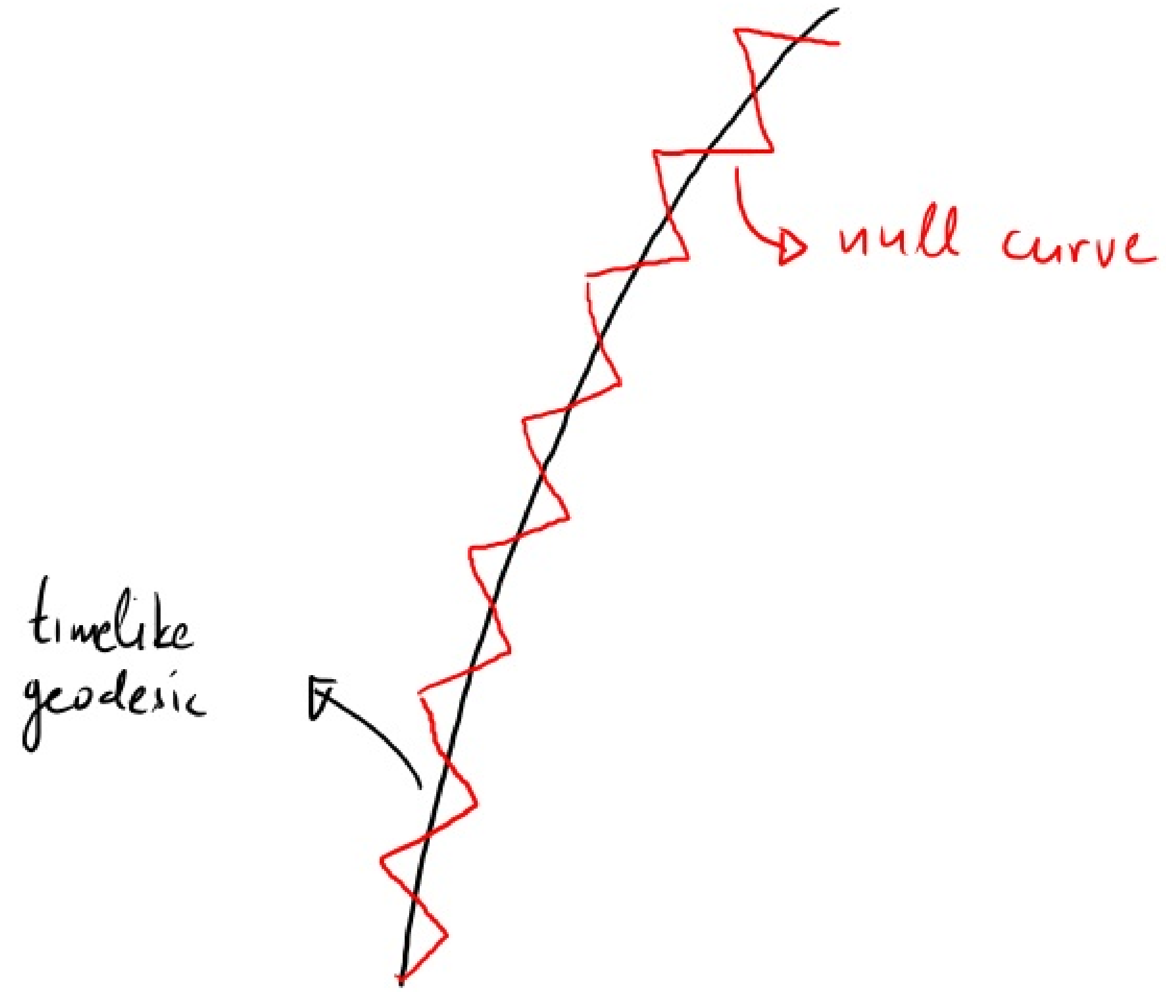
- Free massless particles:

We choose affine parameter λ, s.t. $p^\mu = \frac{dx^\mu}{d\lambda}$

$p^\nu \nabla_\nu p^\mu = 0$

- Timelike geodesics are local maxima of proper time

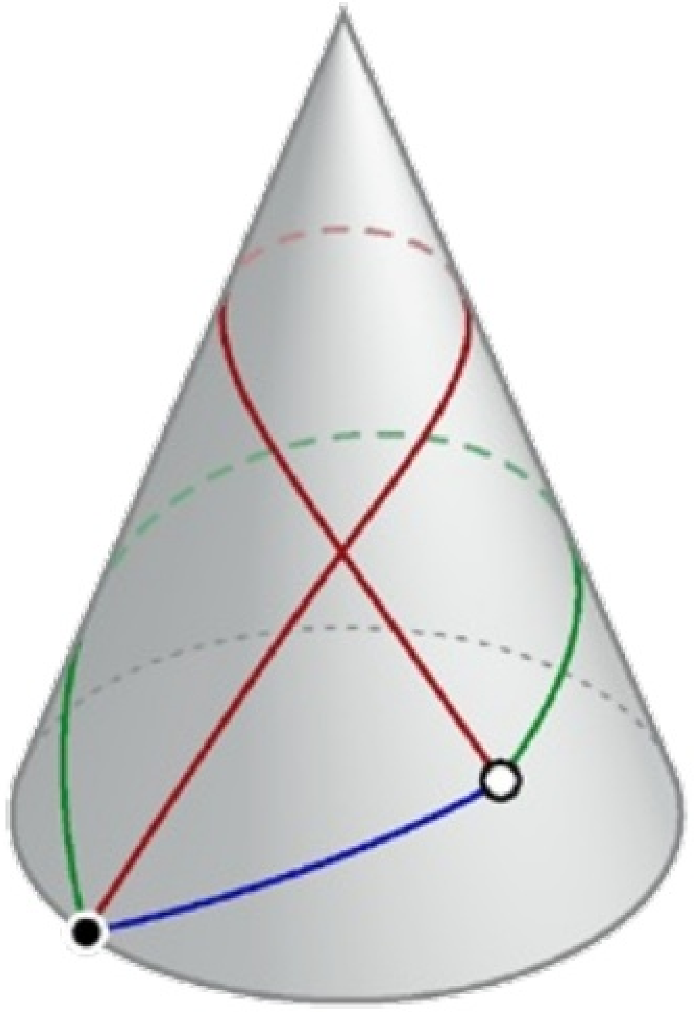
(can't be minima, always arbitrarily close to a null curve – zero length –)



- Timelike geodesics are local maxima of proper time

- Global topology may allow connecting two points with more than one geodesic

  – of different length –



http://www.rdrop.com/~half/Creations/Puzzles/cone.geodesics/index.html

- Timelike geodesics are local maxima of proper time

hence the "local"

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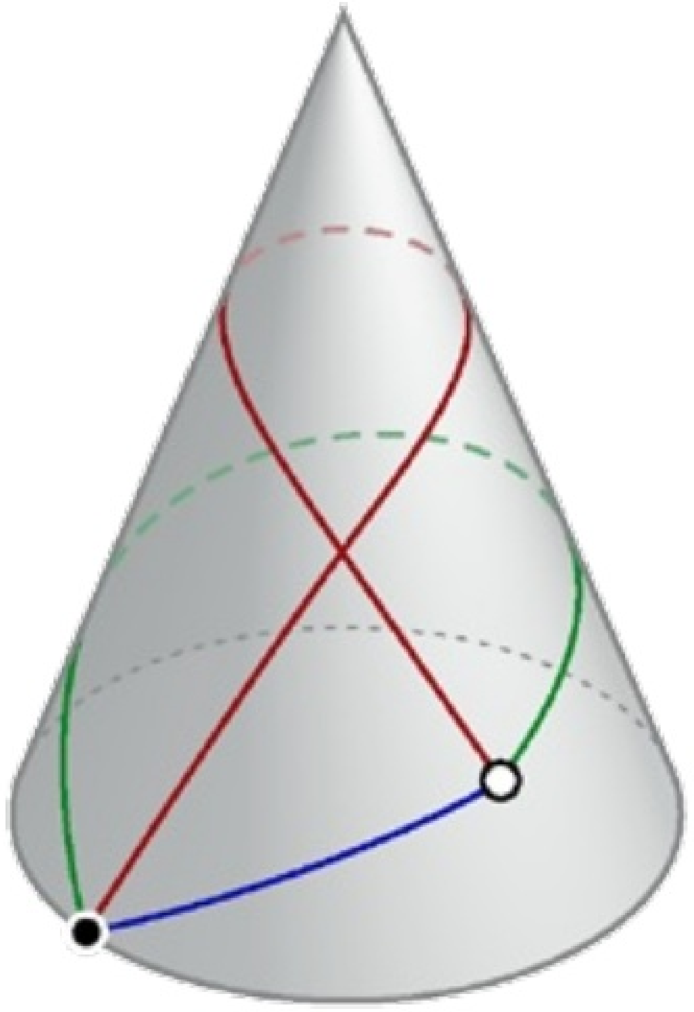

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Extremization of length/proper time → geodesics

Consider a timelike curve, $ds^2 = -d\tau^2 < 0$

$$\tau = \int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

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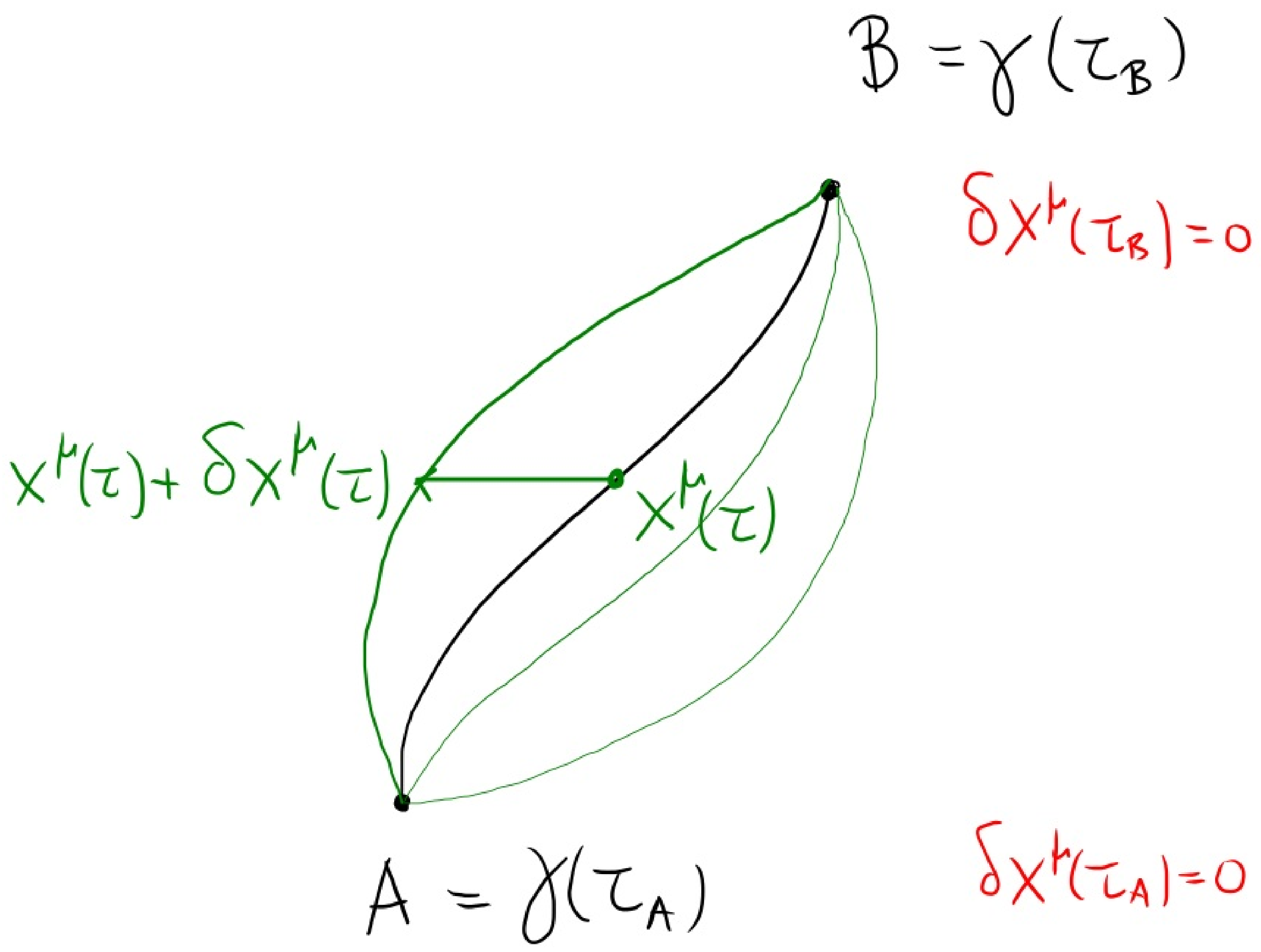

Vary $\gamma(\tau)$ s.t. $\gamma'(\tau)$ is set of points with $x^\mu(\tau) + \delta x^\mu(\tau)$. Then

$$\delta\tau = \int d\tau \, \delta \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

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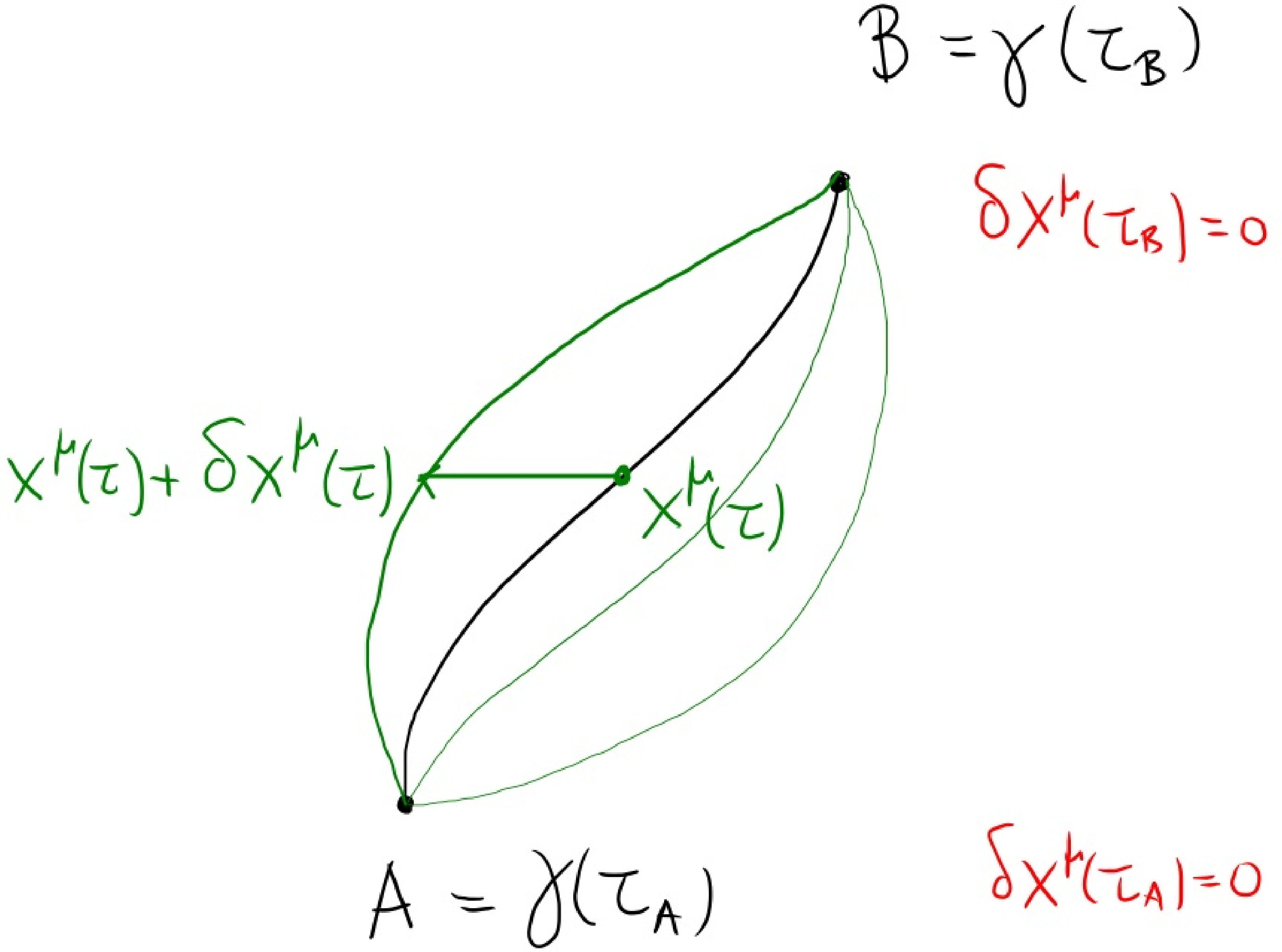

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$$\delta\tau = \int d\tau\, \delta \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

$$= -\frac{1}{2} \int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{-1/2} \delta \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}$$

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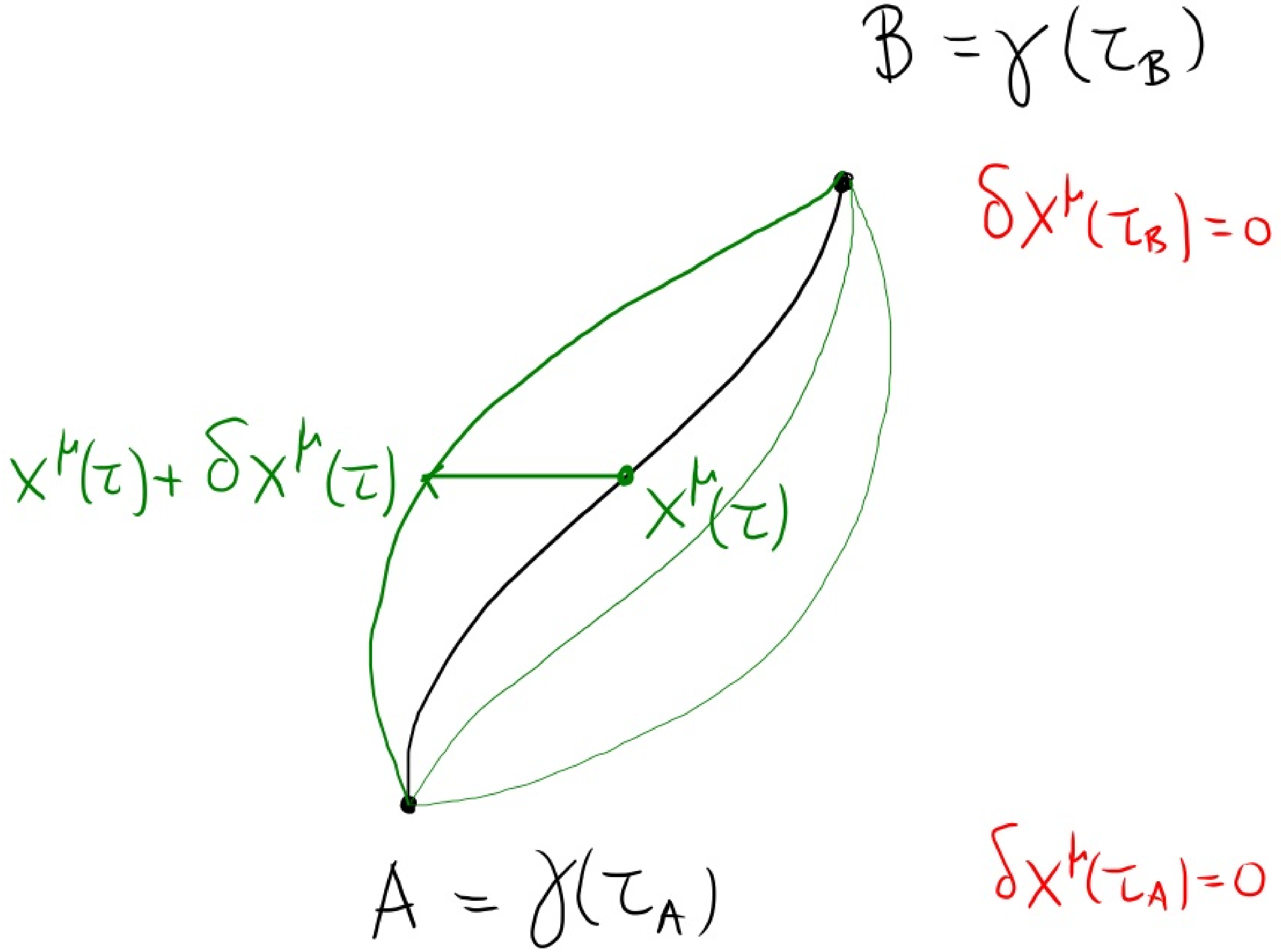

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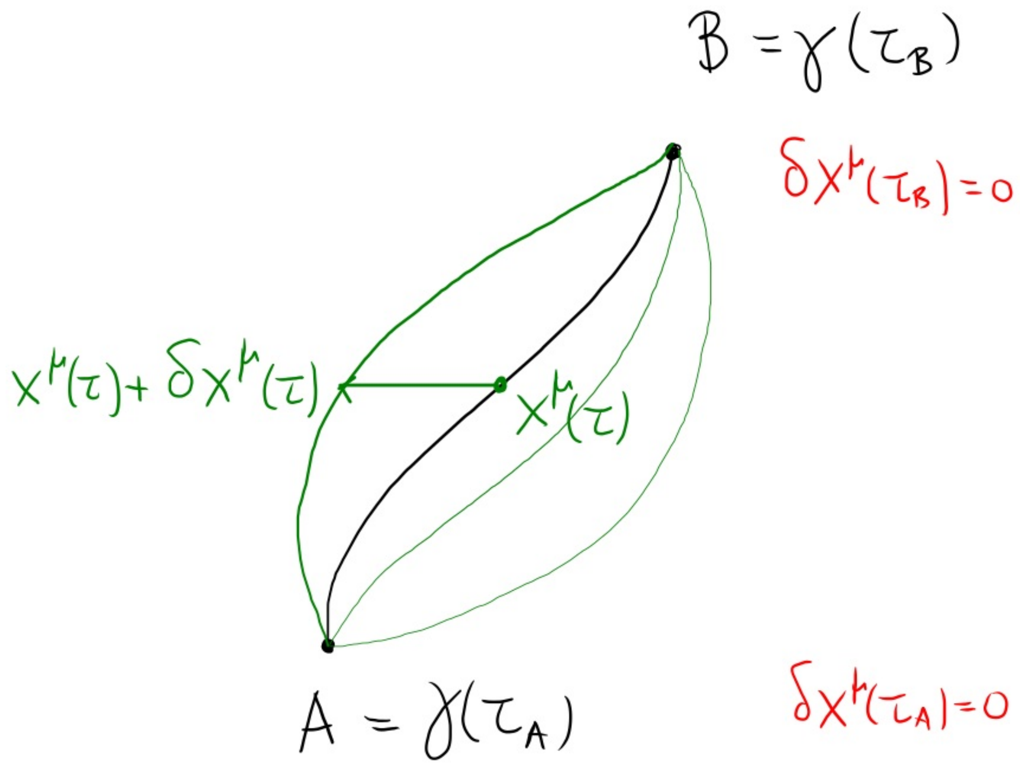
$$= -\frac{1}{2} \int d\tau \, \delta \left\{ g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}, \quad \text{since } g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} U^\mu U^\nu = -1$$

Extremization of length/proper time → geodesics

$$x^\mu \to x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu}\, \delta x^\lambda + \dots$$ (Taylor series expansion)



$$\delta\tau = -\frac{1}{2}\int d\tau\, \delta\left\{ g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right\},$$ since $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}U^\mu U^\nu = -1$

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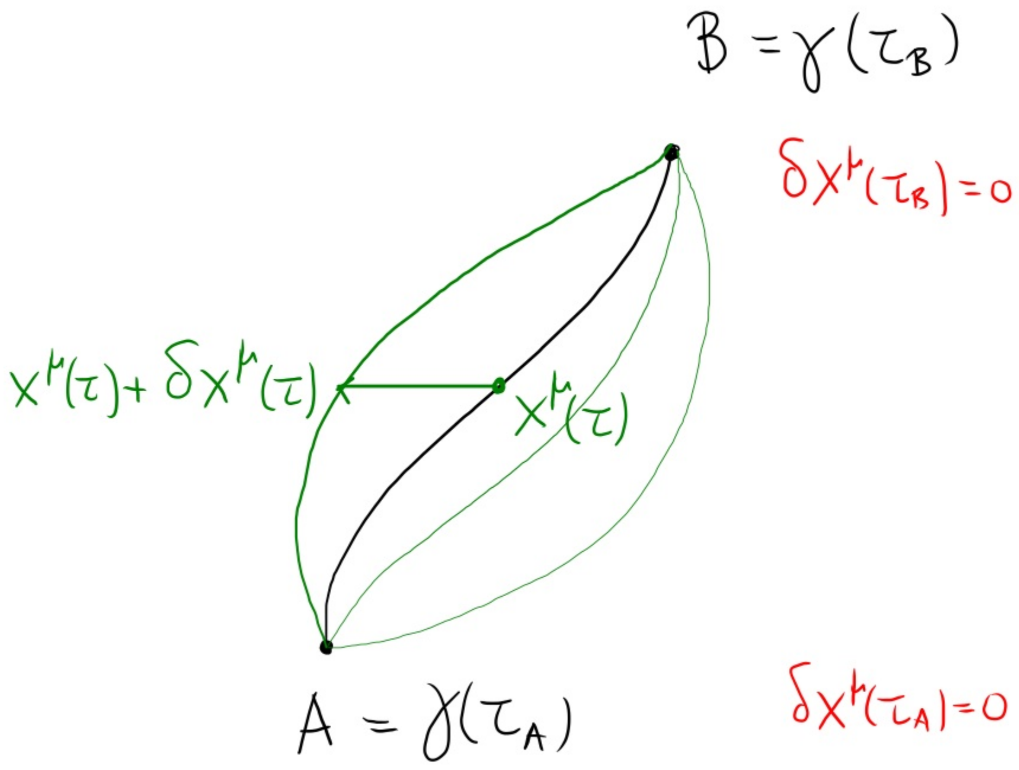

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu}\, \delta x^\lambda + \dots$$ (Taylor series expansion)

$$\int d\tau\, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right) =$$

$$= \int d\tau \left\{ \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu}\, \delta\left(\frac{dx^\mu}{d\tau}\right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau}\, \delta\left(\frac{dx^\nu}{d\tau}\right) \right\}$$

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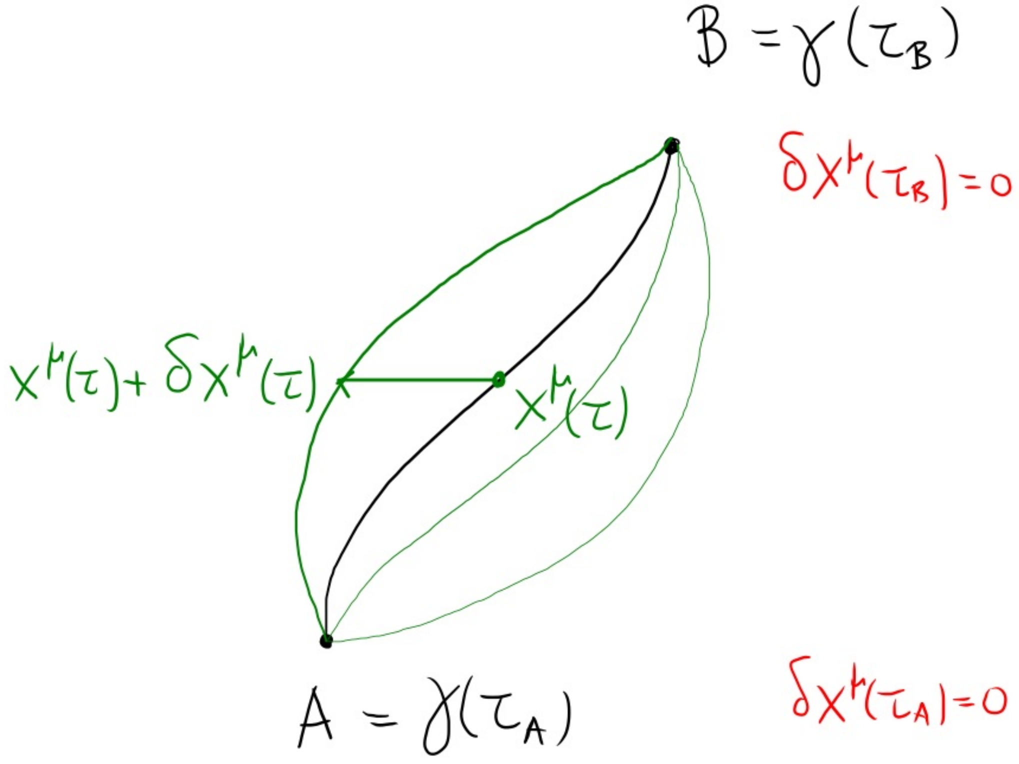

$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu}\, \delta x^\lambda + \dots \qquad \text{(Taylor series expansion)}$$

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- symmetric under $\mu \leftrightarrow \nu$
- $\delta\left(\frac{dx^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\delta x^\mu\right)$

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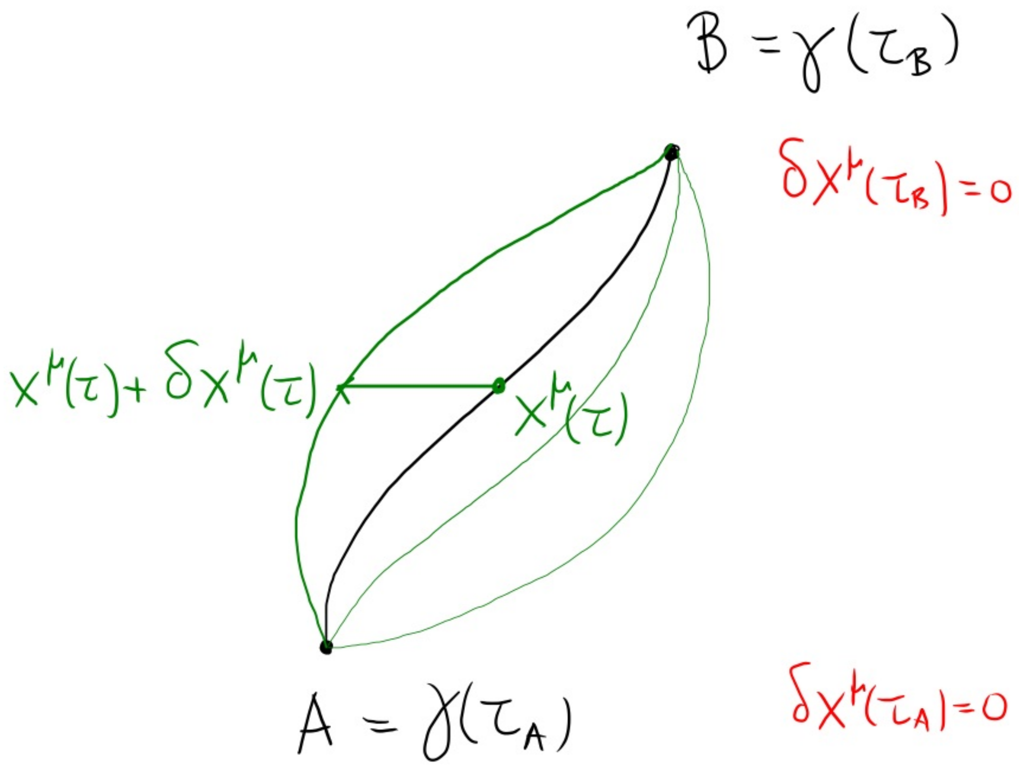

$$\int d\tau\, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) =$$

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$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu) \right\}$$

Extremization of length/proper time → geodesics

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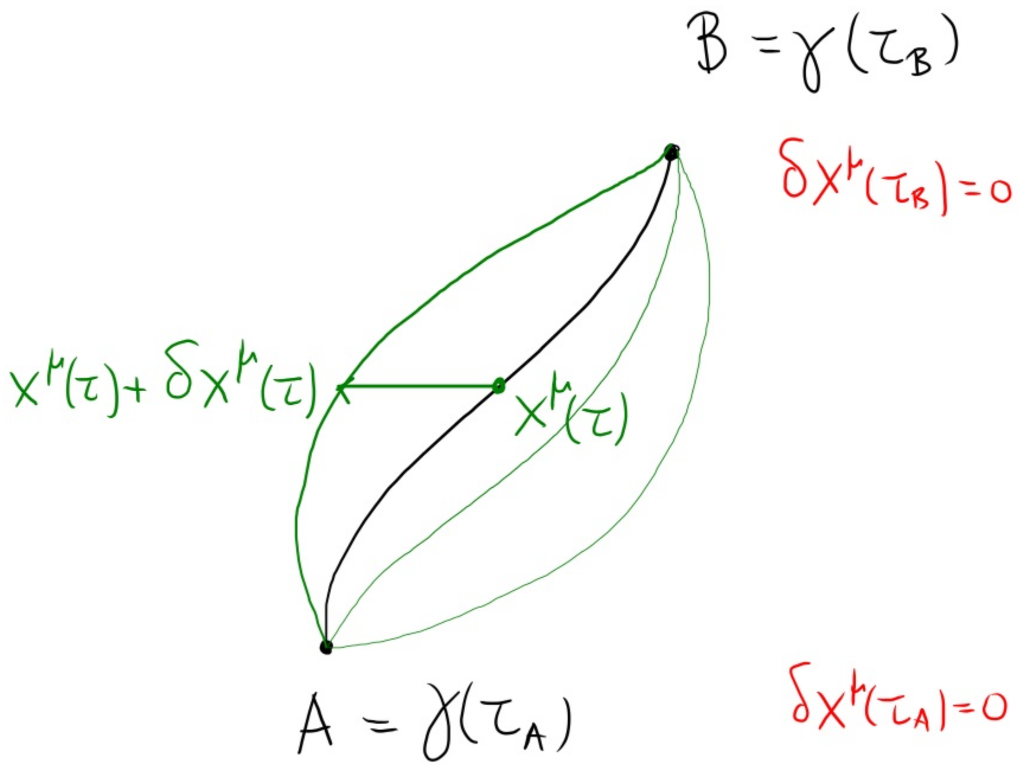

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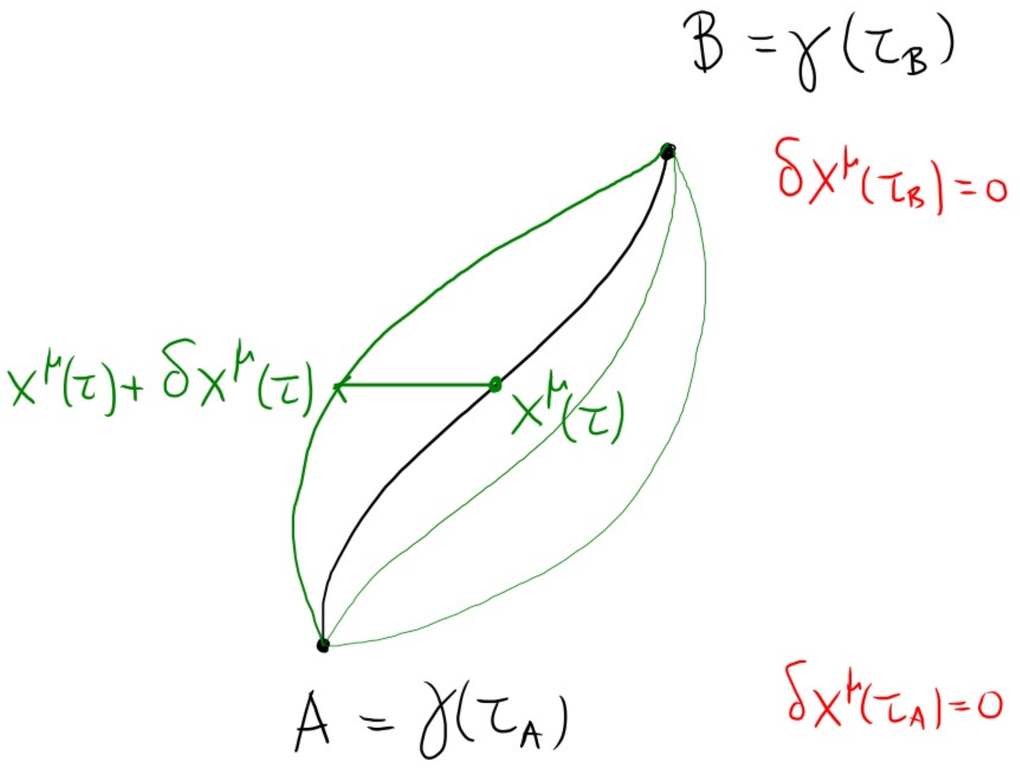
But $2g_{\mu\nu} \dfrac{dx^\mu}{d\tau} \dfrac{d}{d\tau} \delta x^\nu = \dfrac{d}{d\tau}\left[2 g_{\mu\nu} \dfrac{dx^\mu}{d\tau} \delta x^\nu\right] - \dfrac{d}{d\tau}\left[2 g_{\mu\nu} \dfrac{dx^\mu}{d\tau}\right] \delta x^\nu$

Extremization of length/proper time → geodesics

$$x^\mu \to x^\mu + \delta x^\mu$$



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$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu) \right\}$$

$$\int_{\tau_A}^{\tau_B} d\tau\, 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu = \left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\, \delta x^\nu \right]\Bigg|_{\tau_A}^{\tau_B} - \int_{\tau_A}^{\tau_B} \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu\, d\tau$$

Extremization of length/proper time → geodesics

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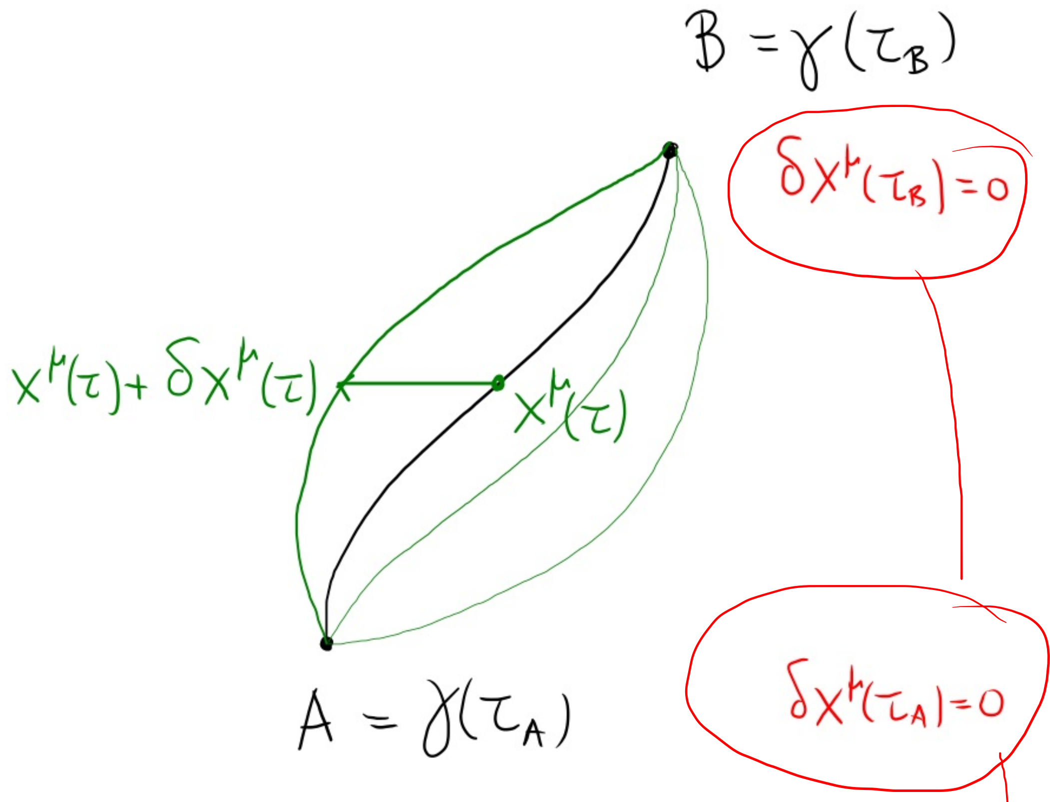


$$g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu}\, \delta x^\lambda + \dots \quad \text{(Taylor series expansion)}$$

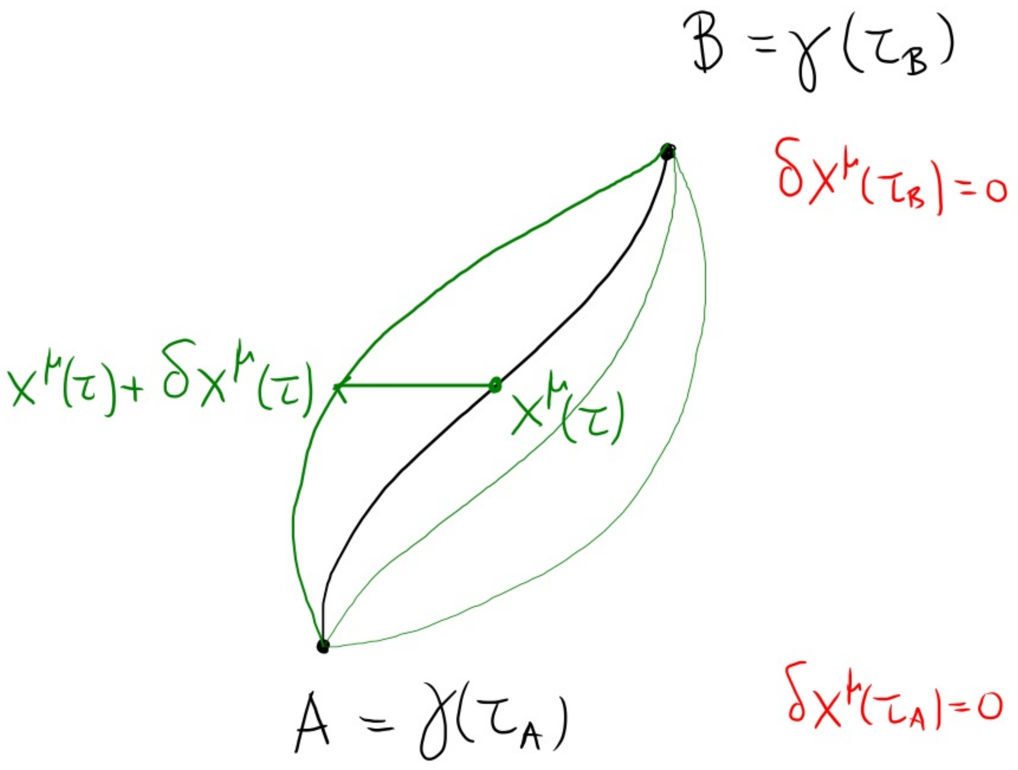
$$\int d\tau\, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\right) =$$

$$= \int d\tau \left\{ \delta g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\, \delta\left(\frac{dx^\mu}{d\tau}\right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau}\, \delta\left(\frac{dx^\nu}{d\tau}\right) \right\}$$

$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu) \right\}$$

$$\int_{\tau_A}^{\tau_B} d\tau\, 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu = \left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \cancel{\delta x^\nu} \right]\Bigg|_{\tau_A}^{\tau_B} - \int_{\tau_A}^{\tau_B} \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu\, d\tau$$

Extremization of length/proper time → geodesics



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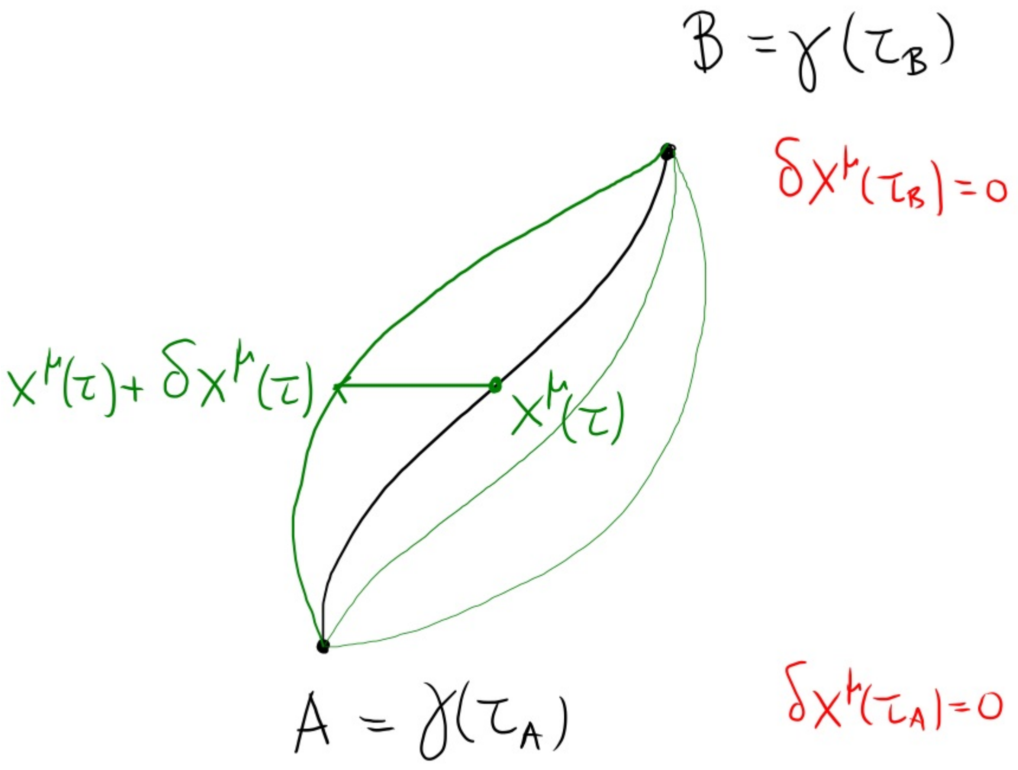
Extremization of length/proper time → geodesics



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$\lambda \leftrightarrow \nu$

$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau}\right] \delta x^\nu \right\}$$

Extremization of length/proper time → geodesics

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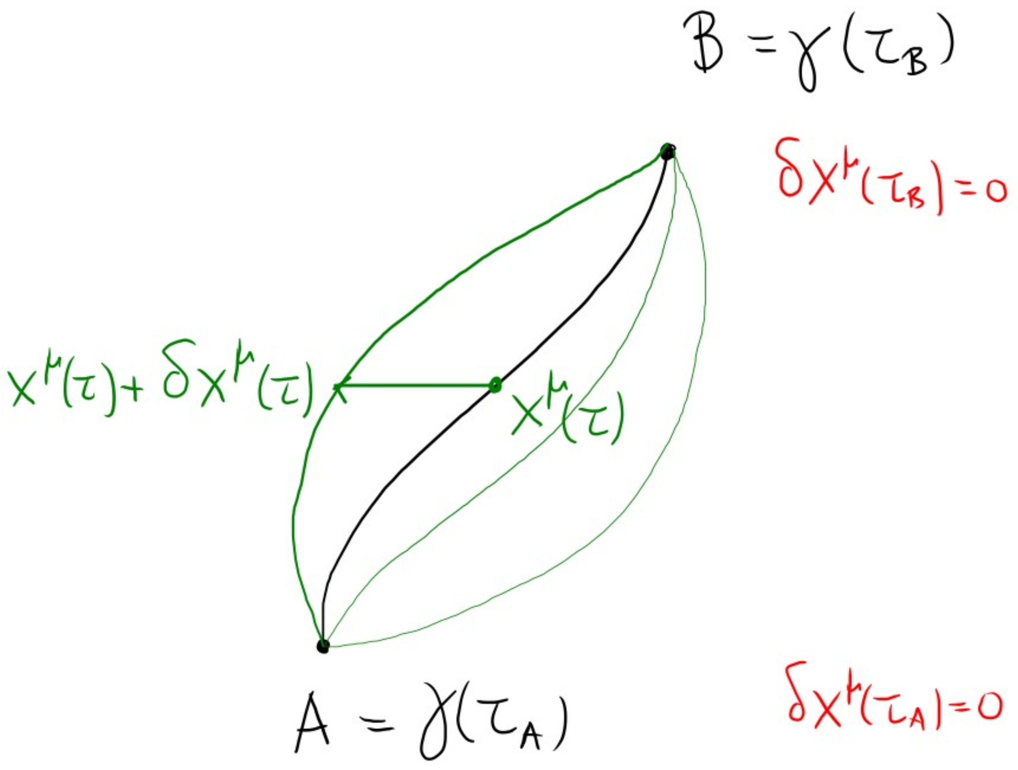

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Extremization of length/proper time → geodesics

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Extremization of length/proper time → geodesics

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Extremization of length/proper time → geodesics

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Extremization $\delta\tau = 0$ for any δx^μ implies the integrant is zero:

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Extremization of length/proper time → geodesics

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$$g^{\sigma\mu} g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} g^{\sigma\mu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right) \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

Extremization of length/proper time → geodesics

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma{}_{\mu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

$$\underbrace{g^{\sigma\mu} g_{\mu\nu}}_{\delta^\sigma{}_\nu} \frac{d^2 x^\mu}{d\tau^2} + \underbrace{\frac{1}{2} g^{\sigma\mu}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right)}_{\Gamma^\sigma{}_{\mu\lambda}} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

Example: Flat space cosmology

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Example: Flat space cosmology

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$$\Rightarrow \quad \begin{aligned} &\Gamma^0{}_{00} = \Gamma^0{}_{0i} = \Gamma^0{}_{i0} = 0 \\ &\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij} \end{aligned}$$

Example: Flat space cosmology

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] \qquad x^i \to x^i + \delta x^i$$

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$i \leftrightarrow j$ symmetric $i \leftrightarrow j$

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$$\delta x^j(\tau_A) = \delta x^j(\tau_B) = 0$$

$$= \int d\tau\; a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\frac{d}{d\tau}\delta x^j \qquad a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\delta x^j\Big|_{\tau_A}^{\tau_B}$$

$$//$$

$$= -\int d\tau \frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\right]\delta x^j + \int d\tau \frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\delta x^j\right]$$

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$$= -\int d\tau\,\frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\right]\delta x^j$$

$$= -\int d\tau\left[2a\frac{da}{d\tau}\frac{dx^i}{d\tau} + a^2\frac{d^2x^i}{d\tau^2}\right]\delta_{ij}\,\delta x^j$$

Example: Flat space cosmology

But $\dfrac{da}{d\tau} = \dfrac{da}{dt}\dfrac{dt}{d\tau} = \dot{a}\dfrac{dt}{d\tau}$, so $\delta I = 0 \ \forall\, \delta x^j$ implies

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Example: Flat space cosmology

$$\Rightarrow \quad \Gamma^i{}_{00} = \Gamma^i{}_{jk} = 0$$

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Null geodesics: consider photons moving on +X axis

$$x^{\mu}(\lambda) = (t(\lambda), x(\lambda), 0, 0)$$

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solution: $\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)}$

- consider comoving observers: $U^\mu = (1,0,0,0)$

$$E_{photon} = -p_\mu U^\mu = -g_{00}\, p^0\, U^0 = -g_{00} \frac{dx^0}{d\lambda} \cdot 1 = +\frac{\omega_0}{a}$$

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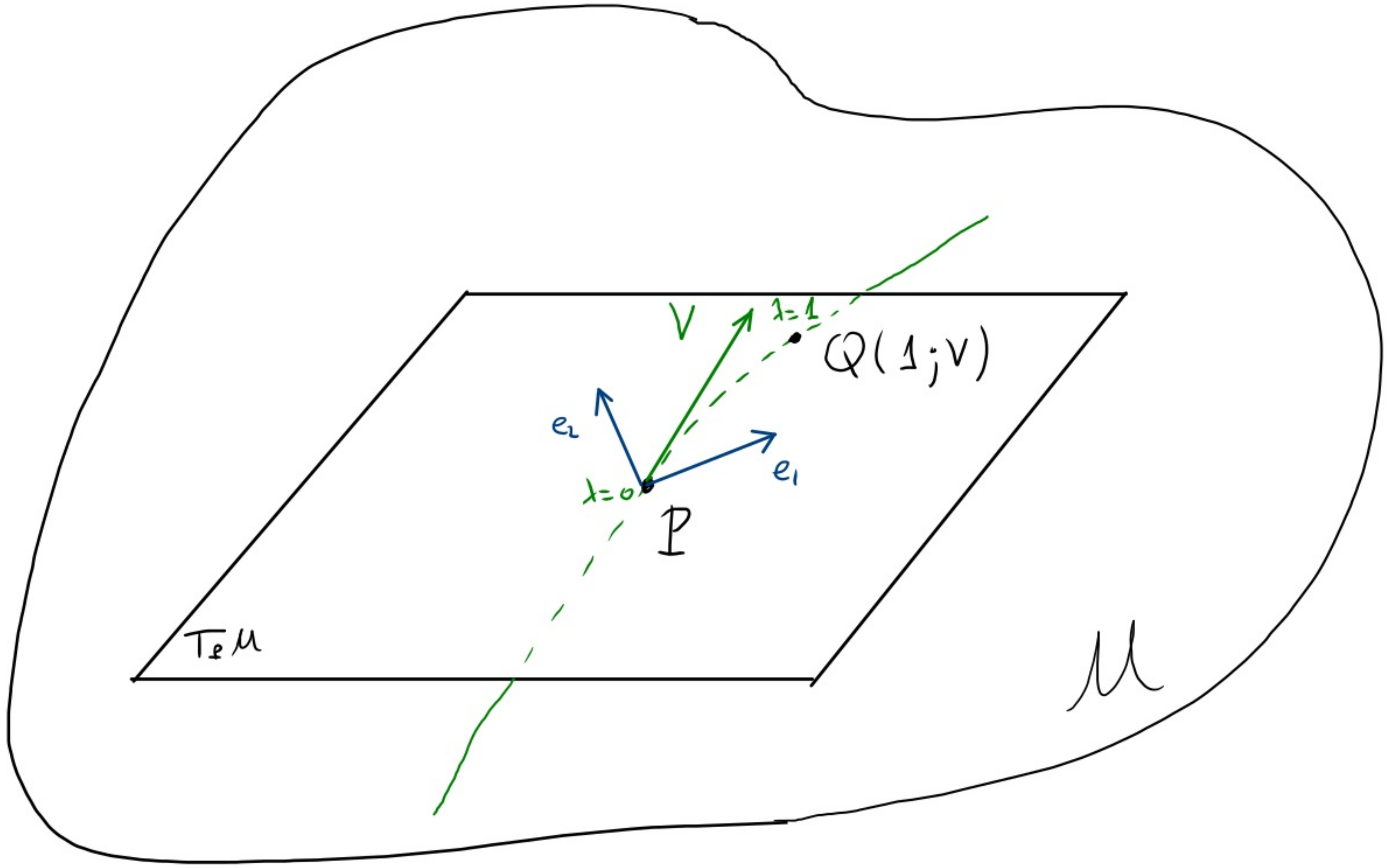
$$\frac{\omega_2}{\omega_1} = \frac{\omega_0/a_2}{\omega_0/a_1} = \frac{a_1}{a_2}$$ cosmological redshift

<u>Note:</u>
- it is the energy of the photon that is reduced
- photon is not "stretched"
- the relation of E and ω is quantum mechanical

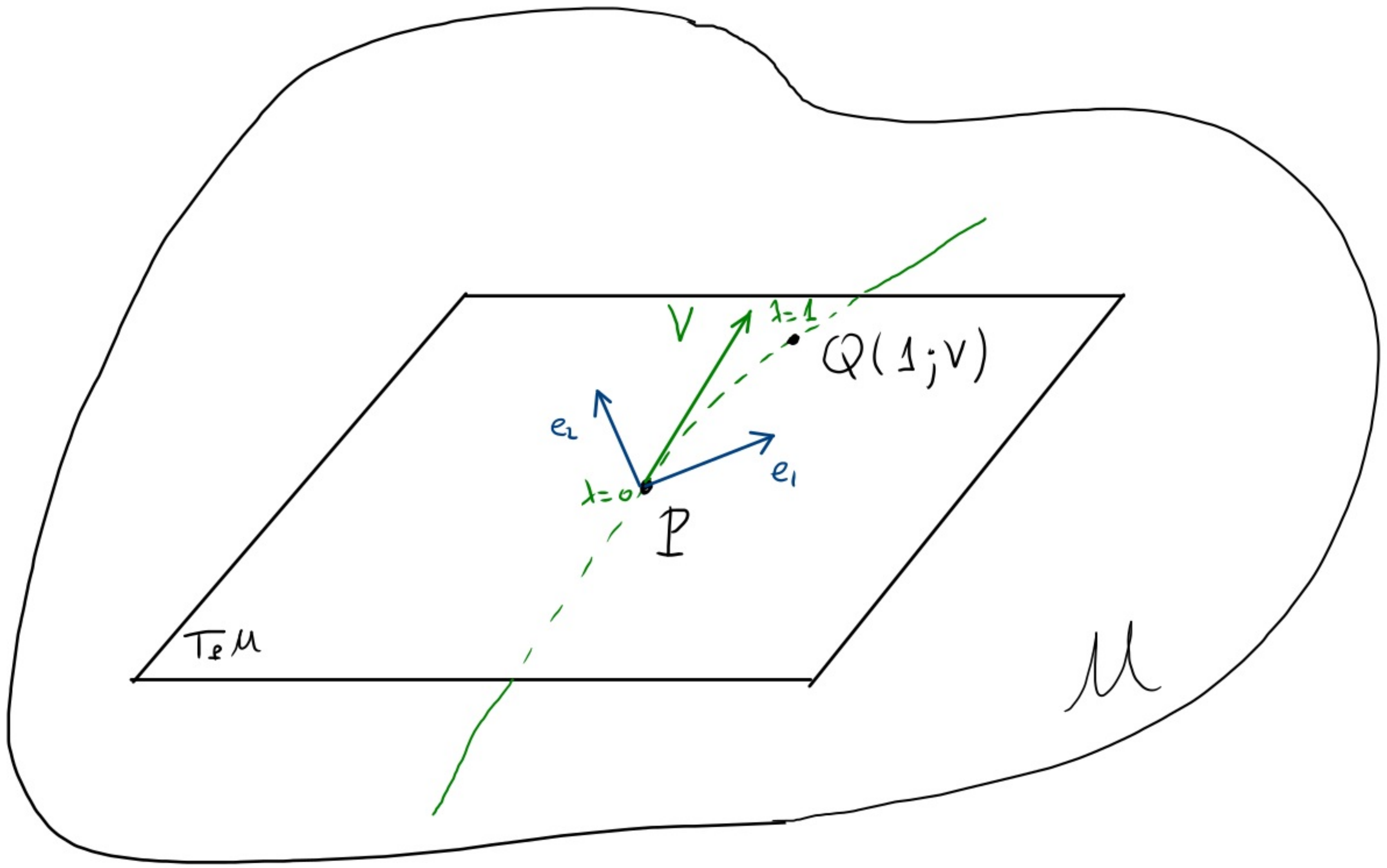
Riemann Normal Coordinates

Construct local inertial frame using geodesics:

- Pick an event P and consider all geodesics through $\underline{P}$



Riemann Normal Coordinates



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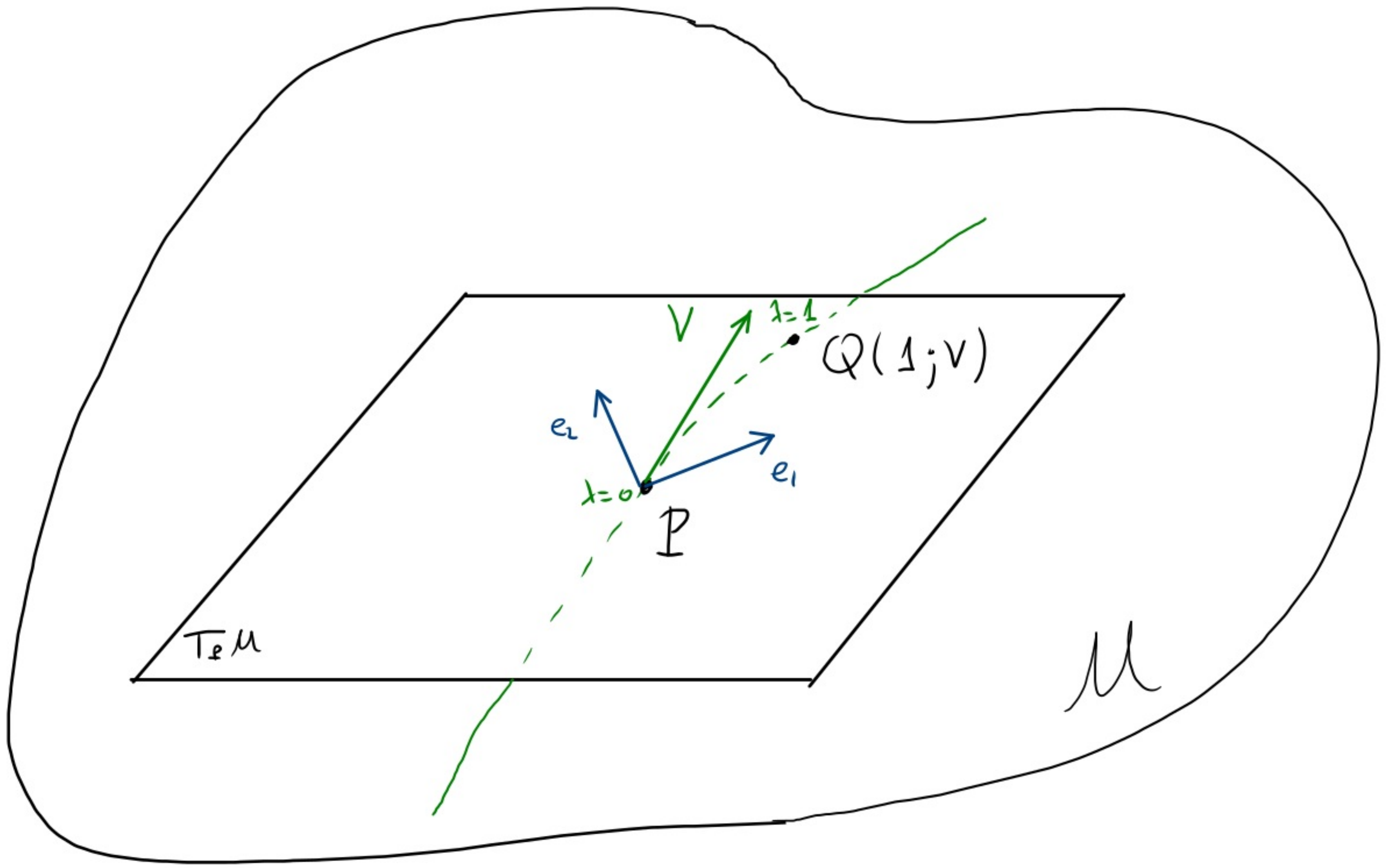
Consider the point $\quad Q = Q(\lambda; V)$

where on geodesic

which geodesic

Riemann Normal Coordinates



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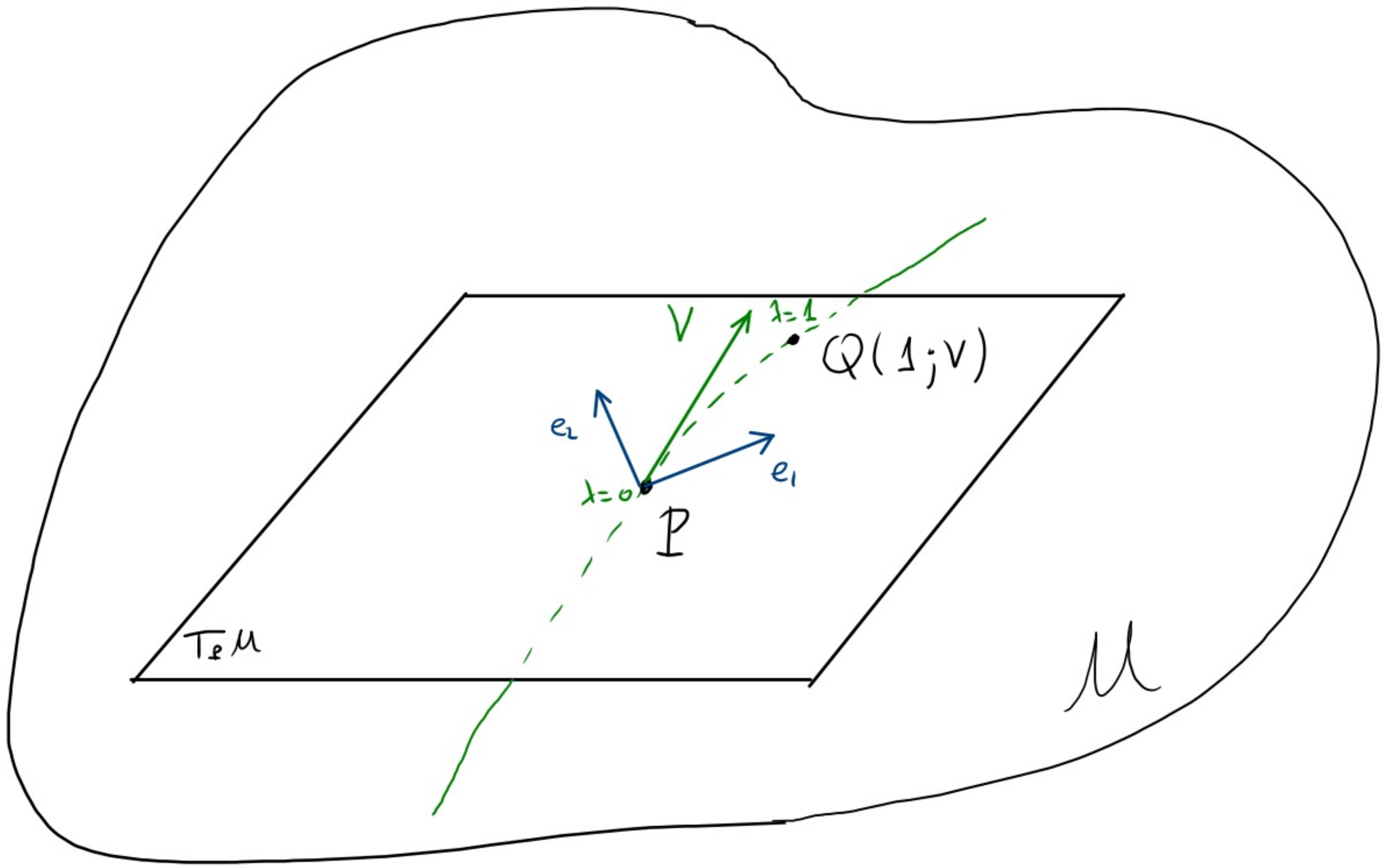

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- As long as geodesics don't intersect, we have 1-1 map of points + vectors

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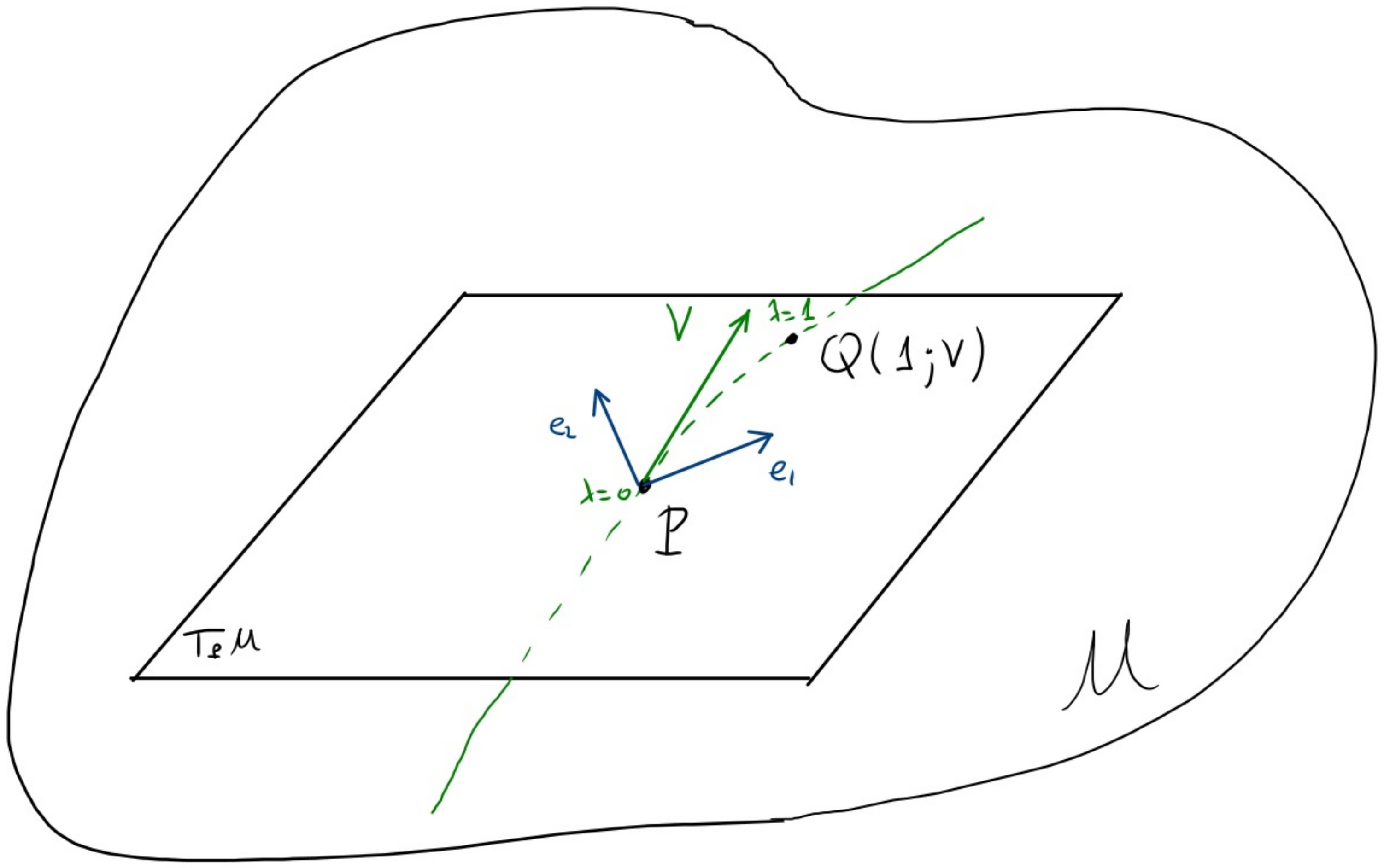
⇒ Choose orthonormal basis $\{e_\mu\}$ at $T_P\mathcal{M}$

If $V = x^\mu e_\mu$, define coordinates of $Q(1;V)$ to be $\{x^\mu\}$



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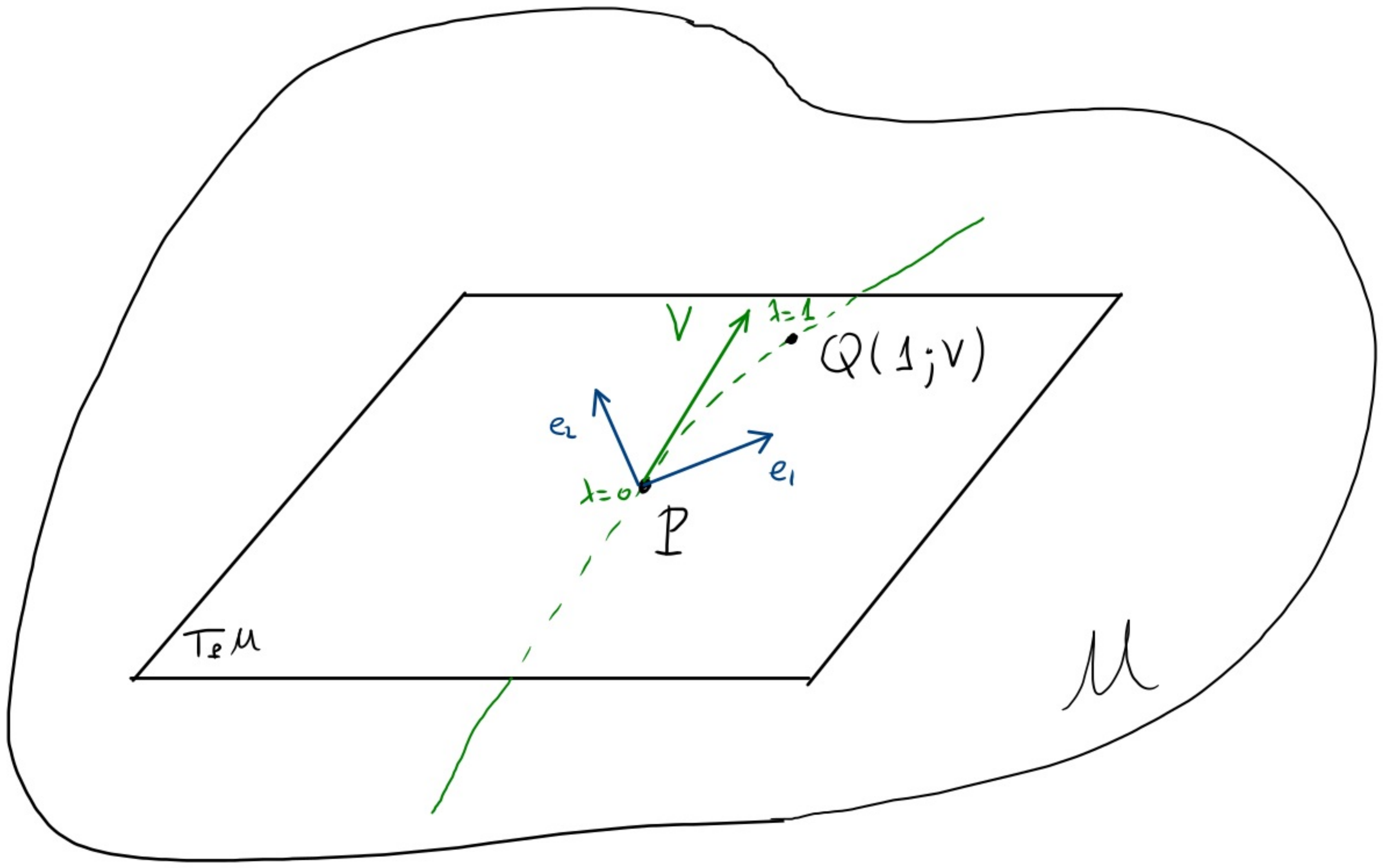

Then:

(a) $e_\mu = \partial_\mu \big|_P$

Riemann Normal Coordinates

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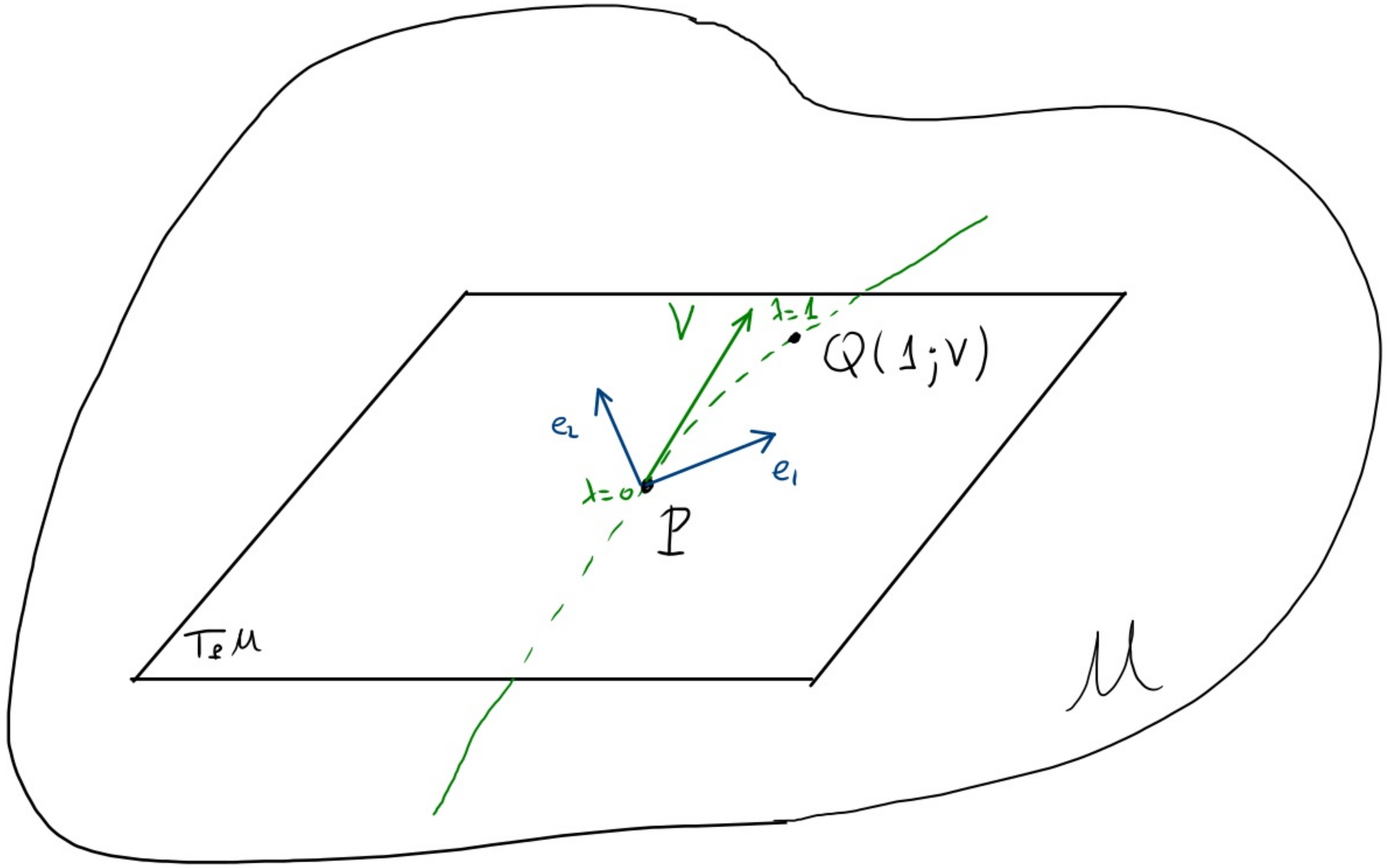

Then:

(a) $e_\mu = \partial_\mu \big|_P$

(b) $\Gamma^\mu{}_{\nu\rho}(P) = 0$

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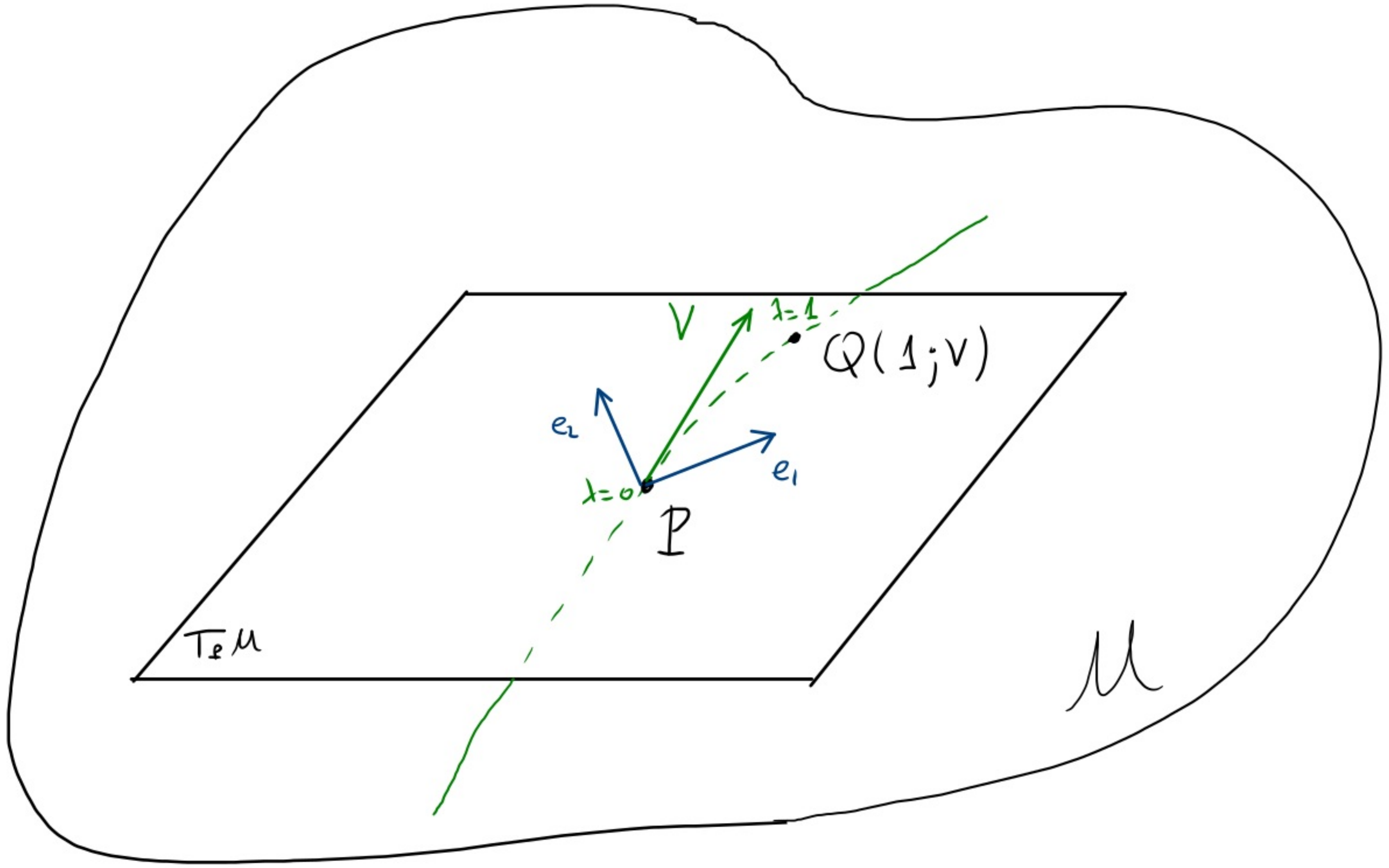

Then:

(a) $e_\mu = \partial_\mu \big|_P$

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(c) $g_{\mu\nu}(P) = \eta_{\mu\nu}$

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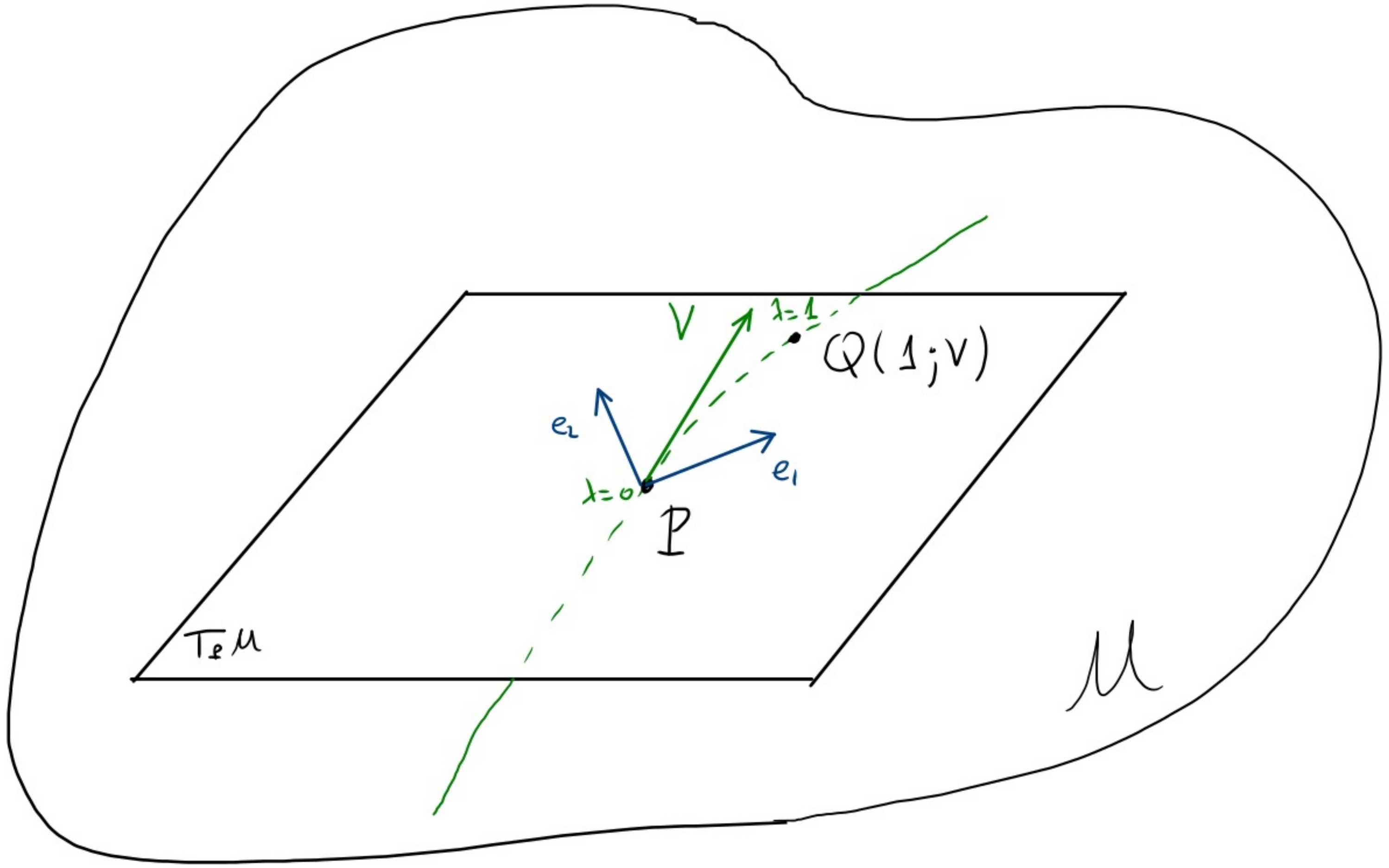

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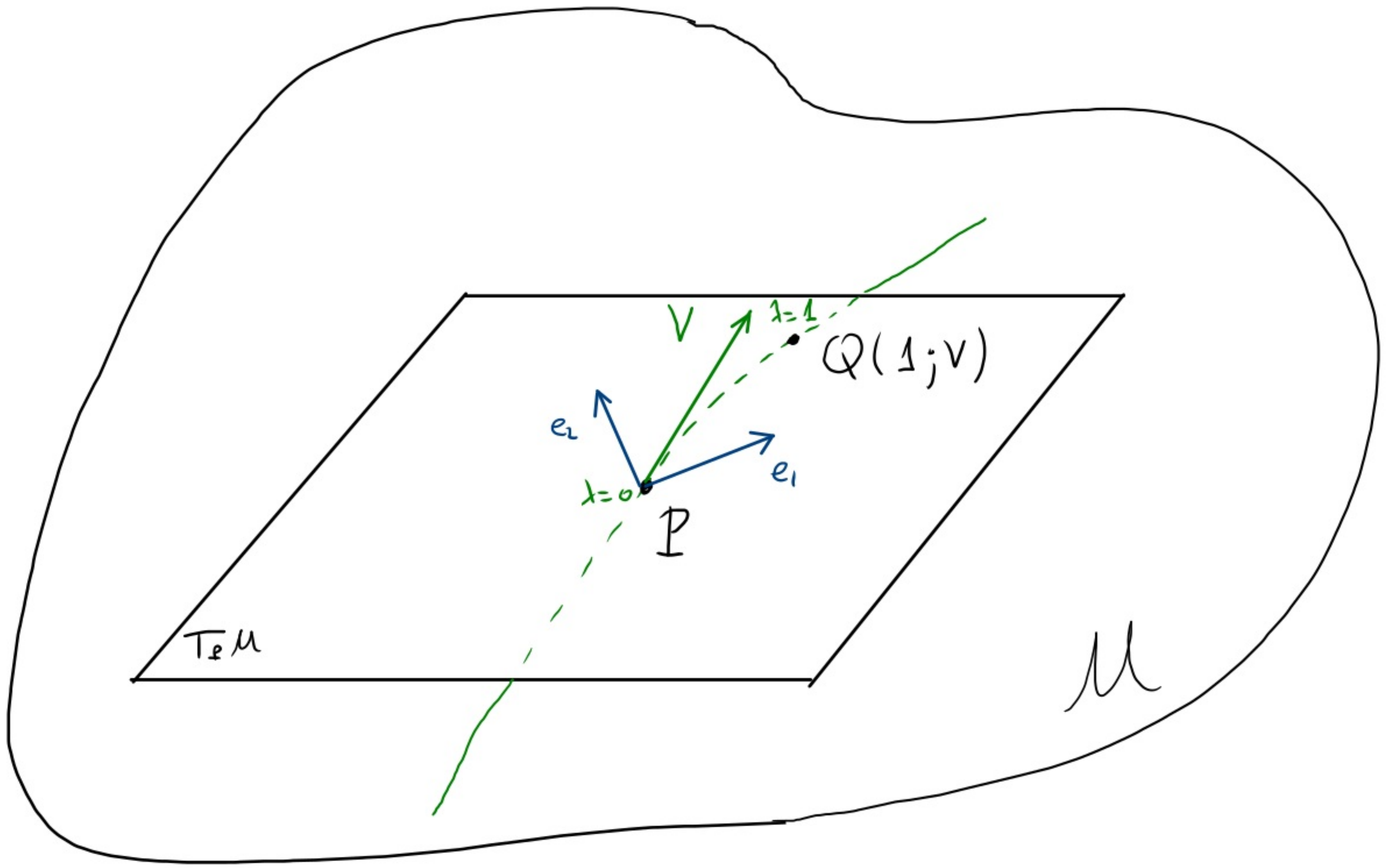
Riemann Normal Coordinates



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Higher derivatives determined by curvature:

$$\partial_\rho \partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right)$$

$$\partial_\sigma \Gamma^\mu{}_{\nu\rho}(P) = -\frac{1}{3}\left(R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\nu\sigma}\right)$$

$$R_{\mu\nu\rho\sigma}(P) = \partial_\rho \partial_\nu g_{\mu\sigma}(P) - \partial_\sigma \partial_\nu g_{\mu\rho}(P)$$

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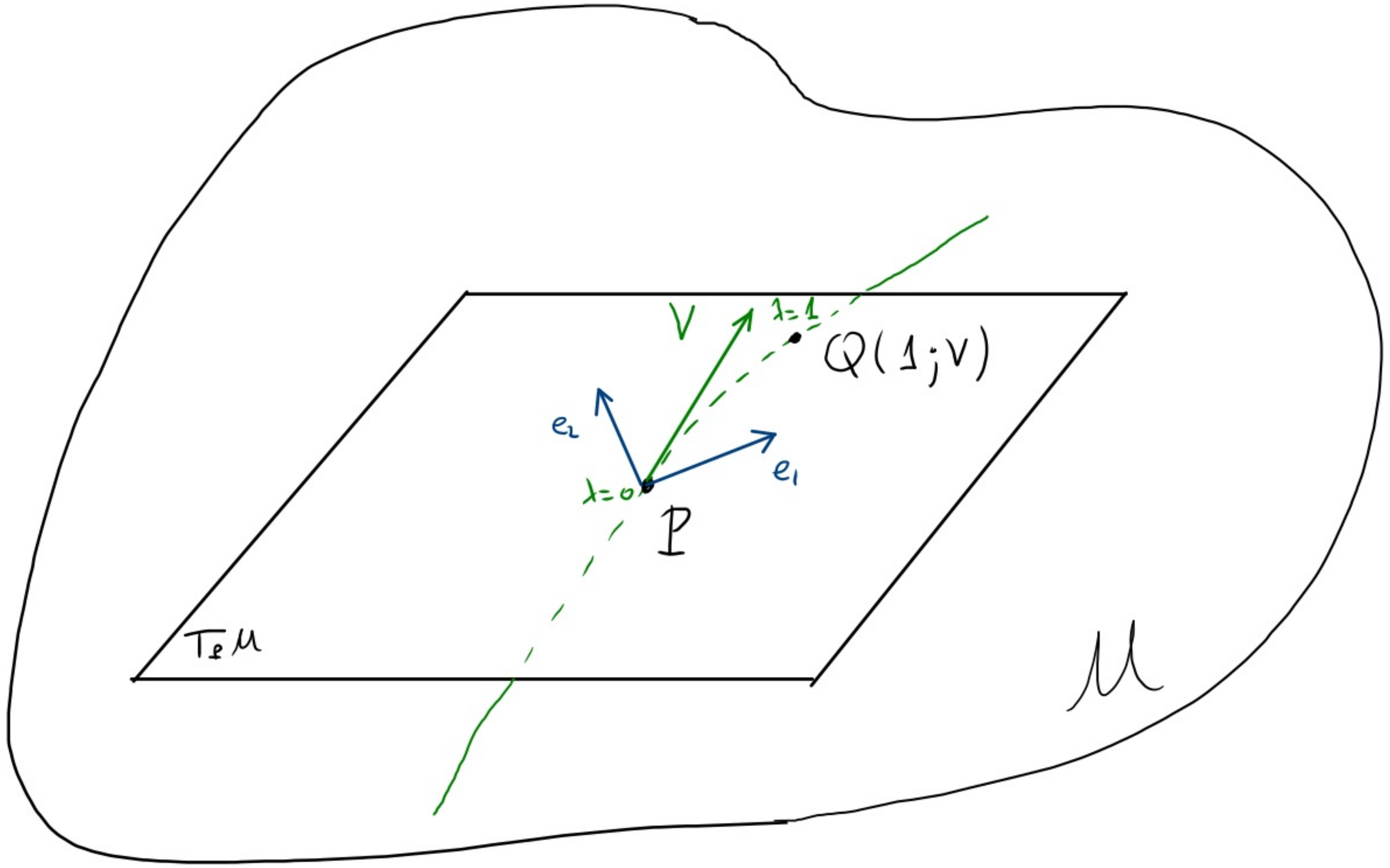

- Any other normal coordinate system same to 2nd order

$$x^{\mu\,\prime}(P) = x^\mu(P) + \mathcal{O}((x^\mu)^3)$$

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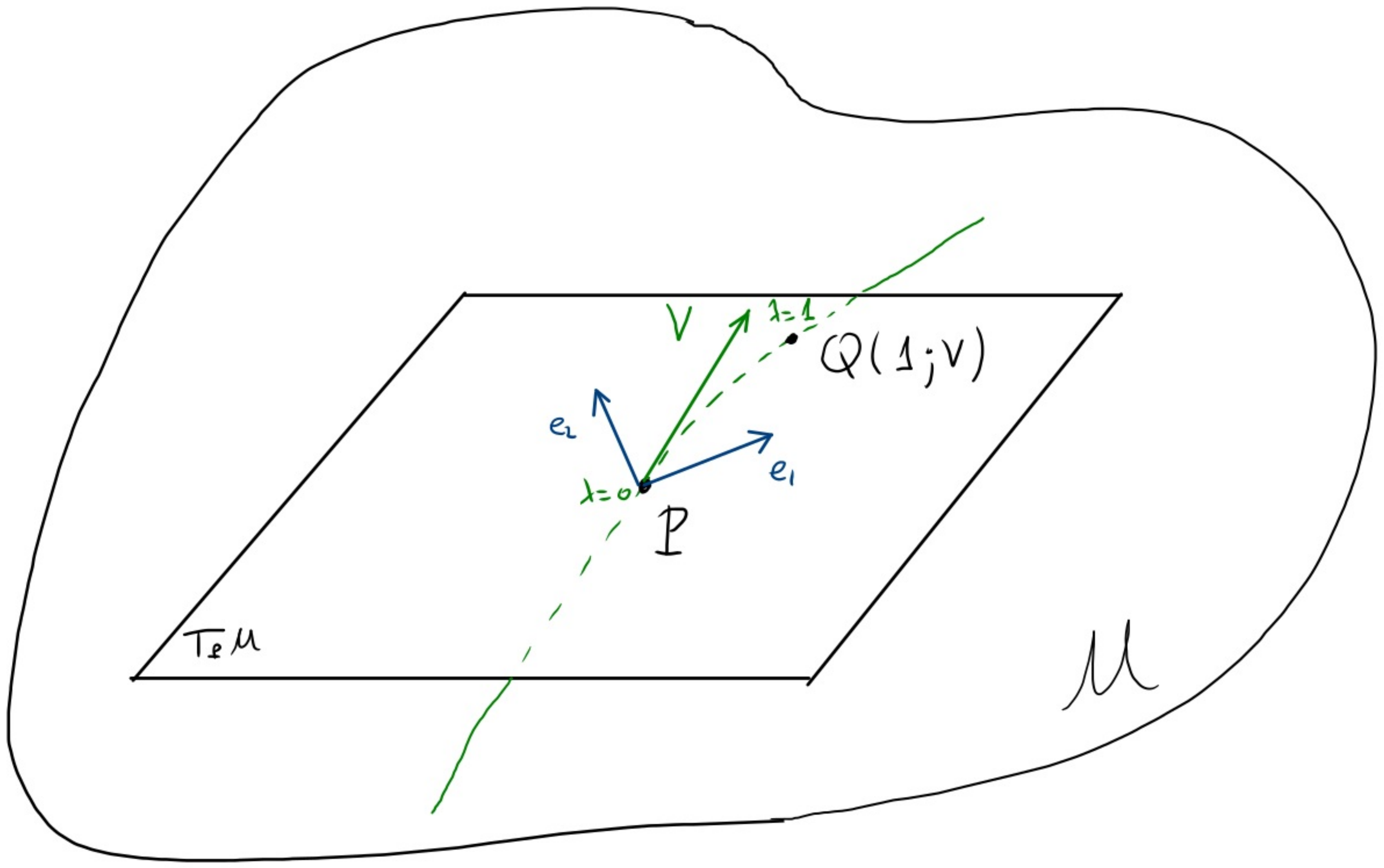

- Any other normal coordinate system same to 2nd order

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- If same to 3rd order, Eqs (1)+(2) are preserved

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Example: Wormhole geometry (Hartle, examples 8.2, 8.3, 8.5)

$$ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$g_{tt} = -1 \qquad g_{rr} = 1$

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Only non-vanishing derivatives:

$$\partial_r g_{\theta\theta} = 2r$$

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easier: only g_{tt}, g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$

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$\mu\nu\rho$ + cyclic

only permutations of $r\theta\theta$, $r\phi\phi$, $\theta\phi\phi$ may appear

$g_{rr} = 1 \qquad g_{\theta\theta} = b^2 + r^2 \qquad g_{\phi\phi} = (b^2+r^2)\sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$

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$g_{rr} = 1 \quad g_{\theta\theta} = b^2+r^2 \quad g_{\phi\phi} = (b^2+r^2)\sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \quad \partial_r g_{\phi\phi} = 2r\sin^2\theta \quad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$

$$g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g\,\Gamma = \frac{1}{2}\left(\partial g + \partial g - \partial g\right)$$

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$$g_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

permute cyclically: start from 3rd term, then 2nd, then 1st

$$g_r \Gamma_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \qquad r \to \theta \to \theta$$

$$g_\theta \Gamma_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \qquad \theta \to r \to \theta$$

$$g_r \Gamma_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \qquad r \to \phi \to \phi$$

$$g_\phi \Gamma_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \qquad \phi \to r \to \phi$$

$$g_\theta \Gamma_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \qquad \theta \to \phi \to \phi$$

$$g_\phi \Gamma_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right) \qquad \phi \to \theta \to \phi$$

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$$g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

no other choice : $g_{\mu\nu}$ is diagonal

$$g_{rr}\Gamma^r_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right)$$

$$g_{\theta\theta}\Gamma^\theta_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$$

$$g_{rr}\Gamma^r_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^\phi_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

$$g_{\theta\theta}\Gamma^\theta_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

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$$g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$$

$$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right)$$

Non-zero terms

$$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$$

$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right)$$

$g_{rr} = 1 \qquad g_{\theta\theta} = b^2 + r^2 \qquad g_{\phi\phi} = (b^2+r^2)\sin^2\theta$

$\partial_r g_{\theta\theta} = 2r \qquad \partial_r g_{\phi\phi} = 2r\sin^2\theta \qquad \partial_\theta g_{\phi\phi} = 2(b^2+r^2)\sin\theta\cos\theta$

$g_{\mu\sigma}\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho}\right)$

$g_{rr}\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1\cdot\Gamma^{r}{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^{r}{}_{\theta\theta} = -r$

$g_{\theta\theta}\Gamma^{\theta}{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right)$

$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$

$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$

$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$

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$$g_{\theta\theta}\Gamma^{\theta}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \Rightarrow (b^2+r^2)\Gamma^{\theta}_{r\theta} = \frac{1}{2}\cdot 2r \Rightarrow \Gamma^{\theta}_{r\theta} = \frac{r}{b^2+r^2}$$

$$g_{rr}\Gamma^{r}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right)$$

$$g_{\phi\phi}\Gamma^{\phi}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right)$$

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$$g_{rr}\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^{r}{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

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$$g_{rr}\Gamma^r{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1\cdot\Gamma^r{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^r{}_{\theta\theta} = -r$$

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$$g_{rr}\Gamma^r{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^r{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\cdot\Gamma^\phi{}_{r\phi} = \frac{1}{2}2r\sin^2\theta \Rightarrow \Gamma^\phi{}_{r\phi} = \frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

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$$g_{\phi\phi}\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\cdot\Gamma^{\phi}{}_{r\phi} = \frac{1}{2}2r\sin^2\theta \Rightarrow \Gamma^{\phi}{}_{r\phi} = \frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \Rightarrow (b^2+r^2)\Gamma^{\theta}{}_{\phi\phi} = \frac{1}{2}(-2(b^2+r^2)\sin\theta\cos\theta) \Rightarrow \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$g_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right)$$

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$$g_{rr}\Gamma^r{}_{\theta\theta} = \frac{1}{2}\left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \Rightarrow 1\cdot\Gamma^r{}_{\theta\theta} = \frac{1}{2}(-2r) \Rightarrow \Gamma^r{}_{\theta\theta} = -r$$

$$g_{\theta\theta}\Gamma^\theta{}_{r\theta} = \frac{1}{2}\left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \Rightarrow (b^2+r^2)\Gamma^\theta{}_{r\theta} = \frac{1}{2}\cdot 2r \Rightarrow \Gamma^\theta{}_{r\theta} = \frac{r}{b^2+r^2}$$

$$g_{rr}\Gamma^r{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \Rightarrow 1\cdot\Gamma^r{}_{\phi\phi} = \frac{1}{2}(-2r\sin^2\theta) \Rightarrow \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{r\phi} = \frac{1}{2}\left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\cdot\Gamma^\phi{}_{r\phi} = \frac{1}{2}2r\sin^2\theta \Rightarrow \Gamma^\phi{}_{r\phi} = \frac{r}{b^2+r^2}$$

$$g_{\theta\theta}\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \Rightarrow (b^2+r^2)\Gamma^\theta{}_{\phi\phi} = \frac{1}{2}(-2(b^2+r^2)\sin\theta\cos\theta) \Rightarrow \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$g_{\phi\phi}\Gamma^\phi{}_{\theta\phi} = \frac{1}{2}\left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi}\right) \Rightarrow (b^2+r^2)\sin^2\theta\,\Gamma^\phi{}_{\theta\phi} = \frac{1}{2}2(b^2+r^2)\sin\theta\cos\theta \Rightarrow \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

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$$\ddot{t} + \Gamma^{t}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \Rightarrow \ddot{t} = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0$$

$$\Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\theta}{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^{t}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^{r}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^{r}{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^{r}{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \Gamma^\theta{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

$$\Gamma^r{}_{\theta\theta} = -r \qquad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{r}{b^2+r^2} \qquad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{r}{b^2+r^2} \qquad \Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

$$\ddot{t} + \Gamma^t{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \Rightarrow$$

$$\ddot{r} + \Gamma^r{}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^r{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0 \Rightarrow$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2 = 0$$

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$$\ddot{\theta} + \Gamma^\theta{}_{r\theta}\dot{r}\dot{\theta} + \Gamma^\theta{}_{\theta r}\dot{\theta}\dot{r} + \Gamma^\theta{}_{\phi\phi}\dot{\phi}\dot{\phi} = 0$$

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Radially falling particle through wormhole:

$\dot{\theta} = \dot{\phi} = 0$

(1) $\Rightarrow$ $t = \alpha \tau + \beta$ (proper time too)

(2) $\Rightarrow$ $\ddot{r} = 0$

$\boxed{\begin{aligned} &\ddot{t} = 0 && (1) \\ &\ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\phi}^2 = 0 && (2) \\ &\ddot{\theta} + \frac{2r}{b^2+r^2}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \\ &\ddot{\phi} + \frac{2r}{b^2+r^2}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0 \end{aligned}}$

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Consider 4-velocity $(m>0)$

$u^\mu = (u^0, u^r, 0, 0)$

$$\ddot{t} = 0 \qquad (1)$$

$$\ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\phi}^2 = 0 \qquad (2)$$

$$\ddot{\theta} + \frac{2r}{b^2+r^2}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

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$\ddot{t} = 0$ (1)

$\ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\phi}^2 = 0$ (2)

$\ddot{\theta} + \frac{2r}{b^2+r^2}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$

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$-(u^0)^2 + (u^r)^2 = -1 \Rightarrow$

$u^0 = \sqrt{1+(u^r)^2}$

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Radially falling particle through wormhole:

$\dot{\theta} = \dot{\phi} = 0$

$(1) \Rightarrow t = \alpha\tau + \beta$

$(2) \Rightarrow \ddot{r} = 0 \qquad (3)$

Consider 4-velocity $(m>0)$

$u^\mu = (u^0, u^r, 0, 0)$

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$u^0 = \sqrt{1+(u^r)^2}$

take $u^\mu = \left(\sqrt{1+U^2}, U, 0, 0\right)$, then

$(3) \Rightarrow r(\tau) = U \cdot \tau$

if $r(0)=0$ (i.e. passing the throat)

$\ddot{t} = 0 \qquad (1)$

$\ddot{r} - r\dot{\theta}^2 - r\sin\theta\,\dot{\phi}^2 = 0 \qquad (2)$

$\ddot{\theta} + \frac{2r}{b^2+r^2}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$

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Radially falling particle through wormhole:

$\dot\theta = \dot\phi = 0$

(1) $\Rightarrow$ $t = \alpha\tau + \beta$

(2) $\Rightarrow$ $\ddot r = 0$ (3)

Consider 4-velocity ($u>0$)

$u^\mu = (u^0, u^r, 0, 0)$

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So, going from $r(\tau_1) = R$ to $r(\tau_2) = -R$, the proper time for the particle is

$$\Delta\tau = \tau_2 - \tau_1 = \frac{2R}{U}$$

Radially falling particle through wormhole:

The spatial distance it travelled was ($dt = d\theta = d\phi = 0$)

$$s = \int ds = \int |dr| = 2R$$

take $u^\mu = (\sqrt{1+U^2}, U, 0, 0)$, then

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if $r(0) = 0$ (i.e. passing the throat)

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