① Consider the metric (not a solution to Einstein's Equations)

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left[-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right]$$

- Calculate the redshift of photons for stationary observers at $r = R_1$ and $r = R_2$
- Go through the analysis of the radial motion of a freely falling particle (both massive & massless case), like we did in class for the Schwarzschild metric. In particular find the respective conserved quantities and the conditions that lead to bound/unbound trajectories and the qualitative properties of their shape.

- Anne and George are twins living on the surface of the "earth" at $r = R_1$.

Anne takes a lift along the radial direction on a (non geodesic) trajectory $r = vt + c_1$ to $r = R_2 > R_1$, and then she immediately comes back along $r = -vt + c_2$ to meet with George again

- calculate c_1 and c_2
- who is going to be older when they meet again, and by how much?
- Calulate Anne's 4-velocity and write down the condition $U^\mu U_\mu = -1$ as a relation between $\left(\frac{dt}{d\tau}\right)^2$, $\left(\frac{dr}{d\tau}\right)^2$ and r.

7. Two particles fall radially in from infinity in the Schwarzschild geometry. One starts with $e = 1$, the other with $e = 2$. A stationary observer at $r = 6M$ measures the speed of each when they pass by. How much faster is the second particle moving at that point?

8. A spaceship is moving without power in a circular orbit about a black hole of mass M. (The exterior geometry is the Schwarzschild geometry.) The Schwarzschild radius of the orbit is $7M$.

(a) What is the period of the orbit as measured by an observer at infinity?

(b) What is the period of the orbit as measured by a clock in the spaceship?

10. Find the linear velocity of a particle in a circular orbit of radius R in the Schwarzschild geometry that would be measured by a stationary observer stationed at one point on the orbit. What is its value at the ISCO?

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(2) Consider the 2-dimensional metric

$$ds^2 = \left(1 - \frac{2M}{r}\right)\left(-dt^2 + dr^2\right)$$

- Compute $\Gamma^{\mu}{}_{\nu\rho}$
- Compute $R^{\mu}{}_{\nu\rho\lambda}$, $R_{\mu\nu}$, R, $G_{\mu\nu}$
- Write down the explicit form of the geodesic equations

$$\ddot{x}^{\mu} = -\Gamma^{\mu}{}_{\nu\rho}\,\dot{x}^{\nu}\dot{x}^{\rho}$$

- Use the conserved quantity e^2, and make appropriate rescalings, to write down the equations in the form:

$$\ddot{r} = -\frac{1}{r^2}\left[\left(1-\frac{2}{r}\right)^{-1}\dot{r}^2 + \left(1-\frac{2}{r}\right)^{-3} e^2\right] \qquad \dot{t} = \left(1-\frac{2}{r}\right)^{-1} e$$

- You may practice going through the same steps for

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)}$$

$$ds^2 = \left(1-\frac{2M}{r}\right)\left(-dt^2 + dr^2 + r^2 d\varphi^2\right) \qquad (3\text{- dimensions})$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\varphi^2 \qquad (\quad " \quad)$$