# Differential Forms with Mathematica

## Use of the intrinsic function TensorWedge

We can compute wedge products of differential forms using TensorWedge:

$\omega \wedge \lambda$ = TensorWedge[$\omega$,$\lambda$]

The wedge like symbols is special, not the ordinary wedge ∧ ([Esc]^[Esc]).
You can type it with [Esc]t^[Esc] or \ [TensorWedge]

First define some differential forms:

```
d = 4; (* dimension of space *)

(*Two forms:*)
A = Array[Subscript[a, #1, #2] &, {d, d}];
B = Array[Subscript[b, #1, #2] &, {d, d}];

(*One Forms *)
Ω = Array[Subscript[ω, #1] &, {d}];
Λ = Array[Subscript[λ, #1] &, {d}];

$Assumptions =
  {(A | B) ∈ Arrays[{d, d}, Reals, Antisymmetric[All]], (Ω | Λ) ∈ Arrays[{d}, Reals]};

Print["A=", A // MatrixForm, "  B=", B // MatrixForm,
 "   Ω=", Ω // MatrixForm, "  Λ=", Λ // MatrixForm]
```

A=$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$ B=$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} \end{pmatrix}$ Ω=$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$ Λ=$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$

Whenever you want the two forms A,B to keep only the independent components, apply the following rule:

In[ ]:=
```
rule = {
    Table[a_{i,j} → -a_{j,i}, {i, 2, d}, {j, i - 1}],  Table[a_{i,i} → 0, {i, d}],
    Table[b_{i,j} → -b_{j,i}, {i, 2, d}, {j, i - 1}],  Table[b_{i,i} → 0, {i, d}]
   } // Flatten;

Print["A=", A /. rule // MatrixForm, "  B=", B /. rule // MatrixForm, "\nrule= ", rule]
```

A= $\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} \\ -a_{1,2} & 0 & a_{2,3} & a_{2,4} \\ -a_{1,3} & -a_{2,3} & 0 & a_{3,4} \\ -a_{1,4} & -a_{2,4} & -a_{3,4} & 0 \end{pmatrix}$ B= $\begin{pmatrix} 0 & b_{1,2} & b_{1,3} & b_{1,4} \\ -b_{1,2} & 0 & b_{2,3} & b_{2,4} \\ -b_{1,3} & -b_{2,3} & 0 & b_{3,4} \\ -b_{1,4} & -b_{2,4} & -b_{3,4} & 0 \end{pmatrix}$

rule= $\{a_{2,1} \to -a_{1,2}, a_{3,1} \to -a_{1,3}, a_{3,2} \to -a_{2,3}, a_{4,1} \to -a_{1,4}, a_{4,2} \to -a_{2,4}, a_{4,3} \to -a_{3,4}, a_{1,1} \to 0, a_{2,2} \to 0, a_{3,3} \to 0, a_{4,4} \to 0, b_{2,1} \to -b_{1,2}, b_{3,1} \to -b_{1,3}, b_{3,2} \to -b_{2,3}, b_{4,1} \to -b_{1,4}, b_{4,2} \to -b_{2,4}, b_{4,3} \to -b_{3,4}, b_{1,1} \to 0, b_{2,2} \to 0, b_{3,3} \to 0, b_{4,4} \to 0\}$

In[ ]:=
```
TensorWedge[Ω ∧ Λ] // Normal // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & \lambda_2\,\omega_1 - \lambda_1\,\omega_2 & \lambda_3\,\omega_1 - \lambda_1\,\omega_3 & \lambda_4\,\omega_1 - \lambda_1\,\omega_4 \\ -\lambda_2\,\omega_1 + \lambda_1\,\omega_2 & 0 & \lambda_3\,\omega_2 - \lambda_2\,\omega_3 & \lambda_4\,\omega_2 - \lambda_2\,\omega_4 \\ -\lambda_3\,\omega_1 + \lambda_1\,\omega_3 & -\lambda_3\,\omega_2 + \lambda_2\,\omega_3 & 0 & \lambda_4\,\omega_3 - \lambda_3\,\omega_4 \\ -\lambda_4\,\omega_1 + \lambda_1\,\omega_4 & -\lambda_4\,\omega_2 + \lambda_2\,\omega_4 & -\lambda_4\,\omega_3 + \lambda_3\,\omega_4 & 0 \end{pmatrix}$$

Same, use the ∧ symbol:

In[ ]:=
```
Ω ∧ Λ // Normal // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & \lambda_2\,\omega_1 - \lambda_1\,\omega_2 & \lambda_3\,\omega_1 - \lambda_1\,\omega_3 & \lambda_4\,\omega_1 - \lambda_1\,\omega_4 \\ -\lambda_2\,\omega_1 + \lambda_1\,\omega_2 & 0 & \lambda_3\,\omega_2 - \lambda_2\,\omega_3 & \lambda_4\,\omega_2 - \lambda_2\,\omega_4 \\ -\lambda_3\,\omega_1 + \lambda_1\,\omega_3 & -\lambda_3\,\omega_2 + \lambda_2\,\omega_3 & 0 & \lambda_4\,\omega_3 - \lambda_3\,\omega_4 \\ -\lambda_4\,\omega_1 + \lambda_1\,\omega_4 & -\lambda_4\,\omega_2 + \lambda_2\,\omega_4 & -\lambda_4\,\omega_3 + \lambda_3\,\omega_4 & 0 \end{pmatrix}$$

Wedge product of 1 form and 2 form:

In[•]:=
```
Ω ∧ A // Normal // MatrixForm
```

Out[•]//MatrixForm=

$$
\begin{pmatrix}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}\,\omega_3\left(a_{1,2}-a_{2,1}\right)-\frac{1}{2}\,\omega_2\left(a_{1,3}-a_{3,1}\right)+\frac{1}{2}\ldots \\ \frac{1}{2}\,\omega_4\left(a_{1,2}-a_{2,1}\right)-\frac{1}{2}\,\omega_2\left(a_{1,4}-a_{4,1}\right)+\frac{1}{2}\ldots \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}\,\omega_3\left(a_{1,2}-a_{2,1}\right)+\frac{1}{2}\,\omega_2\left(a_{1,3}-a_{3,1}\right)-\frac{1}{2}\,\omega_1\left(a_{2,3}-a_{3,2}\right) \\ -\frac{1}{2}\,\omega_4\left(a_{1,2}-a_{2,1}\right)+\frac{1}{2}\,\omega_2\left(a_{1,4}-a_{4,1}\right)-\frac{1}{2}\,\omega_1\left(a_{2,4}-a_{4,2}\right) \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \frac{1}{2}\,\omega_3\left(a_{1,2}-a_{2,1}\right)-\frac{1}{2}\,\omega_2\left(a_{1,3}-a_{3,1}\right)+\frac{1}{2}\,\omega_1\left(a_{2,3}-a_{3,2}\right) \\ 0 \\ -\frac{1}{2}\,\omega_4\left(a_{1,3}-a_{3,1}\right)+\frac{1}{2}\,\omega_3\left(a_{1,4}-a_{4,1}\right)-\frac{1}{2}\,\omega_1\left(a_{3,4}-a_{4,3}\right) \end{pmatrix} &
\begin{pmatrix} -\frac{1}{2}\,\omega_3\left(a_{1,2}-a_{2,1}\right)+\frac{1}{2}\,\omega_2\left(a_{1,3}-a_{3,1}\right)-\frac{1}{2}\ldots \\ 0 \\ 0 \\ -\frac{1}{2}\,\omega_4\left(a_{2,3}-a_{3,2}\right)+\frac{1}{2}\,\omega_3\left(a_{2,4}-a_{4,2}\right)-\frac{1}{2}\ldots \end{pmatrix} \\
\begin{pmatrix} 0 \\ \frac{1}{2}\,\omega_4\left(a_{1,2}-a_{2,1}\right)-\frac{1}{2}\,\omega_2\left(a_{1,4}-a_{4,1}\right)+\frac{1}{2}\,\omega_1\left(a_{2,4}-a_{4,2}\right) \\ \frac{1}{2}\,\omega_4\left(a_{1,3}-a_{3,1}\right)-\frac{1}{2}\,\omega_3\left(a_{1,4}-a_{4,1}\right)+\frac{1}{2}\,\omega_1\left(a_{3,4}-a_{4,3}\right) \\ 0 \end{pmatrix} &
\begin{pmatrix} -\frac{1}{2}\,\omega_4\left(a_{1,2}-a_{2,1}\right)+\frac{1}{2}\,\omega_2\left(a_{1,4}-a_{4,1}\right)-\frac{1}{2}\ldots \\ 0 \\ \frac{1}{2}\,\omega_4\left(a_{2,3}-a_{3,2}\right)-\frac{1}{2}\,\omega_3\left(a_{2,4}-a_{4,2}\right)+\frac{1}{2}\ldots \\ 0 \end{pmatrix}
\end{pmatrix}
$$

Get rid of redundant components of A: apply the rule:

In[•]:=
```
(Ω∧A // Normal) /. rule // MatrixForm
```

Out[•]//MatrixForm=

$$
\begin{pmatrix}
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ \omega_3\,a_{1,2}-\omega_2\,a_{1,3}+\omega_1\,a_{2,3} \\ \omega_4\,a_{1,2}-\omega_2\,a_{1,4}+\omega_1\,a_{2,4} \end{pmatrix} &
\begin{pmatrix} 0 \\ -\omega_3\,a_{1,2}+\omega_2\,a_{1,3}-\omega_1\,a_{2,3} \\ 0 \\ \omega_4\,a_{1,3}-\omega_3\,a_{1,4}+\omega_1\,a_{3,4} \end{pmatrix} &
\begin{pmatrix} -\omega_4\,a_{1,}\ldots \\ -\omega_4\,a_{1,}\ldots \\ \\ \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ -\omega_3\,a_{1,2}+\omega_2\,a_{1,3}-\omega_1\,a_{2,3} \\ -\omega_4\,a_{1,2}+\omega_2\,a_{1,4}-\omega_1\,a_{2,4} \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} \omega_3\,a_{1,2}-\omega_2\,a_{1,3}+\omega_1\,a_{2,3} \\ 0 \\ 0 \\ \omega_4\,a_{2,3}-\omega_3\,a_{2,4}+\omega_2\,a_{3,4} \end{pmatrix} &
\begin{pmatrix} \omega_4\,a_{1,}\ldots \\ \\ -\omega_4\,a_{2,}\ldots \\ \end{pmatrix} \\
\begin{pmatrix} 0 \\ \omega_3\,a_{1,2}-\omega_2\,a_{1,3}+\omega_1\,a_{2,3} \\ 0 \\ -\omega_4\,a_{1,3}+\omega_3\,a_{1,4}-\omega_1\,a_{3,4} \end{pmatrix} &
\begin{pmatrix} -\omega_3\,a_{1,2}+\omega_2\,a_{1,3}-\omega_1\,a_{2,3} \\ 0 \\ 0 \\ -\omega_4\,a_{2,3}+\omega_3\,a_{2,4}-\omega_2\,a_{3,4} \end{pmatrix} &
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} &
\begin{pmatrix} \omega_4\,a_{1,}\ldots \\ \omega_4\,a_{2,}\ldots \\ \\ \end{pmatrix} \\
\begin{pmatrix} 0 \\ \omega_4\,a_{1,2}-\omega_2\,a_{1,4}+\omega_1\,a_{2,4} \\ \omega_4\,a_{1,3}-\omega_3\,a_{1,4}+\omega_1\,a_{3,4} \\ 0 \end{pmatrix} &
\begin{pmatrix} -\omega_4\,a_{1,2}+\omega_2\,a_{1,4}-\omega_1\,a_{2,4} \\ 0 \\ \omega_4\,a_{2,3}-\omega_3\,a_{2,4}+\omega_2\,a_{3,4} \\ 0 \end{pmatrix} &
\begin{pmatrix} -\omega_4\,a_{1,3}+\omega_3\,a_{1,4}-\omega_1\,a_{3,4} \\ -\omega_4\,a_{2,3}+\omega_3\,a_{2,4}-\omega_2\,a_{3,4} \\ 0 \\ 0 \end{pmatrix} &
\end{pmatrix}
$$

Nonzero elements:  ( x === y is SameQ[x,y])

In[ ]:=
```
Select[Flatten[(Ω∧A // Normal) /. rule], (!# === 0) &] // Column
```

Out[ ]=

$\omega_3 a_{1,2} - \omega_2 a_{1,3} + \omega_1 a_{2,3}$
$\omega_4 a_{1,2} - \omega_2 a_{1,4} + \omega_1 a_{2,4}$
$-\omega_3 a_{1,2} + \omega_2 a_{1,3} - \omega_1 a_{2,3}$
$\omega_4 a_{1,3} - \omega_3 a_{1,4} + \omega_1 a_{3,4}$
$-\omega_4 a_{1,2} + \omega_2 a_{1,4} - \omega_1 a_{2,4}$
$-\omega_4 a_{1,3} + \omega_3 a_{1,4} - \omega_1 a_{3,4}$
$-\omega_3 a_{1,2} + \omega_2 a_{1,3} - \omega_1 a_{2,3}$
$-\omega_4 a_{1,2} + \omega_2 a_{1,4} - \omega_1 a_{2,4}$
$\omega_3 a_{1,2} - \omega_2 a_{1,3} + \omega_1 a_{2,3}$
$\omega_4 a_{2,3} - \omega_3 a_{2,4} + \omega_2 a_{3,4}$
$\omega_4 a_{1,2} - \omega_2 a_{1,4} + \omega_1 a_{2,4}$
$-\omega_4 a_{2,3} + \omega_3 a_{2,4} - \omega_2 a_{3,4}$
$\omega_3 a_{1,2} - \omega_2 a_{1,3} + \omega_1 a_{2,3}$
$-\omega_4 a_{1,3} + \omega_3 a_{1,4} - \omega_1 a_{3,4}$
$-\omega_3 a_{1,2} + \omega_2 a_{1,3} - \omega_1 a_{2,3}$
$-\omega_4 a_{2,3} + \omega_3 a_{2,4} - \omega_2 a_{3,4}$
$\omega_4 a_{1,3} - \omega_3 a_{1,4} + \omega_1 a_{3,4}$
$\omega_4 a_{2,3} - \omega_3 a_{2,4} + \omega_2 a_{3,4}$
$\omega_4 a_{1,2} - \omega_2 a_{1,4} + \omega_1 a_{2,4}$
$\omega_4 a_{1,3} - \omega_3 a_{1,4} + \omega_1 a_{3,4}$
$-\omega_4 a_{1,2} + \omega_2 a_{1,4} - \omega_1 a_{2,4}$
$\omega_4 a_{2,3} - \omega_3 a_{2,4} + \omega_2 a_{3,4}$
$-\omega_4 a_{1,3} + \omega_3 a_{1,4} - \omega_1 a_{3,4}$
$-\omega_4 a_{2,3} + \omega_3 a_{2,4} - \omega_2 a_{3,4}$

Nonzero elements:  ( x === y is SameQ[x,y]) with components:

```
list = (Ω∧A // Normal) /. rule;

Select[
  Flatten[
   Table[{Subscript["(Ω∧A)", i, j, k], list[[i, j, k]]}, {i, d}, {j, d}, {k, d}]
   (*make pairs of name+values*)
   , 2]           (*Flatten to level 2 only*)
  , (! #[[2]] === 0) & (*select nonzero elements*)
 ] // Column
```

Out[ ]=

$\{(\Omega\wedge A)_{1,2,3},\ \omega_3\,a_{1,2} - \omega_2\,a_{1,3} + \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{1,2,4},\ \omega_4\,a_{1,2} - \omega_2\,a_{1,4} + \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{1,3,2},\ -\omega_3\,a_{1,2} + \omega_2\,a_{1,3} - \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{1,3,4},\ \omega_4\,a_{1,3} - \omega_3\,a_{1,4} + \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{1,4,2},\ -\omega_4\,a_{1,2} + \omega_2\,a_{1,4} - \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{1,4,3},\ -\omega_4\,a_{1,3} + \omega_3\,a_{1,4} - \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{2,1,3},\ -\omega_3\,a_{1,2} + \omega_2\,a_{1,3} - \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{2,1,4},\ -\omega_4\,a_{1,2} + \omega_2\,a_{1,4} - \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{2,3,1},\ \omega_3\,a_{1,2} - \omega_2\,a_{1,3} + \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{2,3,4},\ \omega_4\,a_{2,3} - \omega_3\,a_{2,4} + \omega_2\,a_{3,4}\}$
$\{(\Omega\wedge A)_{2,4,1},\ \omega_4\,a_{1,2} - \omega_2\,a_{1,4} + \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{2,4,3},\ -\omega_4\,a_{2,3} + \omega_3\,a_{2,4} - \omega_2\,a_{3,4}\}$
$\{(\Omega\wedge A)_{3,1,2},\ \omega_3\,a_{1,2} - \omega_2\,a_{1,3} + \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{3,1,4},\ -\omega_4\,a_{1,3} + \omega_3\,a_{1,4} - \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{3,2,1},\ -\omega_3\,a_{1,2} + \omega_2\,a_{1,3} - \omega_1\,a_{2,3}\}$
$\{(\Omega\wedge A)_{3,2,4},\ -\omega_4\,a_{2,3} + \omega_3\,a_{2,4} - \omega_2\,a_{3,4}\}$
$\{(\Omega\wedge A)_{3,4,1},\ \omega_4\,a_{1,3} - \omega_3\,a_{1,4} + \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{3,4,2},\ \omega_4\,a_{2,3} - \omega_3\,a_{2,4} + \omega_2\,a_{3,4}\}$
$\{(\Omega\wedge A)_{4,1,2},\ \omega_4\,a_{1,2} - \omega_2\,a_{1,4} + \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{4,1,3},\ \omega_4\,a_{1,3} - \omega_3\,a_{1,4} + \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{4,2,1},\ -\omega_4\,a_{1,2} + \omega_2\,a_{1,4} - \omega_1\,a_{2,4}\}$
$\{(\Omega\wedge A)_{4,2,3},\ \omega_4\,a_{2,3} - \omega_3\,a_{2,4} + \omega_2\,a_{3,4}\}$
$\{(\Omega\wedge A)_{4,3,1},\ -\omega_4\,a_{1,3} + \omega_3\,a_{1,4} - \omega_1\,a_{3,4}\}$
$\{(\Omega\wedge A)_{4,3,2},\ -\omega_4\,a_{2,3} + \omega_3\,a_{2,4} - \omega_2\,a_{3,4}\}$

Check graded anticommutativity of wedge product:

In[ ]:=

```
((A ∧ Ω  - Ω∧A) // Normal) == ConstantArray[0, {d, d, d}]
```

Out[ ]=

```
True
```

In[ ]:=
```
Dimensions[list]
```

Out[ ]=
```
{4, 4, 4}
```

Wedge product of two 2-forms: the nonzero elements

In[ ]:=
```
Select[Flatten[(A∧B // Normal) /. rule], (!# === 0) &] // Column
```

Out[ ]=

$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$-a_{3,4} b_{1,2} + a_{2,4} b_{1,3} - a_{2,3} b_{1,4} - a_{1,4} b_{2,3} + a_{1,3} b_{2,4} - a_{1,2} b_{3,4}$
$a_{3,4} b_{1,2} - a_{2,4} b_{1,3} + a_{2,3} b_{1,4} + a_{1,4} b_{2,3} - a_{1,3} b_{2,4} + a_{1,2} b_{3,4}$

# Use of the function HodgeDual to compute duals of forms

Compare a two form and its Hodge Dual:

In[∘]:=
```
A1 = (          A  // Normal) /. rule;
A2 = (HodgeDual[A] // Normal) /. rule;
Print["A= ", A1 // MatrixForm, "    *A= ", A2 // MatrixForm]
```

A= $\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & a_{1,4} \\ -a_{1,2} & 0 & a_{2,3} & a_{2,4} \\ -a_{1,3} & -a_{2,3} & 0 & a_{3,4} \\ -a_{1,4} & -a_{2,4} & -a_{3,4} & 0 \end{pmatrix}$ *A= $\begin{pmatrix} 0 & a_{3,4} & -a_{2,4} & a_{2,3} \\ -a_{3,4} & 0 & a_{1,4} & -a_{1,3} \\ a_{2,4} & -a_{1,4} & 0 & a_{1,2} \\ -a_{2,3} & a_{1,3} & -a_{1,2} & 0 \end{pmatrix}$

The Hodge dual of a 4-form is a scalar:

In[∘]:=
```
HodgeDual[A∧B] /. rule
```

Out[∘]= $a_{3,4}\, b_{1,2} - a_{2,4}\, b_{1,3} + a_{2,3}\, b_{1,4} + a_{1,4}\, b_{2,3} - a_{1,3}\, b_{2,4} + a_{1,2}\, b_{3,4}$

The Hodge dual of a 3-form is a vector:

In[∘]:=
```
(HodgeDual[A∧Ω] // Normal) /. rule // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\omega_4\, a_{2,3} + \omega_3\, a_{2,4} - \omega_2\, a_{3,4} \\ \omega_4\, a_{1,3} - \omega_3\, a_{1,4} + \omega_1\, a_{3,4} \\ -\omega_4\, a_{1,2} + \omega_2\, a_{1,4} - \omega_1\, a_{2,4} \\ \omega_3\, a_{1,2} - \omega_2\, a_{1,3} + \omega_1\, a_{2,3} \end{pmatrix}$$

# Work with abstract forms without explicit components

Define 10 dimensional forms

In[∘]:=
```
Dim = 10;
$Assumptions = {
   (ξ | ξ1 | ξ2 | ξ3) ∈ Arrays[{Dim, Dim, Dim}, Reals, Antisymmetric[{1, 2, 3}]],
   σ ∈ Arrays[{Dim}, Reals, Antisymmetric[{1}]]
  }
```

Out[∘]=
```
{(ξ | ξ1 | ξ2 | ξ3) ∈ Arrays[{10, 10, 10}, ℝ, Antisymmetric[{1, 2, 3}]], σ ∈ Arrays[{10}, ℝ, {}]}
```

This is left unevaluated:

In[∘]:=
```
ξ ∧ σ + σ∧ξ
```

Out[∘]= ξ∧σ + σ∧ξ

TensorReduce puts tensors in canonical form. Here it applies the anticommutative property of the wedge product:

```
In[ ]:= ξ ∧ σ + σ∧ξ // TensorReduce
Out[ ]= 0
```

Linear combinations and wedge product of those: Here they are left unevaluated:

```
In[ ]:= (a1 ξ1 + a2 ξ2) ∧ (b1 ξ1 + b2 ξ2)
Out[ ]= (a1 ξ1 + a2 ξ2)∧(b1 ξ1 + b2 ξ2)
```

The products are expanded:

```
In[ ]:= (a1 ξ1 + a2 ξ2) ∧ (b1 ξ1 + b2 ξ2) // TensorExpand
Out[ ]= a1 b2 ξ1∧ξ2 + a2 b1 ξ2∧ξ1
```

A differential form of order higher than Dim is zero:

```
In[ ]:= ξ ∧ ξ1 ∧ ξ2 ∧ ξ3
Out[ ]= 0
```