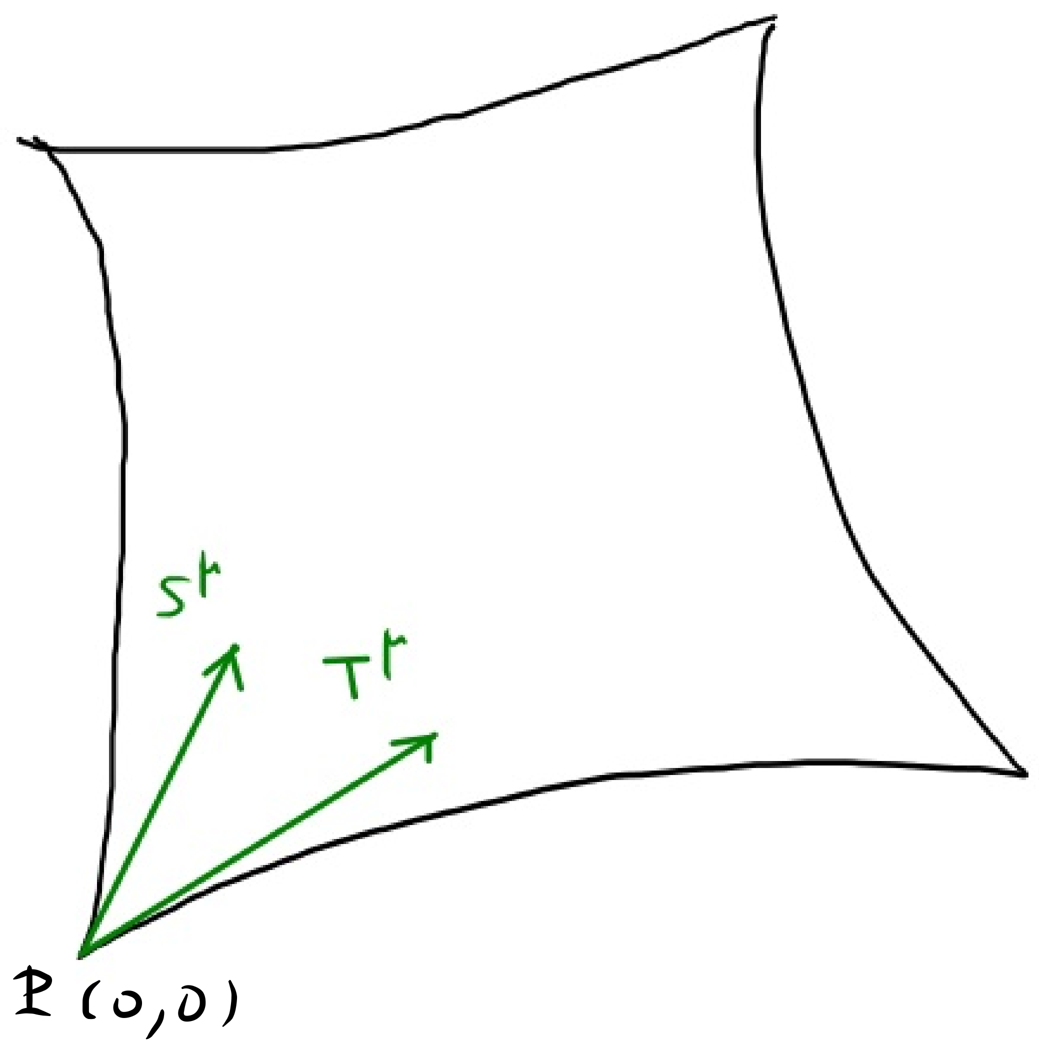
Parallel transport along (infinitesimal) closed curve:

- consider 2d surface:
  (t, s) coordinates, origin at P

  T^μ, S^μ coordinate vectors



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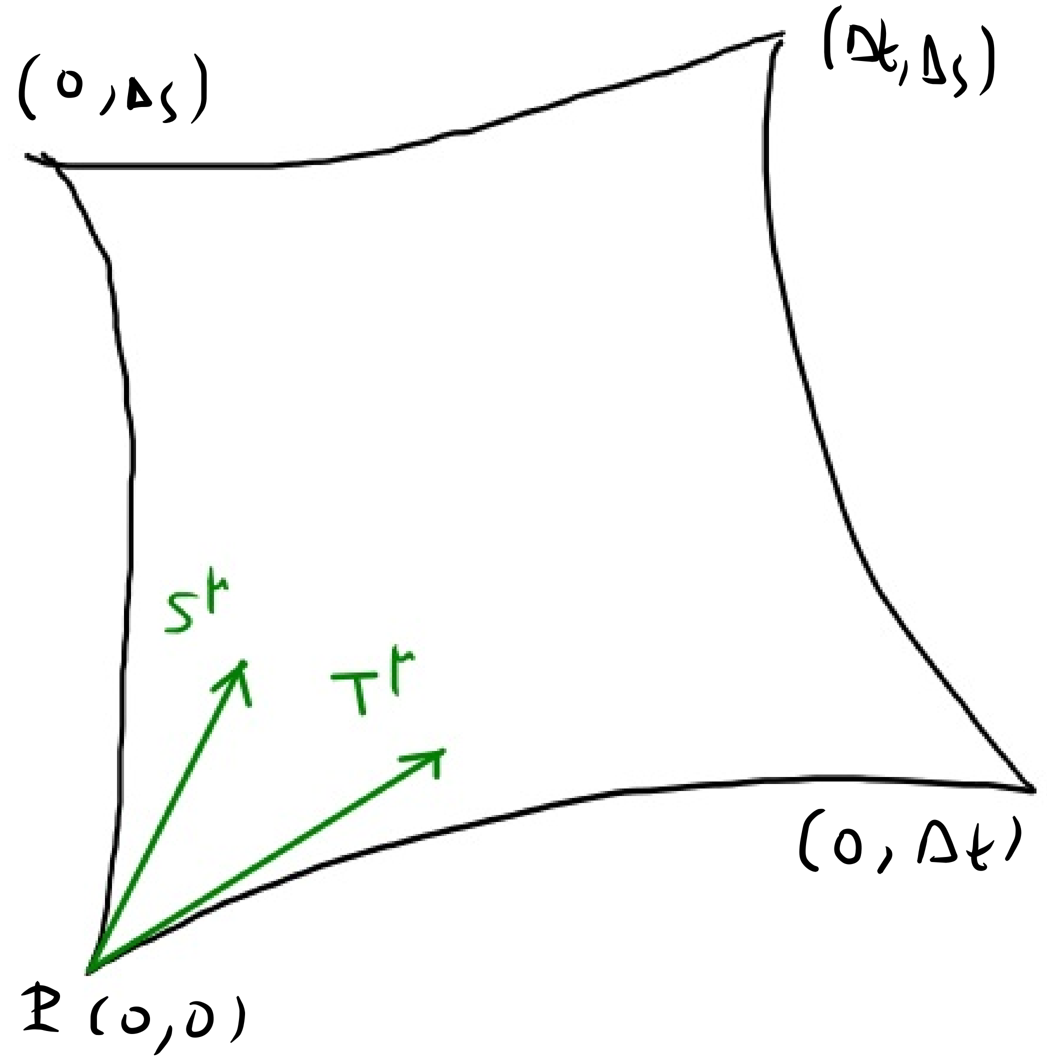
- consider closed curve along integral curves
  of T^μ, S^μ shown here:



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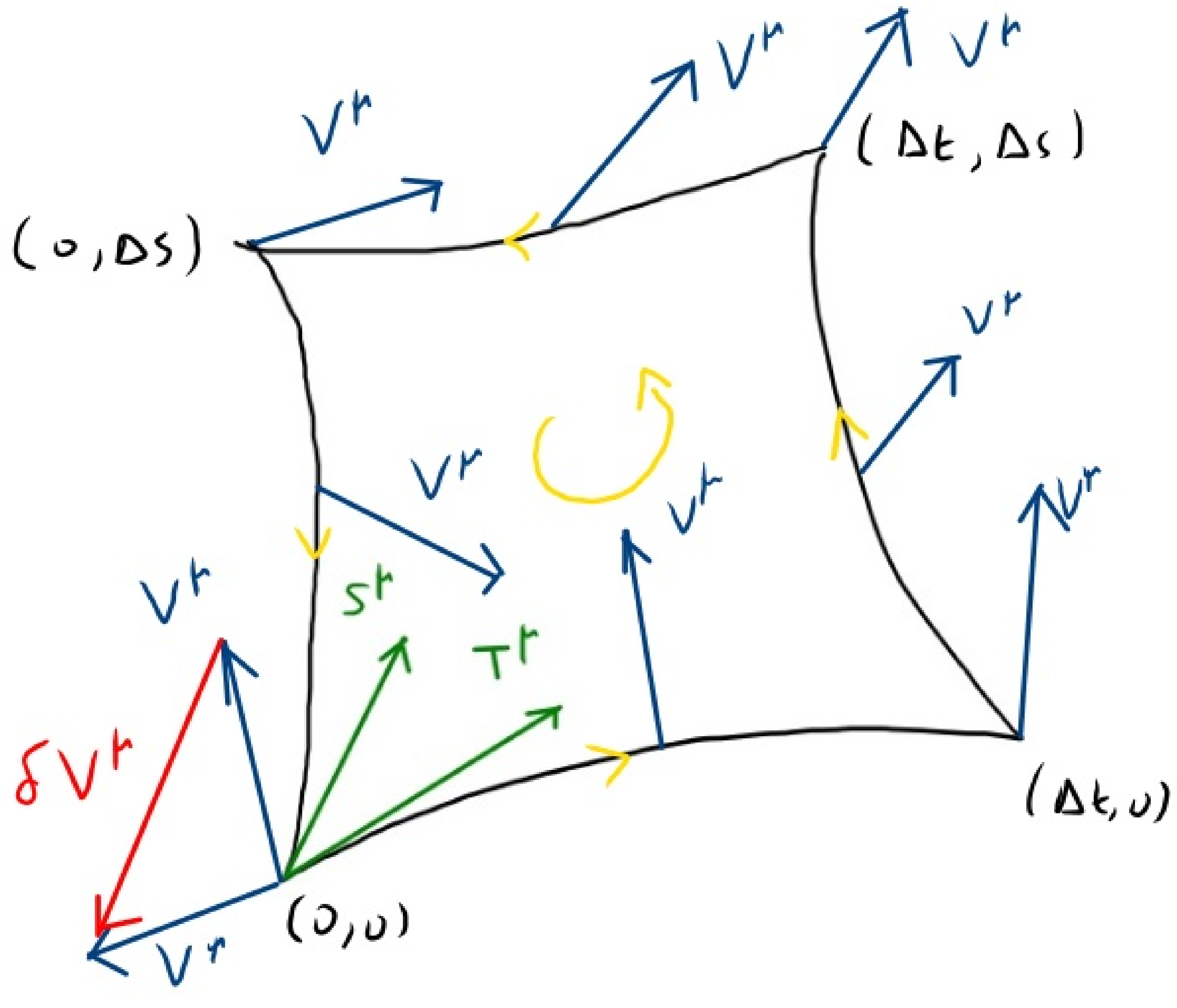

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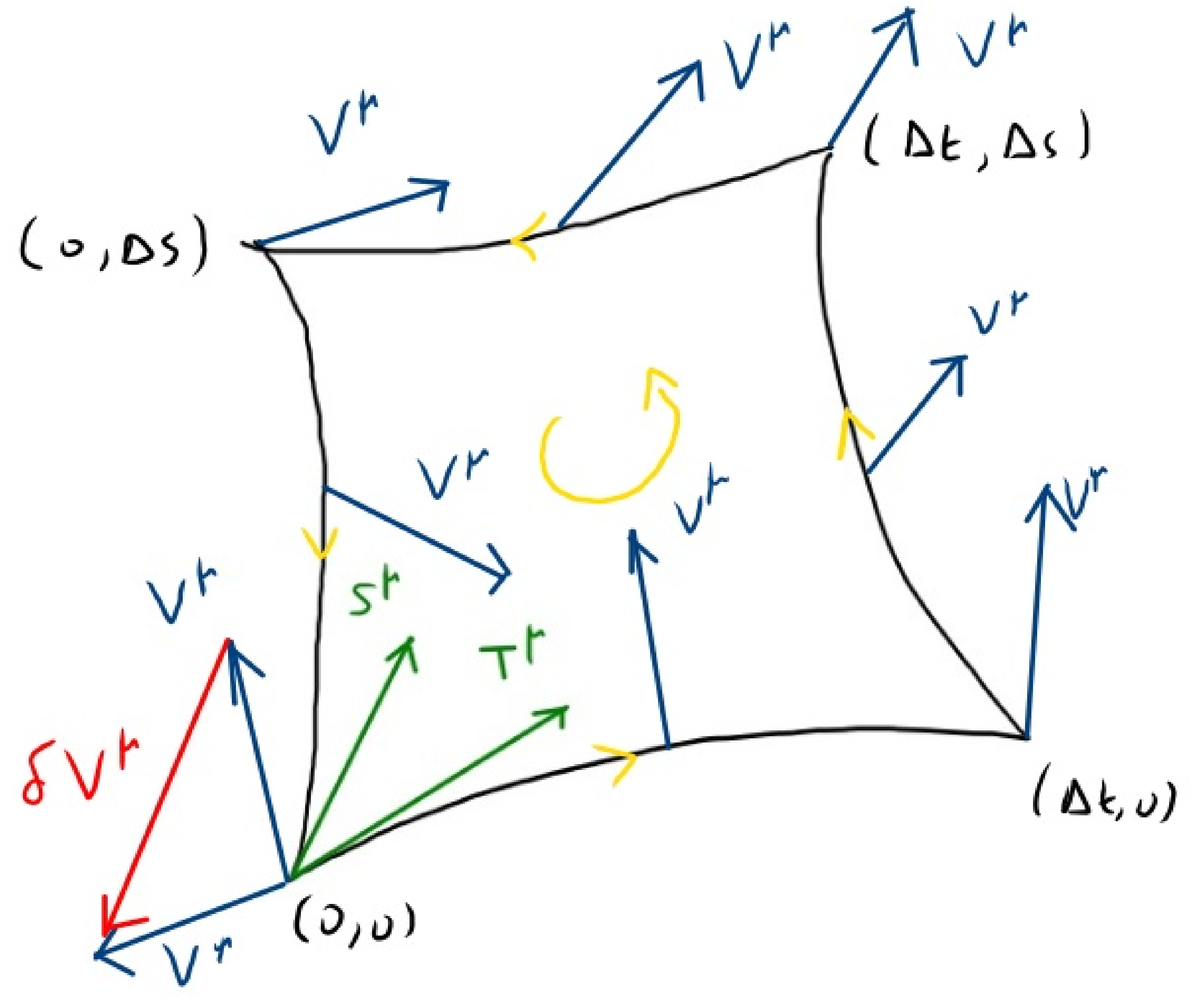


- consider vector V^μ at $(0,0)$ parallel transported along the curve

- consider 1-form field ω_μ (specific values at each point!), and the change in

$$\delta(\omega_\mu V^\mu) = \underbrace{\cancel{\delta\omega_\mu}}_{0} V^\mu + \omega_\mu \delta V^\mu = \omega_\mu \delta V^\mu$$

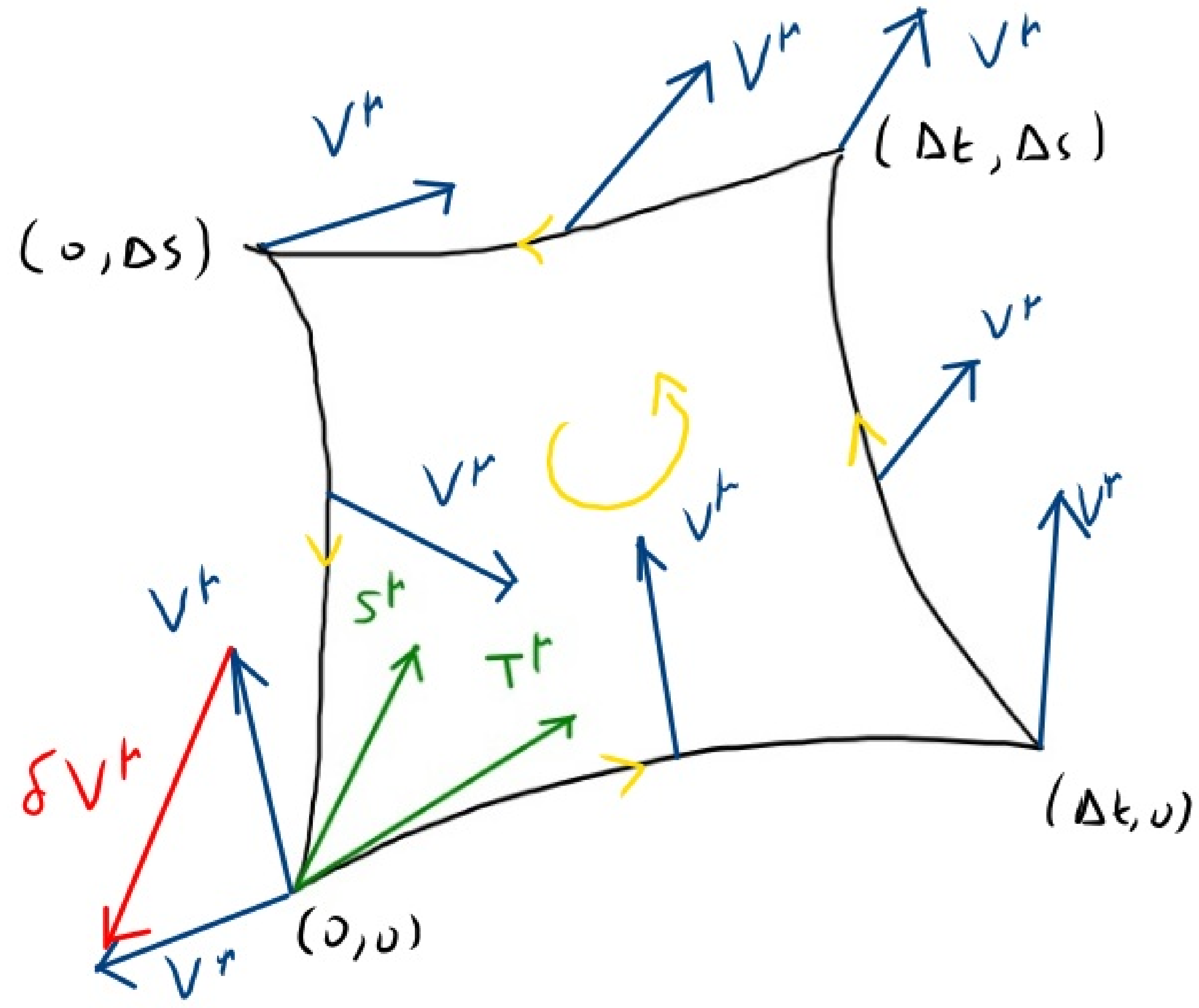
Parallel transport along (infinitesimal) closed curve:

• $\omega_\mu V^\mu$ is a function on the curve

Remember that for any function $f(t)$ we have:

$$f(t+\epsilon) = f\left(t+\frac{\epsilon}{2}\right) + \frac{\epsilon}{2} f'\left(t+\frac{\epsilon}{2}\right) + \frac{(\epsilon/2)^2}{2!} f''\left(t+\frac{\epsilon}{2}\right) + O(\epsilon^3)$$

$$f(t) = f\left(t+\frac{\epsilon}{2}\right) - \frac{\epsilon}{2} f'\left(t+\frac{\epsilon}{2}\right) + \frac{[-\epsilon/2]^2}{2!} f''\left(t+\frac{\epsilon}{2}\right) - O(\epsilon^3)$$



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$\hookrightarrow \delta\omega_\mu = 0$

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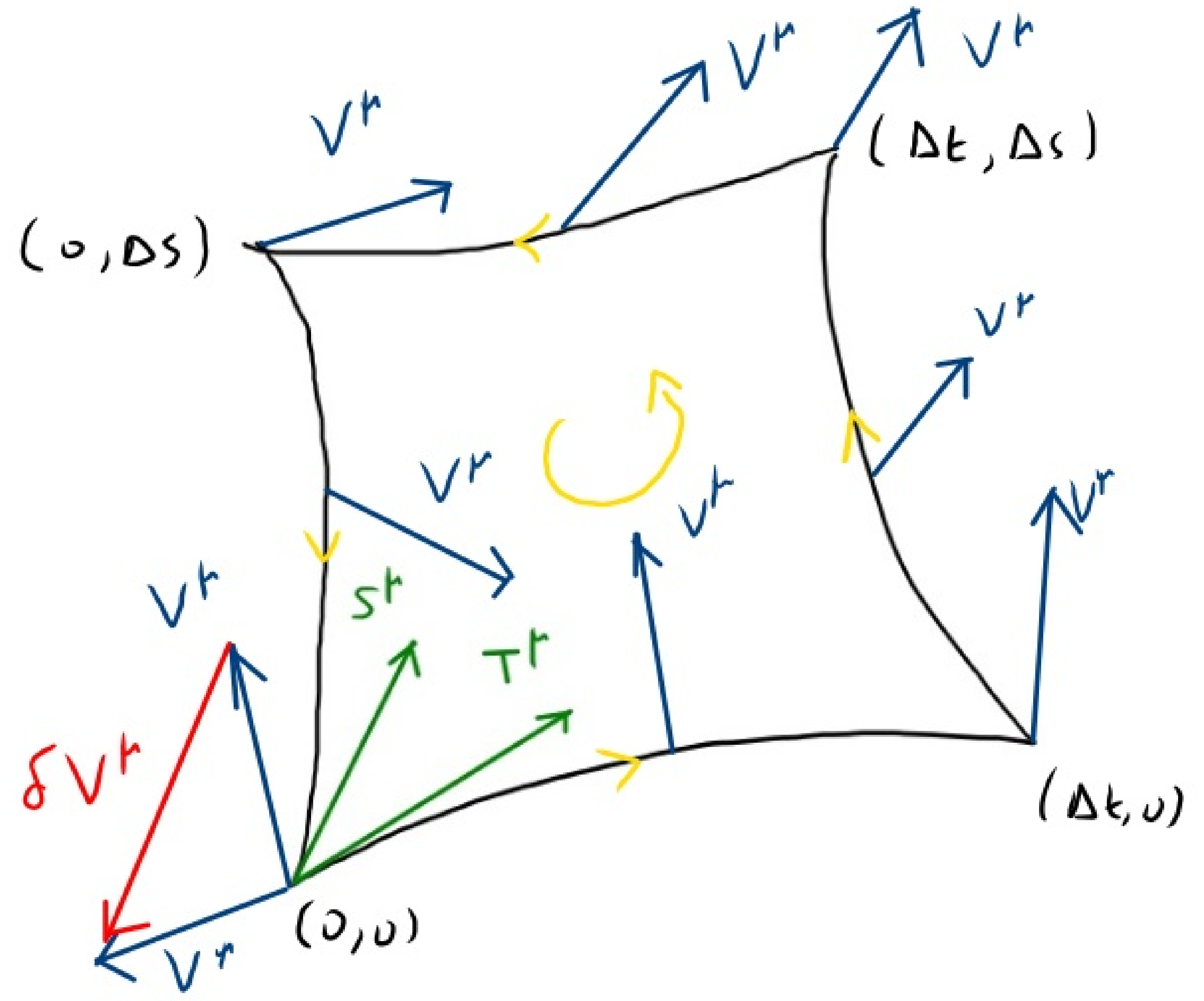
$$\Downarrow$$

$$f(t+\epsilon) - f(t) = \epsilon f'(t+\tfrac{\epsilon}{2}) + O(\epsilon^3)$$

better approximation!

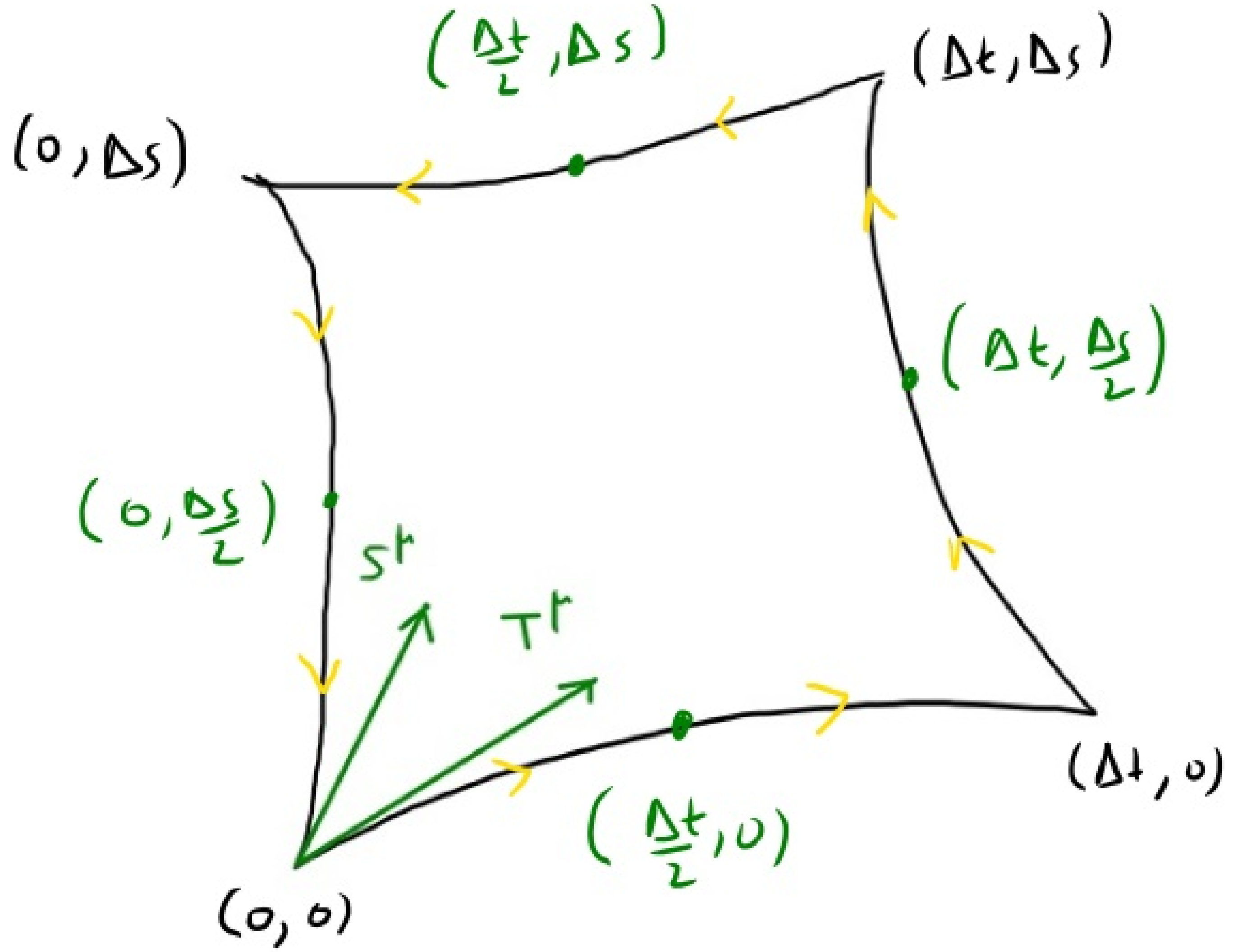


Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

(1) $(0,0) \to (\Delta t, 0)$

$$\delta_1 = \Delta t \, \frac{\partial}{\partial t}\left(V^\mu \omega_\mu\right)\Big|_{\left(\frac{\Delta t}{2},0\right)} + O(\Delta t^3)$$



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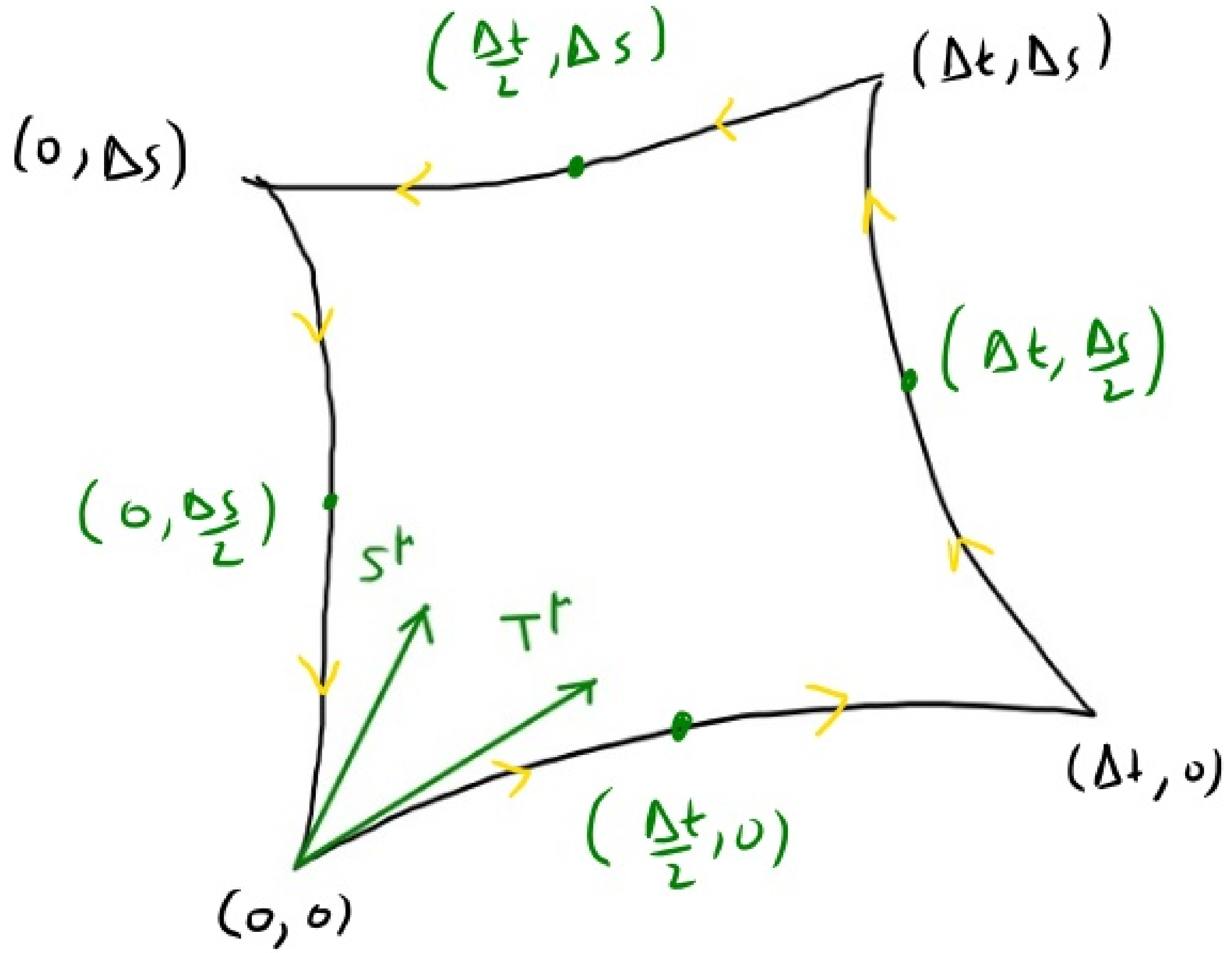

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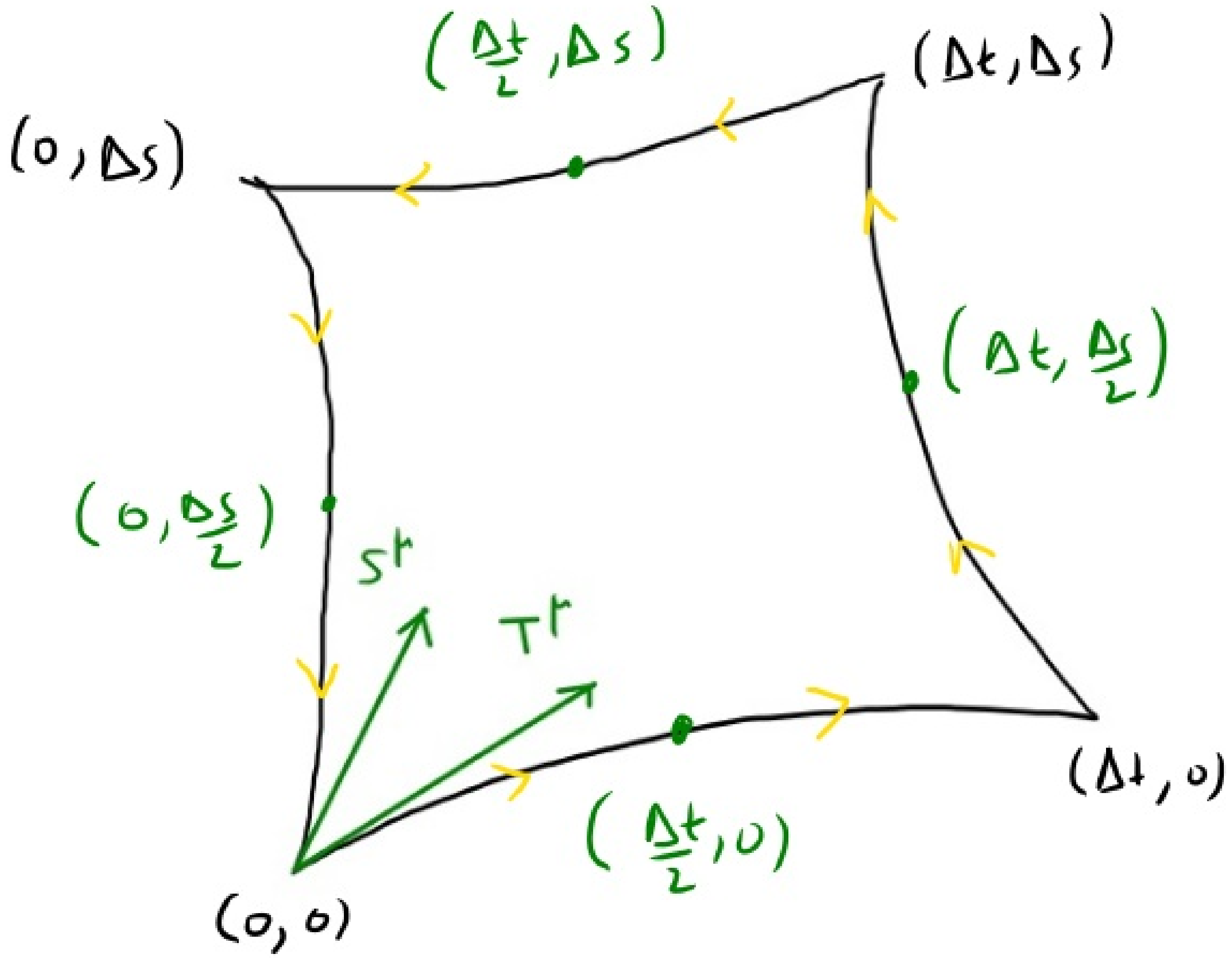

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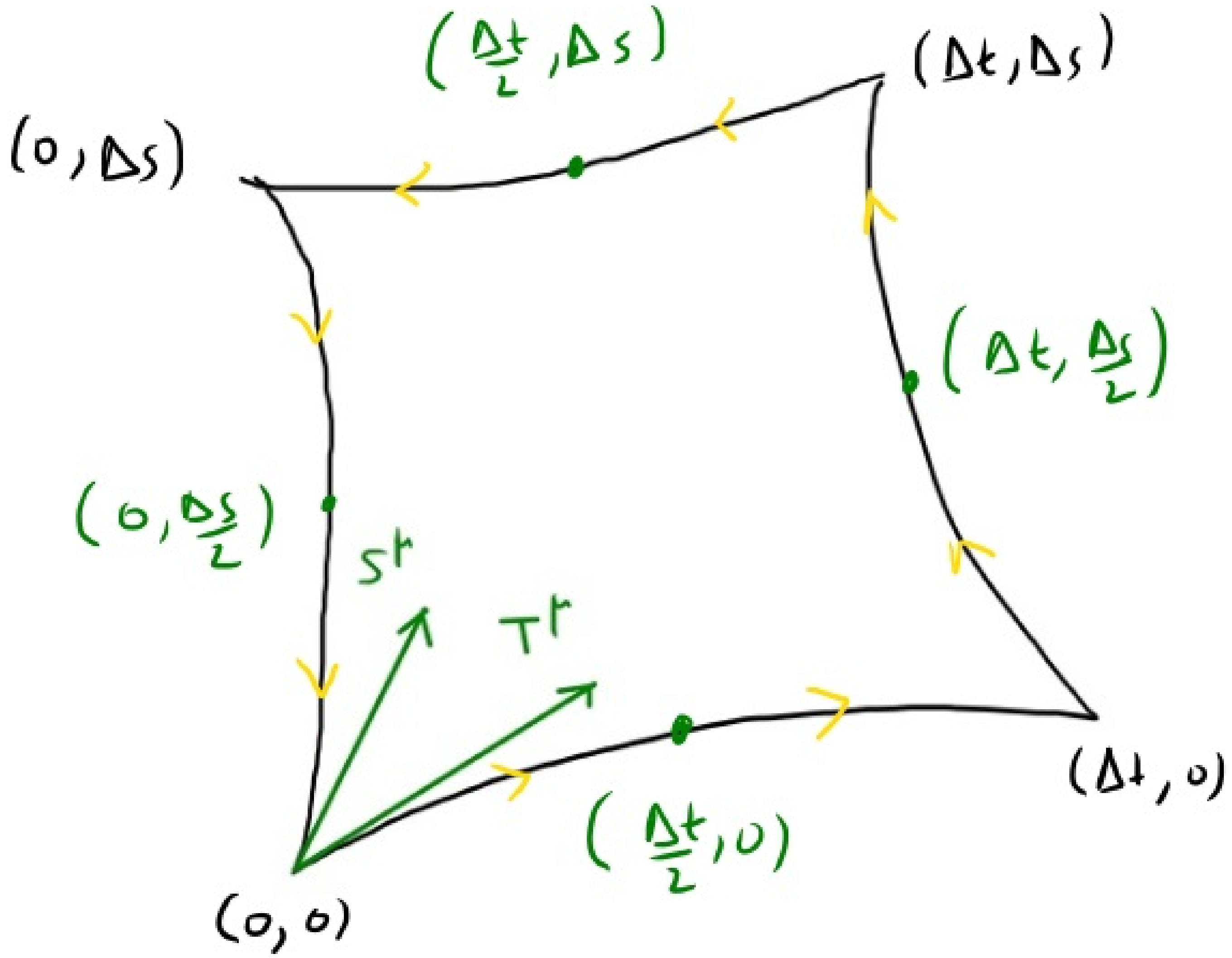

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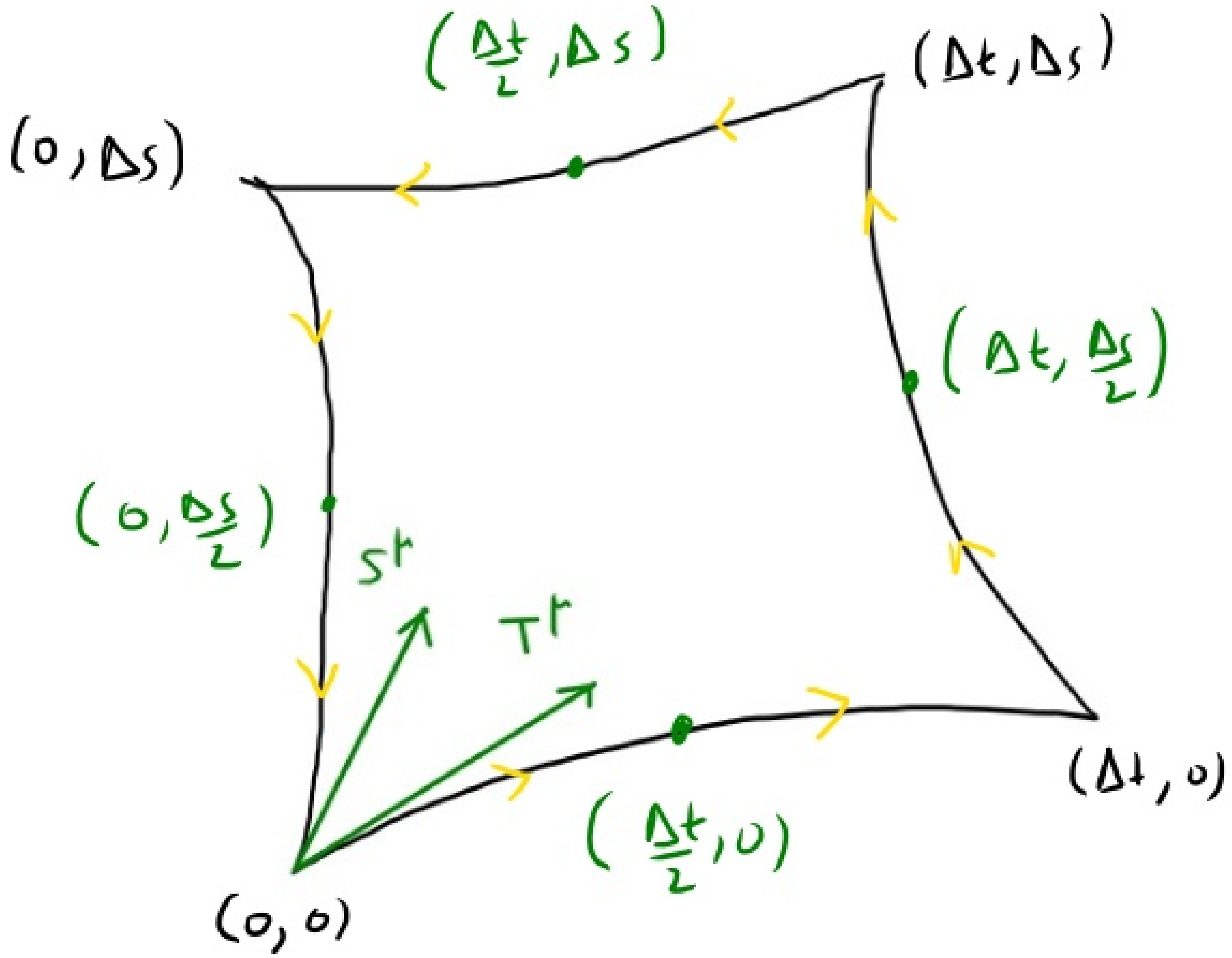

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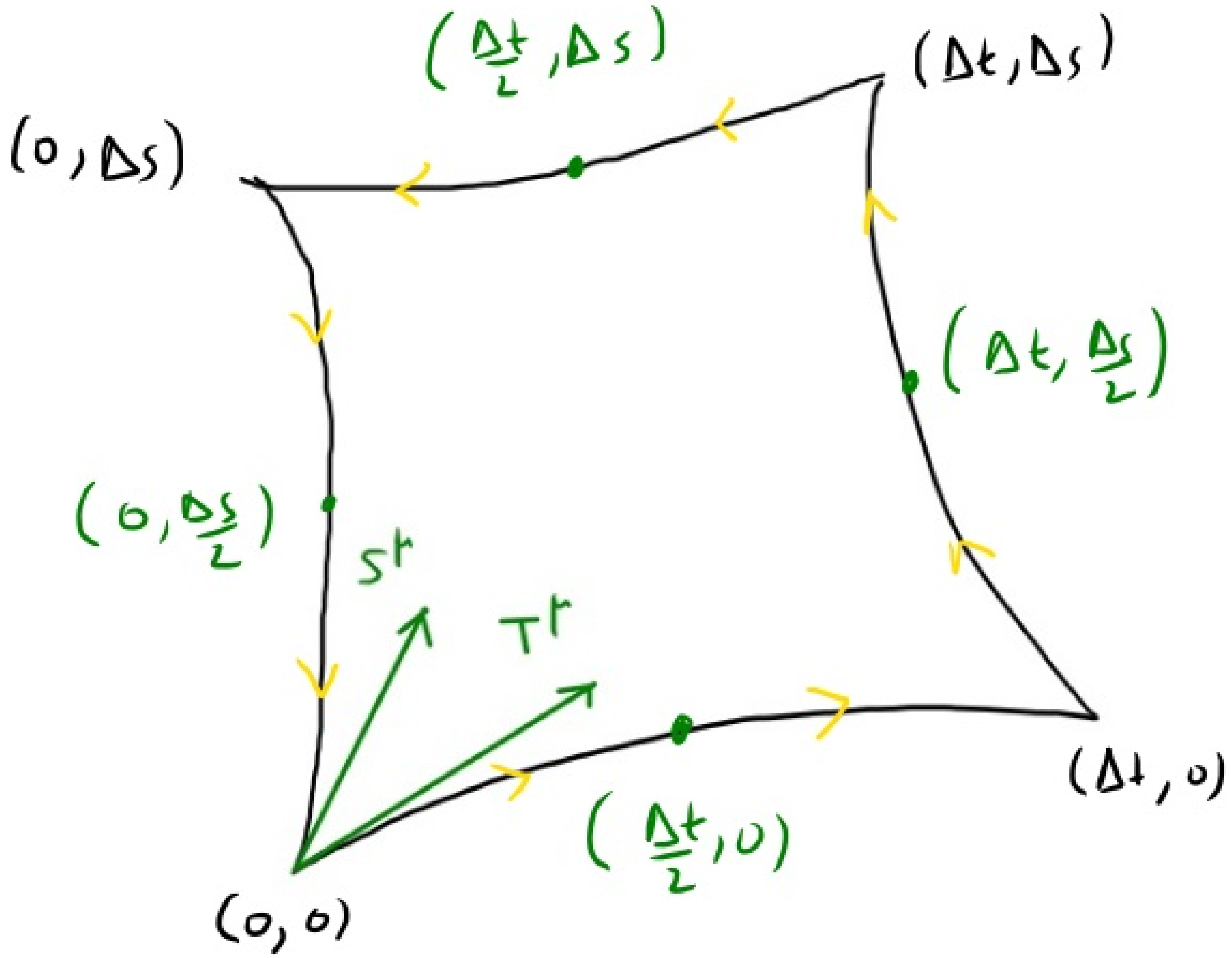

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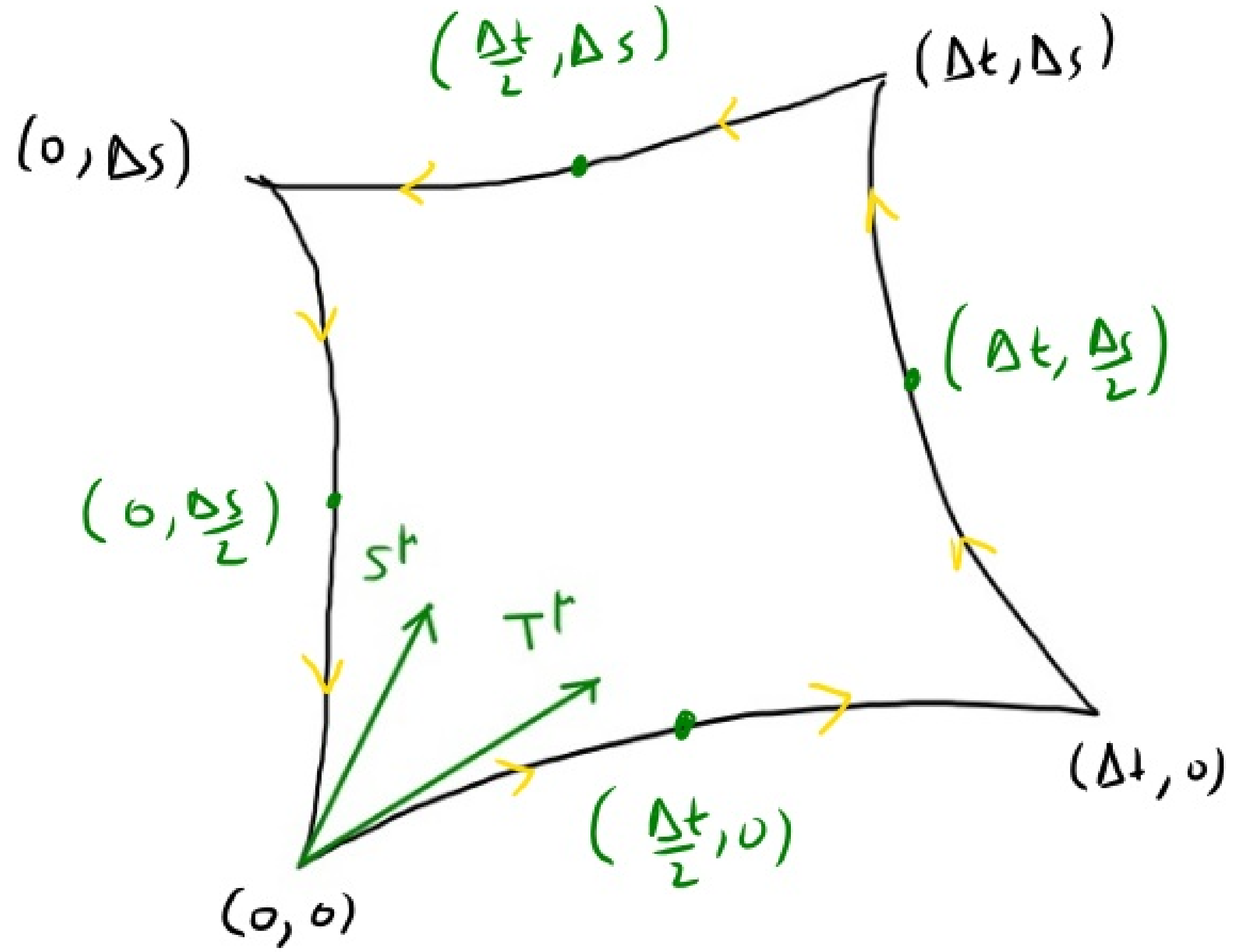

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Parallel transport along (infinitesimal) closed curve:

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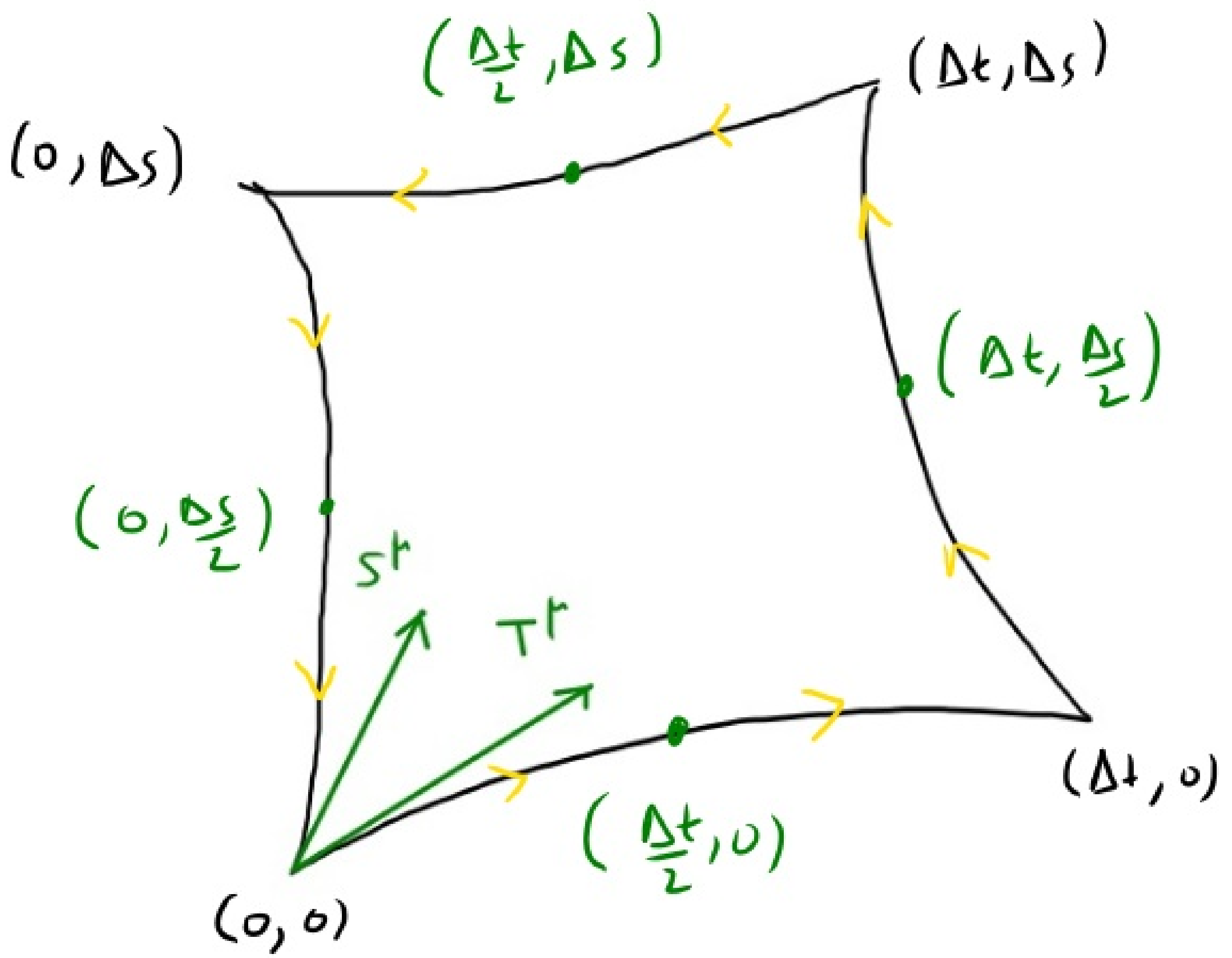

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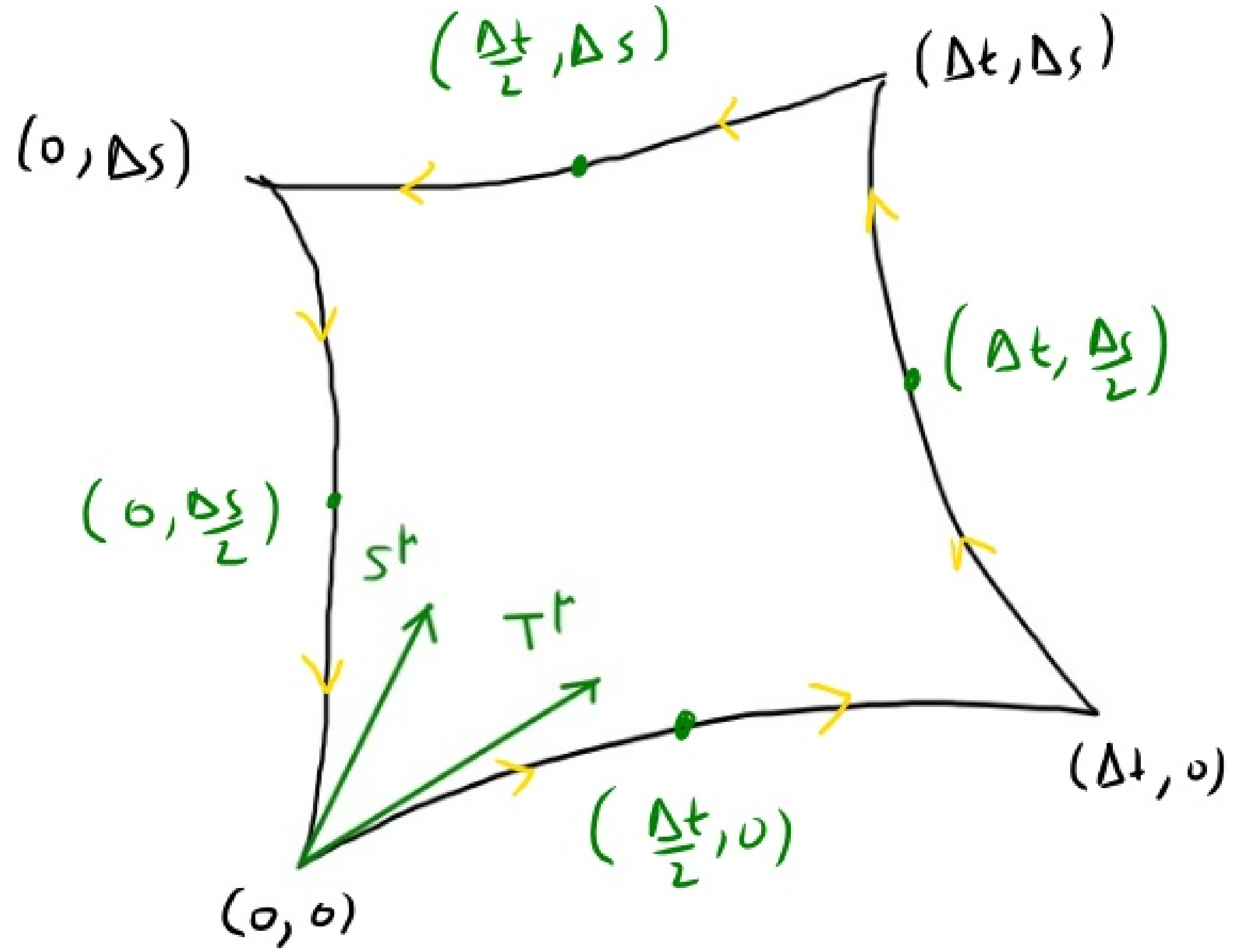

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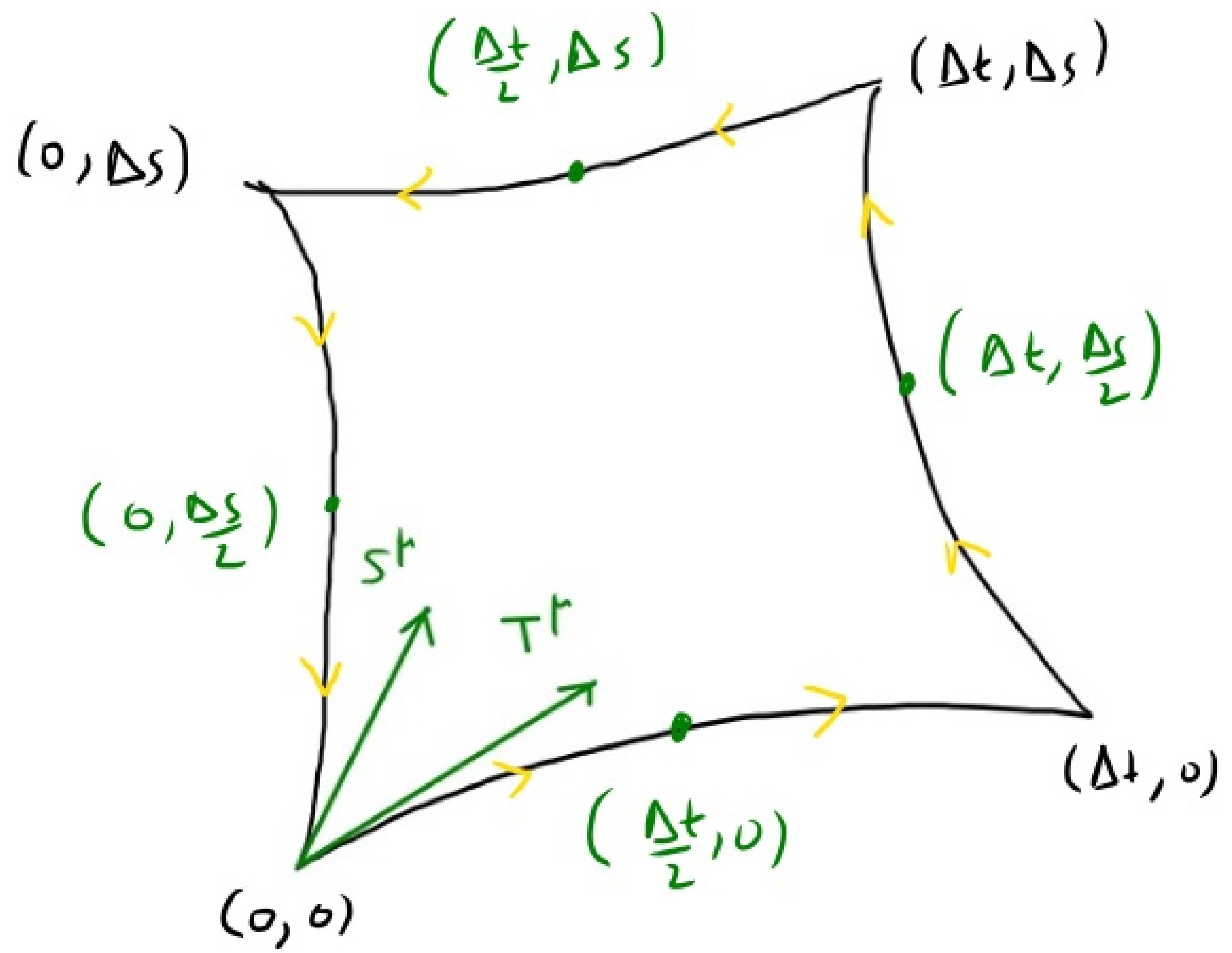

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Add (1)-(4):

$$\delta_1+\delta_2+\delta_3+\delta_4 =$$

$$= \Delta t \left\{ V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)} \right\}$$

$$+ \Delta s \left\{ V^\mu S^\nu \nabla_\nu w_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu w_\mu \Big|_{(0,\frac{\Delta s}{2})} \right\}$$

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- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

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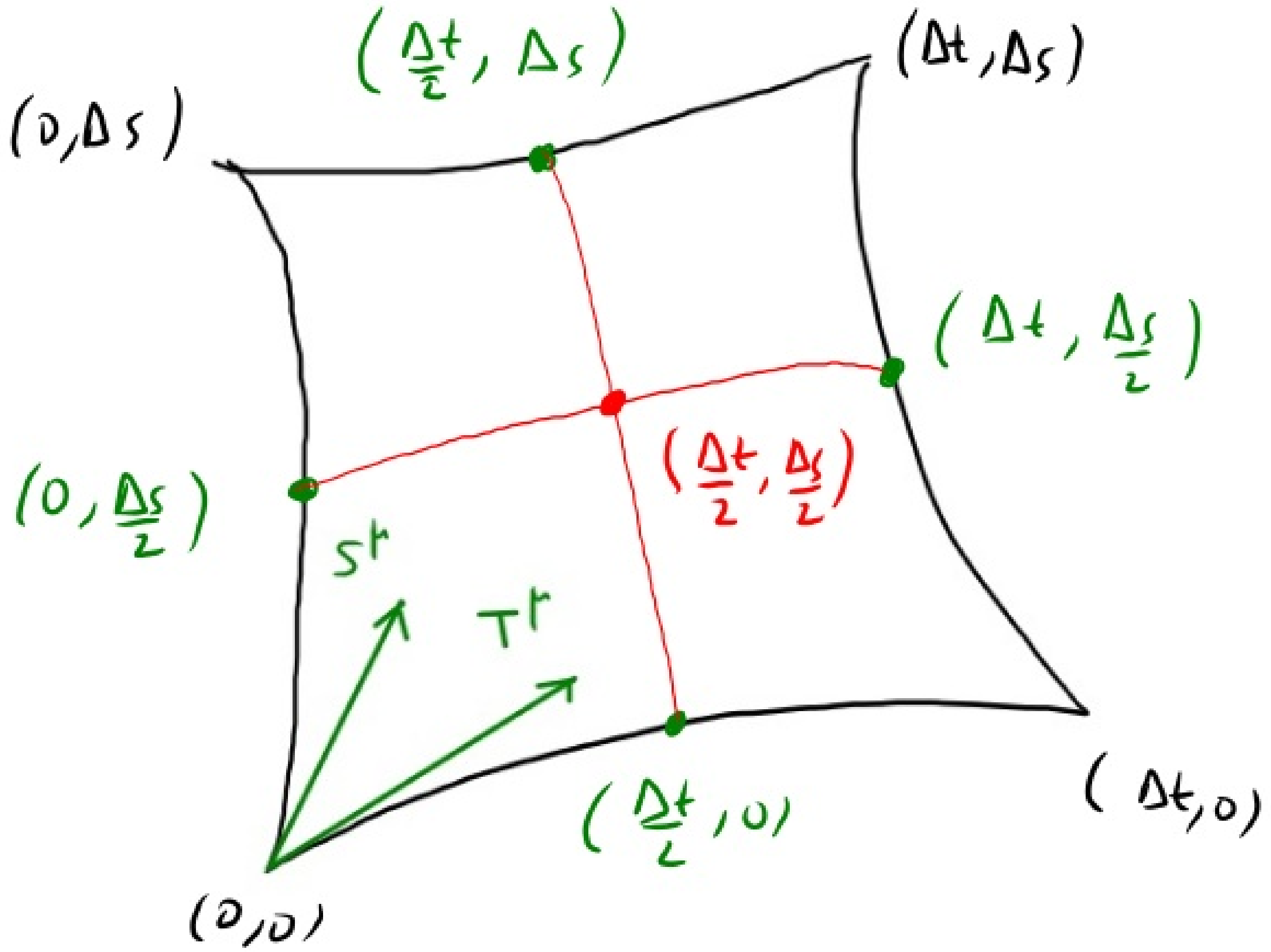

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But function @ $\Delta s = 0$ function @ Δs

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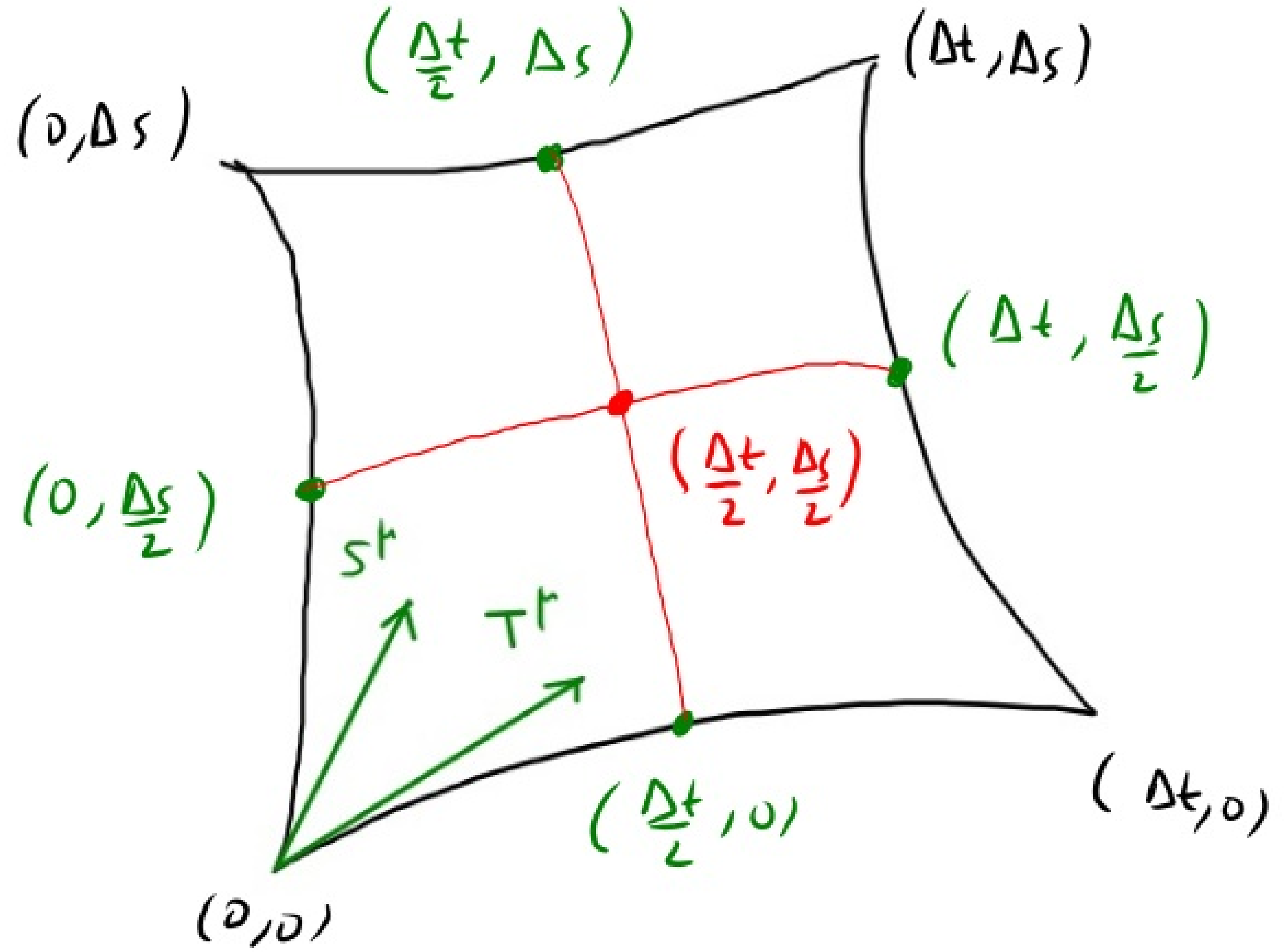

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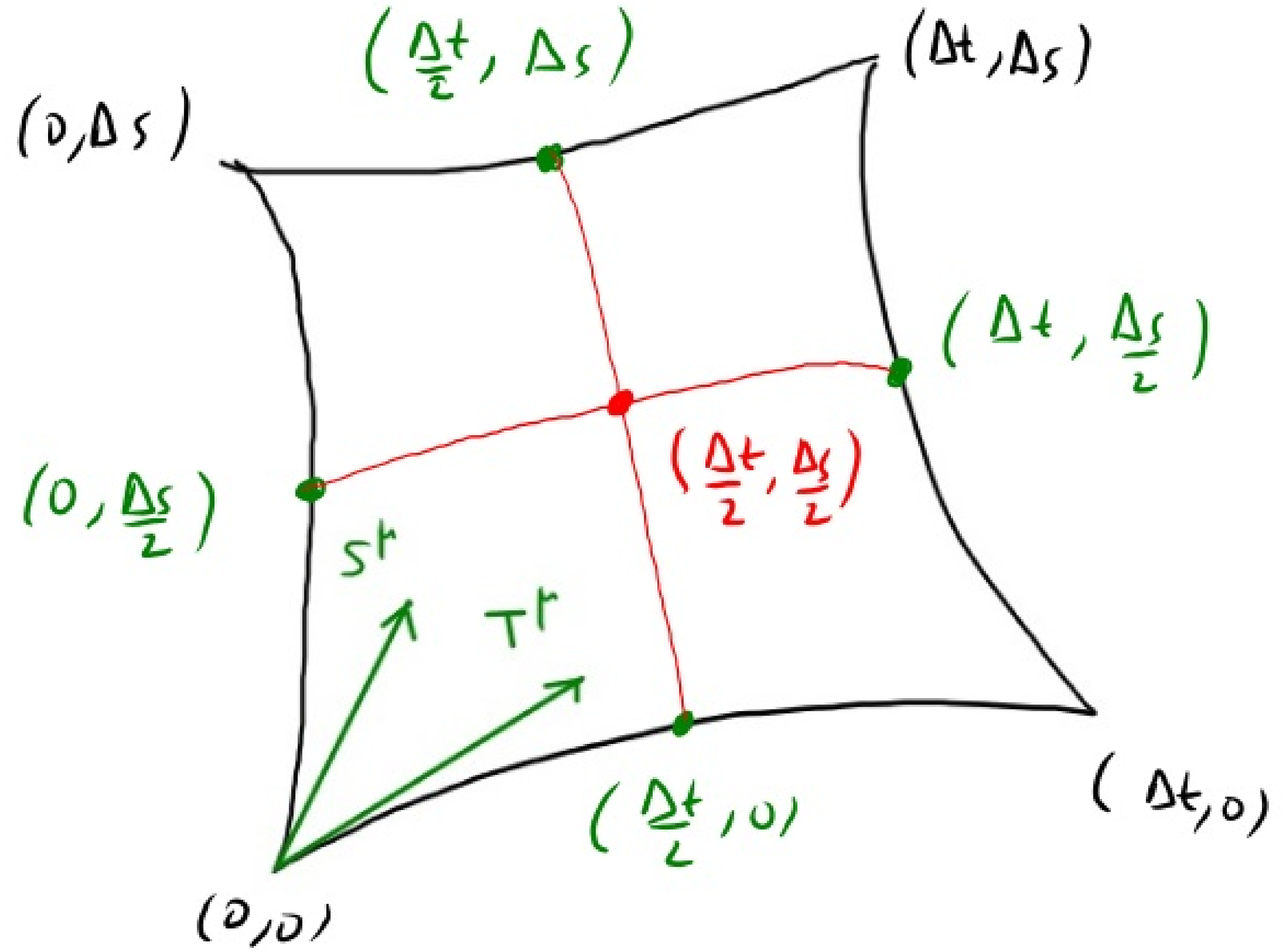

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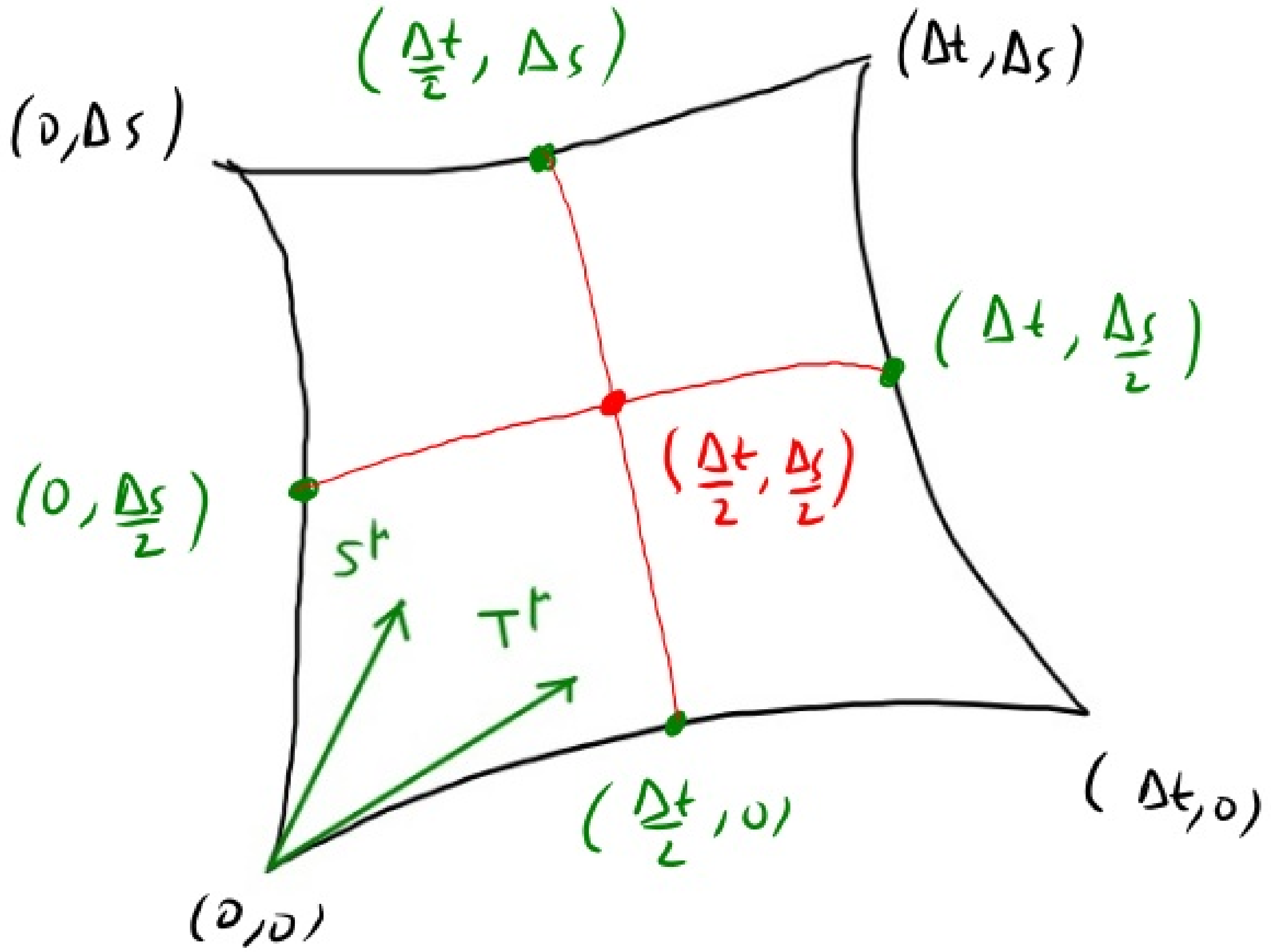

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$$\delta_1 + \delta_2 + \delta_3 + \delta_4 =$$

$$= \Delta t \left\{ V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)} \right\}$$

$$+ \Delta s \left\{ V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0, \frac{\Delta s}{2})} \right\}$$

Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)}$$

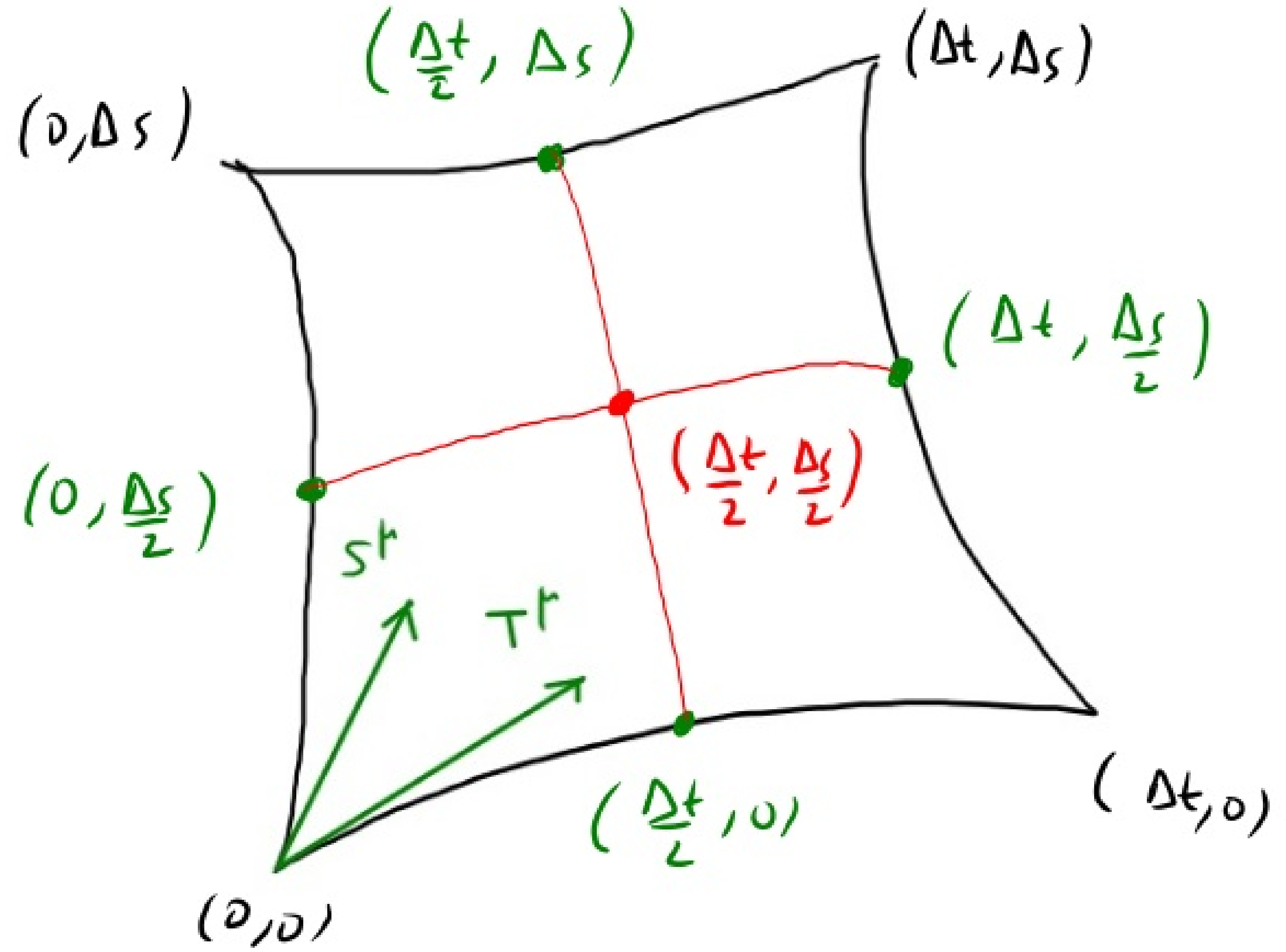
$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) \Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + O(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) \Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0, \frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) \Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) \Big\} \Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})}$$



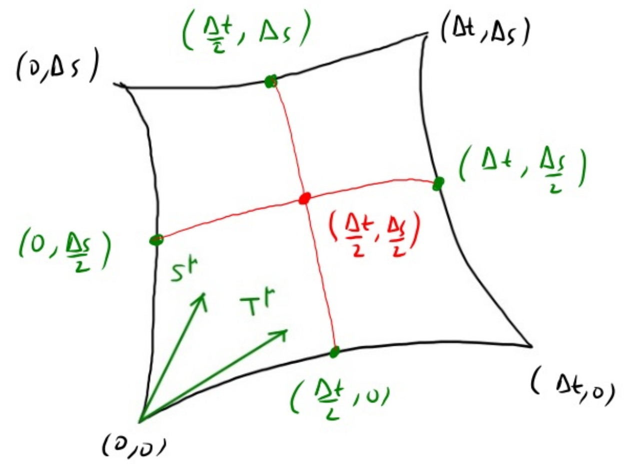
Add (1)-(4):

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 =$$

$$= \Delta t \Big\{ V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)} \Big\}$$

$$+ \Delta s \Big\{ V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0, \frac{\Delta s}{2})} \Big\}$$

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But function @ $\Delta s=0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} (V^\mu T^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) - S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu) \Big\} \Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

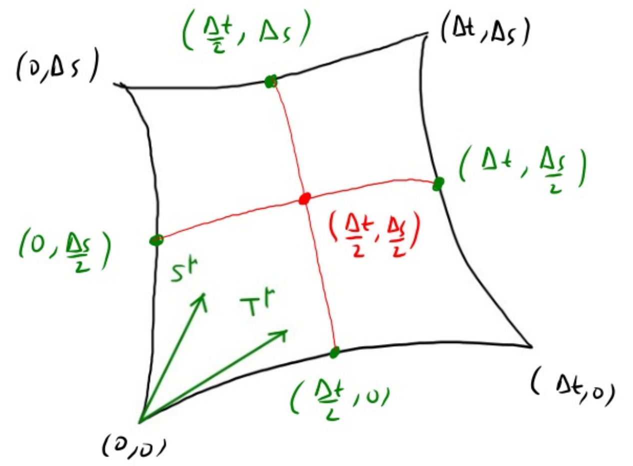
Everything evaluated @ $(\frac{\Delta t}{2},\frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) =$$

$$T^\rho \cancel{\nabla_\rho} V^\mu S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2},\frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \cancel{\nabla_\rho} V^\mu S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

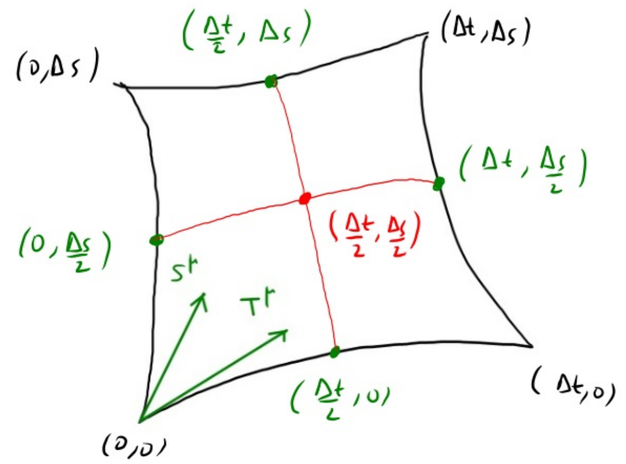
parallel transported

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$S^\rho \cancel{\nabla_\rho} V^\mu T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu \nabla_\rho T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu T^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + O(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

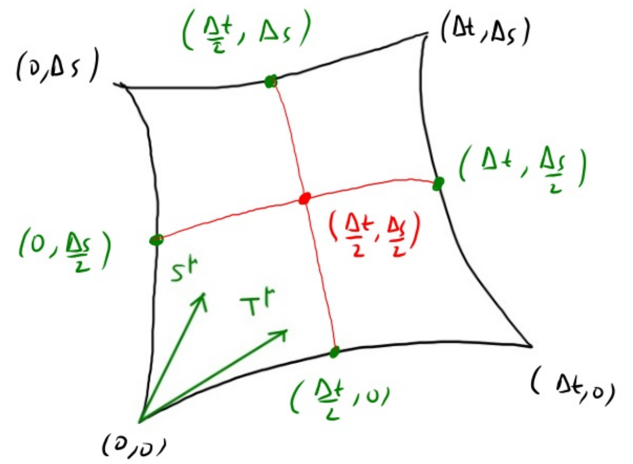
$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$T^\rho \underbrace{\cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \nabla_\rho S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$S^\rho \underbrace{\cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu \nabla_\rho T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu T^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$



Parallel transport along (infinitesimal) closed curve:

- $\omega_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

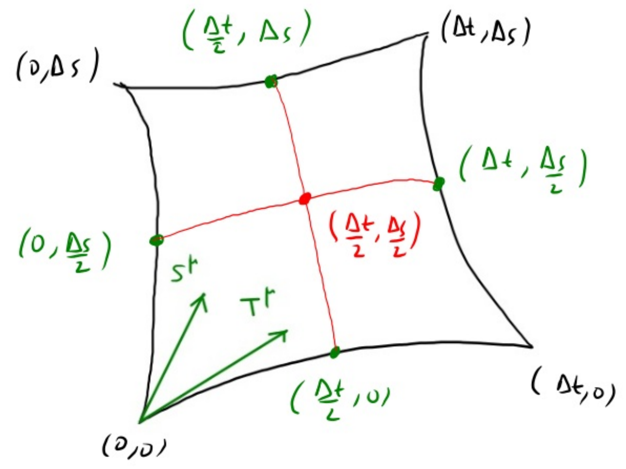
$$\underbrace{T^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho S^\nu}} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$\underbrace{S^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + \underline{S^\rho} V^\mu \underline{\cancel{\nabla_\rho T^\nu}} \nabla_\nu \omega_\mu + S^\rho V^\mu T^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But — function @ $\Delta s=0$ — function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + O(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2},\frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

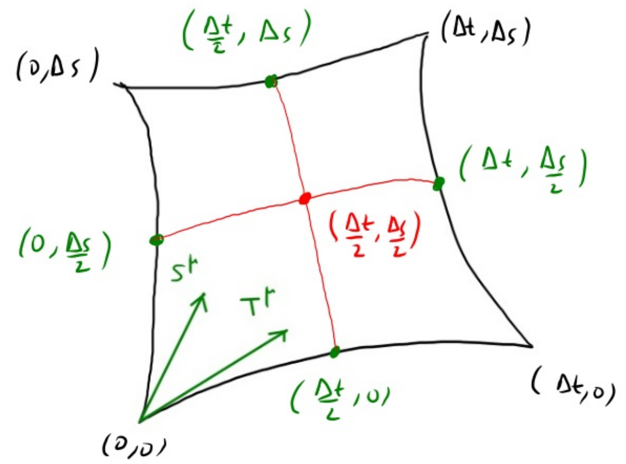
$$\underbrace{T^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho S^\nu}} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$\underbrace{S^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + \underline{S^\rho} V^\mu \underline{\cancel{\nabla_\rho T^\nu}} \nabla_\nu \omega_\mu + \underset{\rho\to\nu}{S^\rho} V^\mu \overset{\nu\to\rho}{T^\nu} \nabla_\rho \nabla_\nu \omega_\mu$$

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

Parallel transport along (infinitesimal) closed curve:



- $w_\mu V^\mu$ is a function on the curve

But

function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2}, 0)} - V^\mu T^\nu \nabla_\nu w_\mu \Big|_{(\frac{\Delta t}{2}, \Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu w_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + O(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu w_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu w_\mu \Big|_{(\Delta t, \frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu w_\mu \Big|_{(0, \frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu w_\mu\right)\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})} + \cdots$$

$$\delta(w_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu w_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu w_\mu\right) \Big\}\Big|_{(\frac{\Delta t}{2}, \frac{\Delta s}{2})}$$

Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu w_\mu\right) =$$

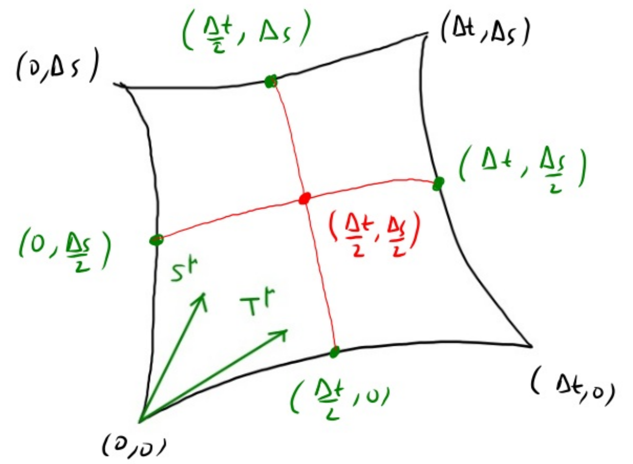
$$\underbrace{T^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu w_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho} S^\nu} \nabla_\nu w_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu w_\mu$$

$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu w_\mu\right) =$$

$$\underbrace{S^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} T^\nu \nabla_\nu w_\mu + \underline{S^\rho} V^\mu \underline{\cancel{\nabla_\rho} T^\nu} \nabla_\nu w_\mu + S^\nu V^\mu T^\rho \nabla_\nu \nabla_\rho w_\mu$$

(t,s) coord system $\Rightarrow [T, S] = 0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But — function @ $\Delta s=0$ — function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho \nabla_\rho} \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + O(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) - S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right)\Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2},\frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho \left(V^\mu S^\nu \nabla_\nu \omega_\mu\right) =$$

$$\underbrace{T^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \cancel{\underline{\nabla_\rho S^\nu}} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

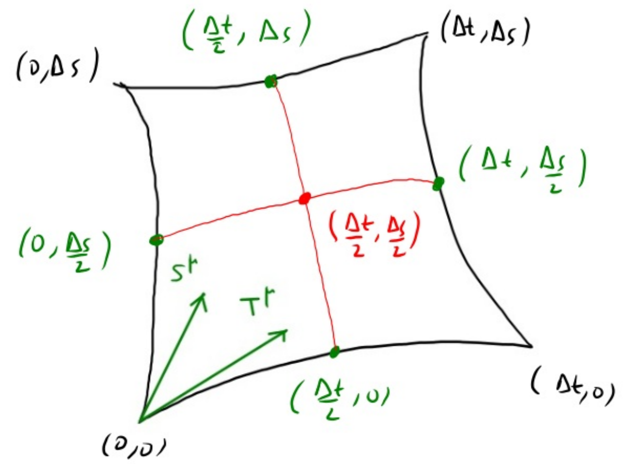
$$S^\rho \nabla_\rho \left(V^\mu T^\nu \nabla_\nu \omega_\mu\right) =$$

$$\underbrace{S^\rho \cancel{\nabla_\rho} V^\mu}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + \underline{S^\rho} V^\mu \cancel{\underline{\nabla_\rho T^\nu}} \nabla_\nu \omega_\mu + S^\nu V^\mu T^\rho \nabla_\nu \nabla_\rho \omega_\mu$$

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s\, V^\mu T^\rho S^\nu \left(\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho\right) \omega_\mu$$

Parallel transport along (infinitesimal) closed curve:



- $\omega_\mu V^\mu$ is a function on the curve

But: function @ $\Delta s = 0$ function @ Δs

$$V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},0)} - V^\mu T^\nu \nabla_\nu \omega_\mu \Big|_{(\frac{\Delta t}{2},\Delta s)}$$

$$= -\Delta s \underbrace{\frac{\partial}{\partial s}}_{\nabla_s = S^\rho\nabla_\rho} (V^\mu T^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \mathcal{O}(\Delta s^3)$$

$$= -\Delta s\, S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(\Delta t,\frac{\Delta s}{2})} - V^\mu S^\nu \nabla_\nu \omega_\mu \Big|_{(0,\frac{\Delta s}{2})}$$

$$= \Delta t\, T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu)\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})} + \cdots$$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s \Big\{ T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) - S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu) \Big\}\Big|_{(\frac{\Delta t}{2},\frac{\Delta s}{2})}$$

Everything evaluated @ $(\frac{\Delta t}{2},\frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) =$$

$$T^\rho \underbrace{\cancel{\nabla_\rho V^\mu}}_{\text{parallel transported}} S^\nu \nabla_\nu \omega_\mu + T^\rho V^\mu \cancel{\nabla_\rho S^\nu} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

$$S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu) =$$

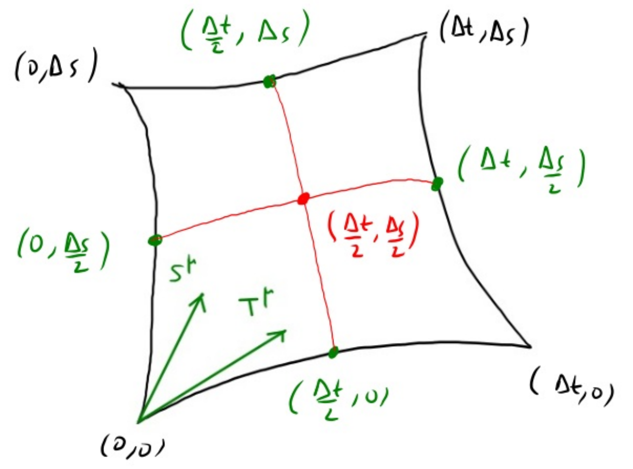
$$S^\rho \underbrace{\cancel{\nabla_\rho V^\mu}}_{\text{parallel transported}} T^\nu \nabla_\nu \omega_\mu + S^\rho V^\mu \cancel{\nabla_\rho T^\nu} \nabla_\nu \omega_\mu + S^\nu V^\mu T^\rho \nabla_\nu \nabla_\rho \omega_\mu$$

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

$$\delta(\omega_\mu V^\mu) = \Delta t\, \Delta s\, V^\mu T^\rho S^\nu (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho)\, \omega_\mu$$

$$= \Delta t\, \Delta s\, V^\mu T^\rho S^\nu \left[-R^\lambda{}_{\mu\rho\nu}\, \omega_\lambda \right]$$

Parallel transport along (infinitesimal) closed curve:



$$\delta(\omega_\mu V^\mu) = \underset{\substack{\parallel \\ 0}}{\delta\omega_\mu} V^\mu + \omega_\mu \delta V^\mu$$

since 1-form field, unique value at (0,0)

Everything evaluated @ $(\frac{\Delta t}{2}, \frac{\Delta s}{2})$:

$$T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) =$$

$$T^\rho \cancel{\nabla_\rho} V^\mu S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho} S^\nu} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

$$S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu) =$$

$$S^\rho \cancel{\nabla_\rho} V^\mu T^\nu \nabla_\nu \omega_\mu + \underline{S^\rho} V^\mu \underline{\cancel{\nabla_\rho} T^\nu} \nabla_\nu \omega_\mu + S^\nu V^\mu T^\rho \nabla_\nu \nabla_\rho \omega_\mu$$

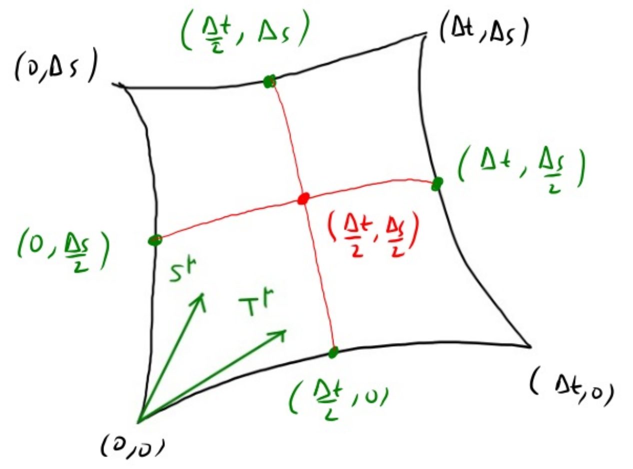
parallel transported

(t,s) coord system $\Rightarrow [T,S]=0 \Rightarrow T^\rho \nabla_\rho S^\nu - S^\rho \nabla_\rho T^\nu = 0$

$$\delta(\omega_\mu V^\mu) = \Delta t \, \Delta s \, V^\mu T^\rho S^\nu (\nabla_\rho \nabla_\nu - \nabla_\nu \nabla_\rho)\, \omega_\mu$$

$$= \Delta t \, \Delta s \, V^\mu T^\rho S^\nu \left[- R^\lambda{}_{\mu\rho\nu}\, \omega_\lambda \right]$$

Parallel transport along (infinitesimal) closed curve:



$$\delta(\omega_\mu V^\mu) = \delta\omega_\mu V^\mu + \omega_\mu \delta V^\mu = \omega_\mu \delta V^\mu$$

RHS:

$$-\Delta t\, \Delta s\, R^\lambda{}_{\mu\rho\nu}\, \omega_\lambda V^\mu T^\rho S^\nu =$$

Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

$$T^\rho \nabla_\rho (V^\mu S^\nu \nabla_\nu \omega_\mu) =$$

$$T^\rho \cancel{\nabla_\rho V^\mu} S^\nu \nabla_\nu \omega_\mu + \underline{T^\rho} V^\mu \underline{\cancel{\nabla_\rho S^\nu}} \nabla_\nu \omega_\mu + T^\rho V^\mu S^\nu \nabla_\rho \nabla_\nu \omega_\mu$$

parallel transported

$$S^\rho \nabla_\rho (V^\mu T^\nu \nabla_\nu \omega_\mu) =$$

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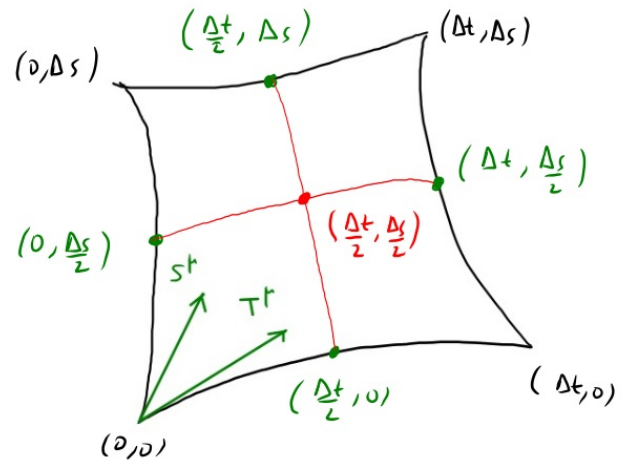

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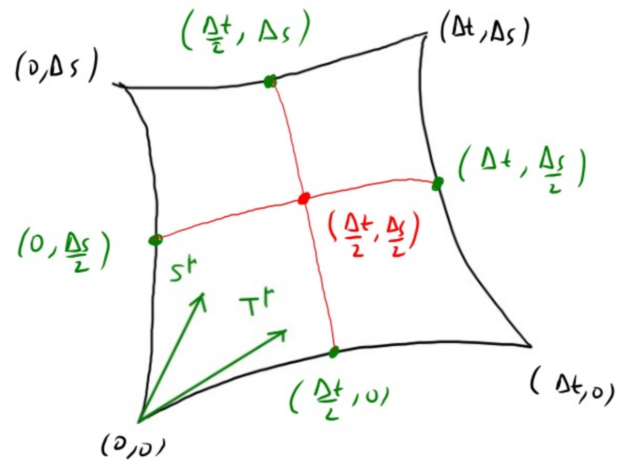

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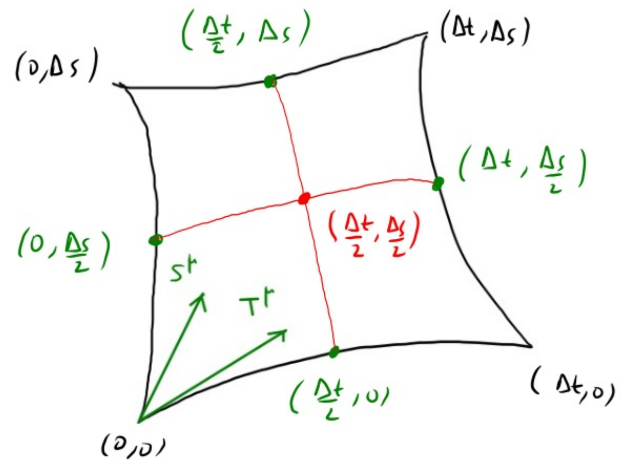
Parallel transport along (infinitesimal) closed curve:



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But $R^\mu{}_{\lambda\rho\nu} = -R^\mu{}_{\lambda\nu\rho}$

$$R^\mu{}_{\lambda\rho\nu}\,(\Delta t\, T^\rho)\,(\Delta s\, S^\nu) = R^\mu{}_{\lambda\rho\nu}\,\underbrace{\delta A^{\rho\nu}}_{\text{area element}}$$

Everything evaluated @ $\left(\frac{\Delta t}{2}, \frac{\Delta s}{2}\right)$:

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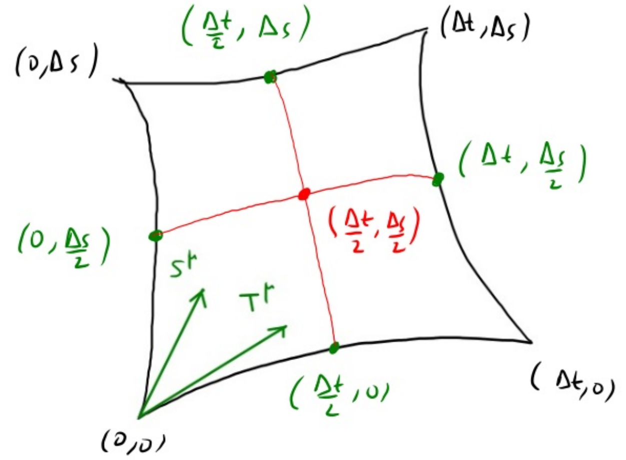

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