- (Anti) symmetrization of tensors
- ϵ - algebra (Levi-Civita symbol)

Problems:

- (Anti) symmetrization of tensors

e.g. $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$ $\downarrow$ antisymmetric

$A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$ $\downarrow$ symmetric

Problems:

- (Anti) symmetrization of tensors

e.g. $A_{[\mu\nu]} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right)$ $\quad$ $A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right)$

$$A_{(\mu_1 \mu_2 \dots \mu_k)} = \frac{1}{k!} \sum_{\sigma} A_{\sigma(\mu_1)\sigma(\mu_2)\dots\sigma(\mu_k)}$$

$$A_{[\mu_1 \mu_2 \dots \mu_k]} = \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma) \, A_{\sigma(\mu_1)\sigma(\mu_2)\dots\sigma(\mu_k)}$$

$$\sigma = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ \sigma(\mu_1) & \sigma(\mu_2) & \cdots & \sigma(\mu_k) \end{pmatrix}$$

1-1 map of $\{\mu_k\}$ to themselves

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1-1 map of $\{\mu_k\}$ to themselves

$$\mathrm{sign}(\sigma) = (-1)^{\left(\begin{smallmatrix}\#\text{ of exchanges } \mu_i \leftrightarrow \mu_j \\ \text{that result in } \sigma\end{smallmatrix}\right)} = \begin{cases} +1 & \#\text{ is even} \\ -1 & \#\text{ is odd} \end{cases}$$

$$A_{(\mu_1 \mu_2 \ldots \mu_k)} = \frac{1}{k!} \sum_{\sigma} A_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)}$$

$$A_{[\mu_1 \mu_2 \ldots \mu_k]} = \frac{1}{k!} \sum_{\sigma} \mathrm{sign}(\sigma)\, A_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)}$$

Levi-Civita symbol (or epsilon)

A fully antisymmetric symbol $\epsilon_{\mu_1 \cdots \mu_k}$, with nonzero elements:

$$\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(k)} = \mathrm{sign}(\sigma)\, \underset{\substack{\parallel \\ 1}}{\epsilon_{12\cdots k}} = \mathrm{sign}(\sigma)$$

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example: $\epsilon_{11} = 0$ $\epsilon_{12} = +1$

$\epsilon_{21} = -1$ $\epsilon_{22} = 0$

Levi-Civita symbol (or epsilon)

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$$\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(k)} = \operatorname{sign}(\sigma)\, \epsilon_{12\cdots k} = \operatorname{sign}(\sigma)$$

* How to compute all permutations of indices?
* How to compute $\operatorname{sign}(\sigma)$?

The $k=3$ case

* first nontrivial case
* appears often in 3-d physics

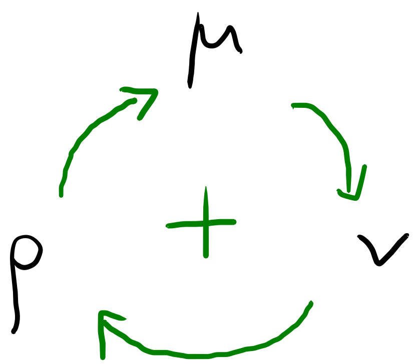
The $k=3$ case

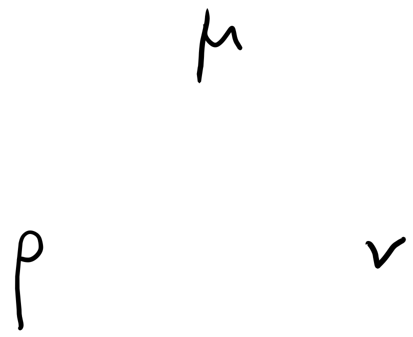
* An easy mnemonic rule:

$$\begin{array}{ccc} & \mu & \\ \rho & & \nu \end{array} \qquad\qquad \begin{array}{ccc} & \mu & \\ \rho & & \nu \end{array}$$

The $k=3$ case

* An easy mnemonic rule:





$$\begin{matrix} \mu & \nu & \rho \\ \rho & \mu & \nu \\ \nu & \rho & \mu \end{matrix}$$

$\mathrm{sign}(\sigma) = +1$

The $k=3$ case

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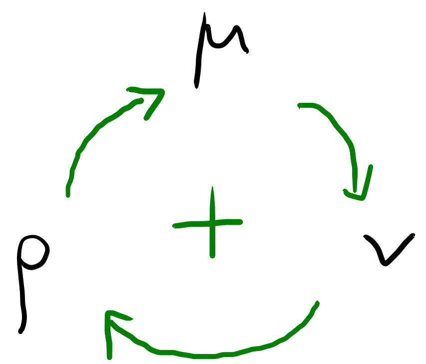

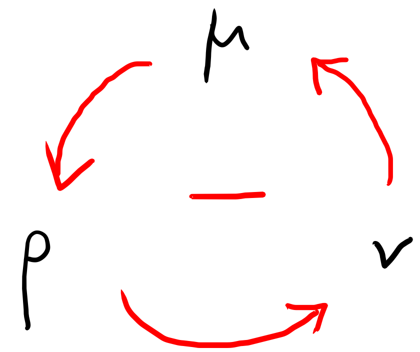

$$\begin{matrix} \mu & \nu & \rho \\ \rho & \mu & \nu \\ \nu & \rho & \mu \end{matrix} \qquad\qquad \begin{matrix} \mu & \rho & \nu \\ \nu & \mu & \rho \\ \rho & \nu & \mu \end{matrix}$$

$\mathrm{sign}(\sigma) = +1$ $\qquad\qquad$ $\mathrm{sign}(\sigma) = -1$

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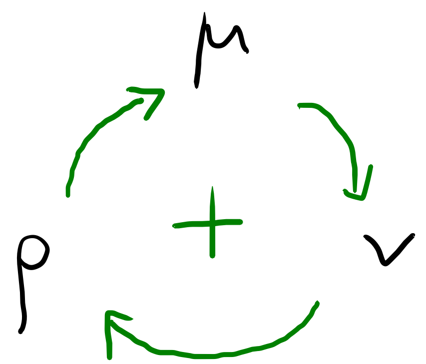

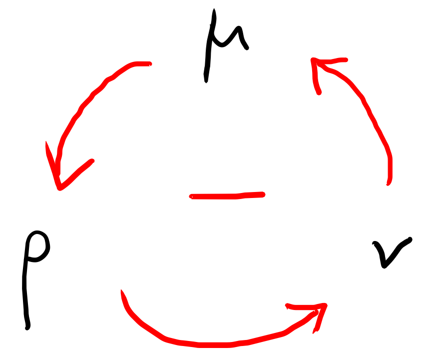

$$\begin{array}{ccc} \mu & \nu & \rho \\ \rho & \mu & \nu \\ \nu & \rho & \mu \end{array} \qquad\qquad \begin{array}{ccc} \mu & \rho & \nu \\ \nu & \mu & \rho \\ \rho & \nu & \mu \end{array}$$

$$\mathrm{sign}(\sigma) = +1 \qquad\qquad \mathrm{sign}(\sigma) = -1$$

* Examples:

(α) $$A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(A \quad A \quad A$$
$$A \quad A \quad A \Big)$$

The k=3 case

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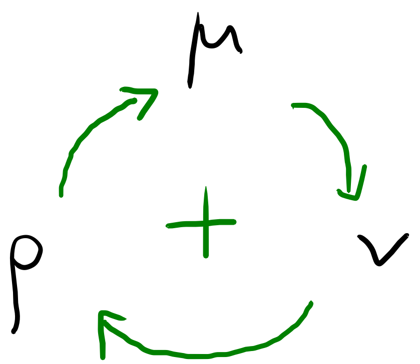

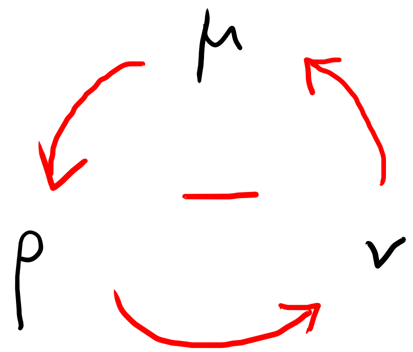

μ	ν	ρ
ρ	μ	ν
ν	ρ	μ

$\mathrm{sign}(\sigma) = +1$

μ	ρ	ν
ν	μ	ρ
ρ	ν	μ

$\mathrm{sign}(\sigma) = -1$

* Examples:

(α) $$A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(+A \quad +A \quad +A \quad -A \quad -A \quad -A \Big)$$

The $k=3$ case

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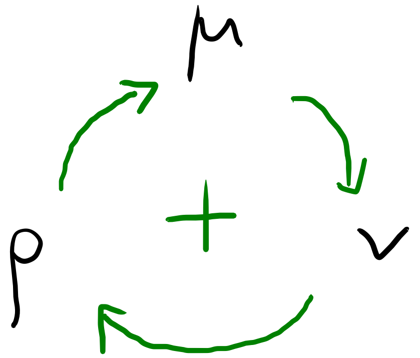

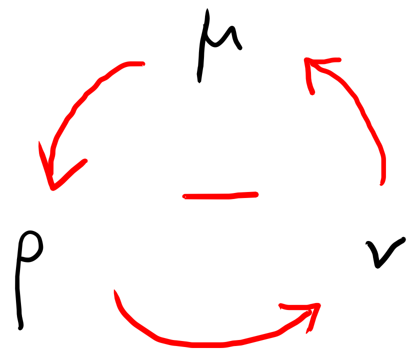

$$\begin{array}{ccc} \mu & \nu & \rho \\ \rho & \mu & \nu \\ \nu & \rho & \mu \end{array} \qquad\qquad \begin{array}{ccc} \mu & \rho & \nu \\ \nu & \mu & \rho \\ \rho & \nu & \mu \end{array}$$

$$\mathrm{sign}(\sigma) = +1 \qquad\qquad \mathrm{sign}(\sigma) = -1$$

* Examples:

$$(\alpha)\quad A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(+A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \Big)$$

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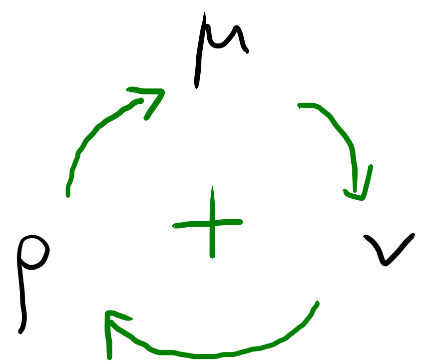

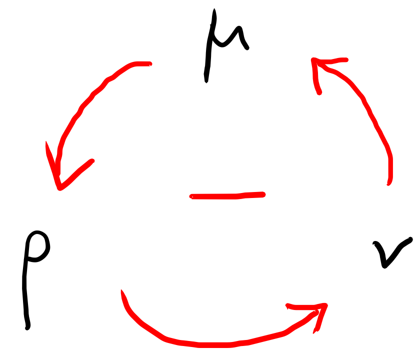

μ	ν	ρ		μ	ρ	ν
ρ	μ	ν		ν	μ	ρ
ν	ρ	μ		ρ	ν	μ

$\mathrm{sign}(\sigma) = +1$ $\qquad\qquad$ $\mathrm{sign}(\sigma) = -1$

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(β) $$\vec{F} = q\,\vec{v}\times\vec{B} \Leftrightarrow f_i = \frac{F_i}{q} = \epsilon_{ijk}\, v_j B_k$$

The k=3 case

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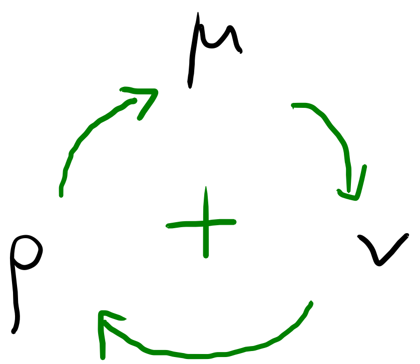

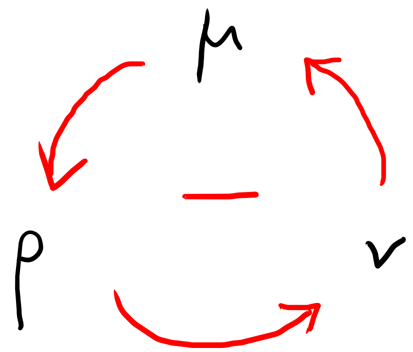

μ	ν	ρ
ρ	μ	ν
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ν	μ	ρ
ρ	ν	μ

$\mathrm{sign}(\sigma) = +1$ $\qquad$ $\mathrm{sign}(\sigma) = -1$

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$$(\beta)\quad \vec{F} = q\,\vec{v}\times\vec{B} \iff f_i = \frac{F_i}{q} = \epsilon_{ijk}\, v_j B_k$$

$$f_1 = \quad v\,B \qquad v\,B$$

$$f_2 = \quad v\,B \qquad v\,B$$

$$f_3 = \quad v\,B \qquad v\,B$$

The $k=3$ case

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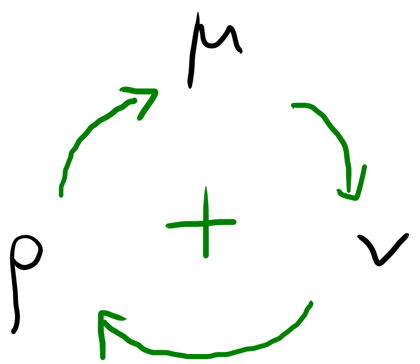

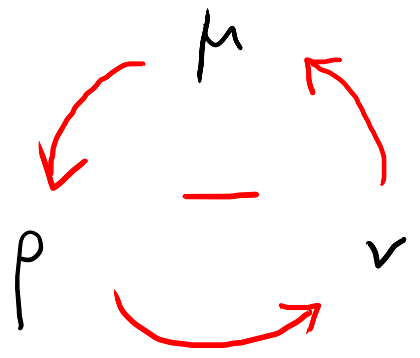

μ	ν	ρ
ρ	μ	ν
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$\text{sign}(\sigma) = +1$

μ	ρ	ν
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(α)
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$$f_1 = +\, v\ B - \ v\ B$$
$$f_2 = +\, v\ B - \ v\ B$$
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The $k=3$ case

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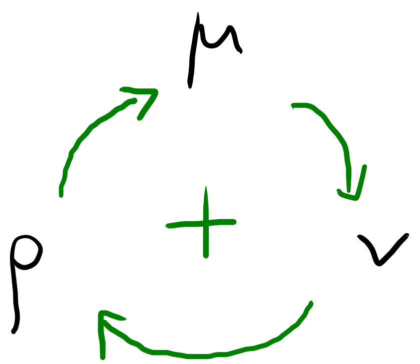

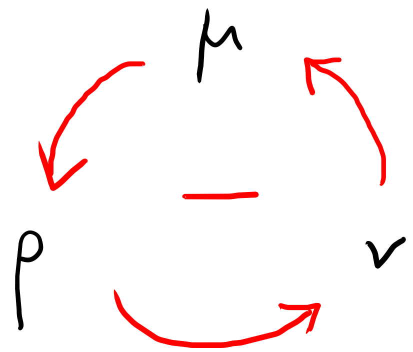

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(β) $$\vec{F} = q\,\vec{v}\times\vec{B} \iff f_i = \frac{F_i}{q} = \epsilon_{ijk} v_j B_k$$

$$\begin{array}{lll} f_1 = +v_2 B_3 - \quad v\ B & \qquad & \epsilon_{123} \\ f_2 = +v_3 B_1 - \quad v\ B & \qquad & \epsilon_{231} \\ f_3 = +v_1 B_2 - \quad v\ B & \qquad & \epsilon_{312} \end{array}$$

The $k=3$ case

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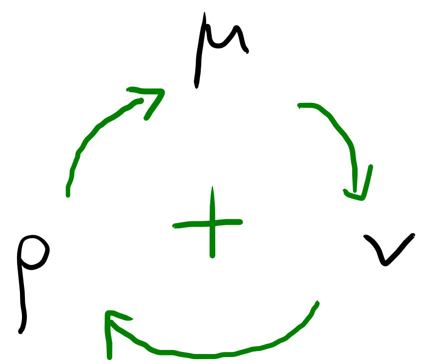

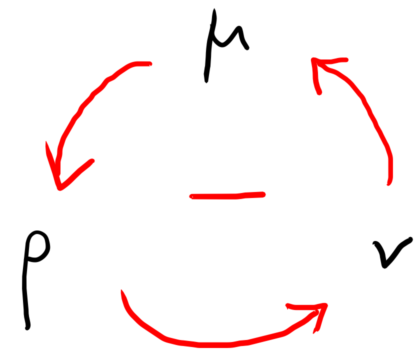

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(β) $$\vec{F} = q\,\vec{v} \times \vec{B} \iff f_i = \frac{F_i}{q} = \epsilon_{ijk}\, v_j B_k$$

$$\begin{aligned} f_1 &= +\, v_2 B_3 - v_3 B_2 \qquad & \epsilon_{132} \\ f_2 &= +\, v_3 B_1 - v_1 B_3 \qquad & \epsilon_{213} \\ f_3 &= +\, v_1 B_2 - v_2 B_1 \qquad & \epsilon_{321} \end{aligned}$$

The $k=3$ case

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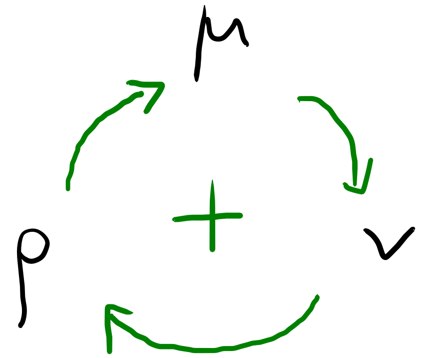

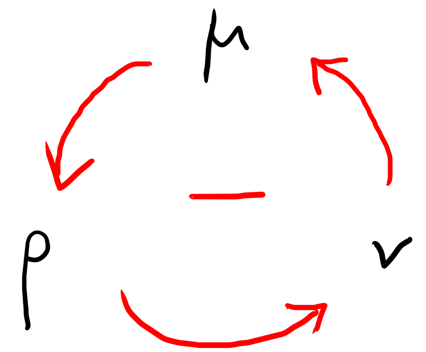

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$$f_1 = + v_2 B_3 - v_3 B_2$$
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The k=3 case

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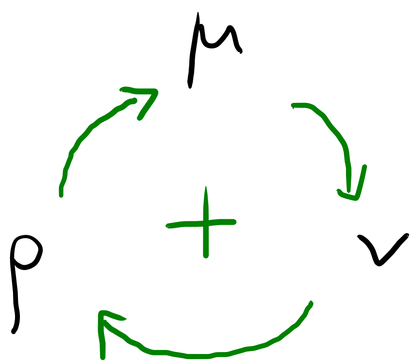

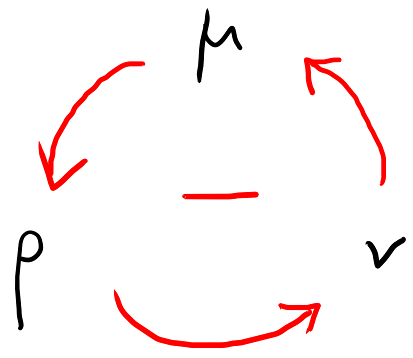

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The $k=3$ case

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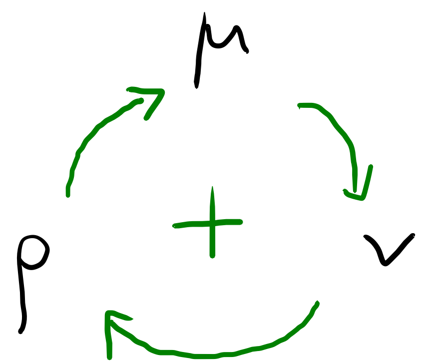

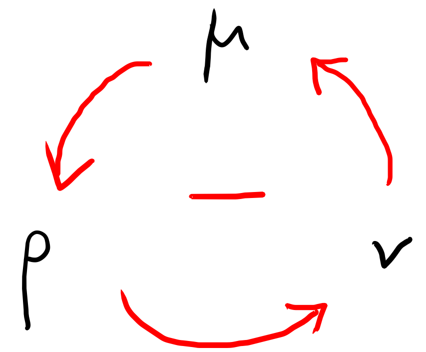

μ	ν	ρ
ρ	μ	ν
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The $k=3$ case

Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

* Examples:

(α) $$A_{[\mu\nu\rho]} = \frac{1}{3!}\left(+ A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \right)$$

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$$\sigma_i \sigma_j = \mathbb{1}_{2\times 2} \, \delta_{ij} + i \, \epsilon_{ijk} \, \sigma_k$$

* Examples:

(α) $$A_{[\mu\nu\rho]} = \frac{1}{3!}\left(+ A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \right)$$

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$$B_1 = \partial_2 A_3 - \partial_3 A_2$$
$$B_2 = \partial_3 A_1 - \partial_1 A_3$$
$$B_3 = \partial_1 A_2 - \partial_2 A_1$$

The $k=3$ case

Pauli Matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \mathbb{1}_{2\times 2} \delta_{ij} + i \epsilon_{ijk} \sigma_k$$

$$\sigma_2 \sigma_3 = + i \sigma_1 \qquad 2 \to 3 \to 1$$

* Examples:

(α) $$A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(+ A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \Big)$$

(β) $$\vec{F} = q\,\vec{v} \times \vec{B} \iff f_i = \frac{F_i}{q} = \epsilon_{ijk} v_j B_k$$

$$f_1 = + v_2 B_3 - v_3 B_2$$
$$f_2 = + v_3 B_1 - v_1 B_3$$
$$f_3 = + v_1 B_2 - v_2 B_1$$

(γ) $$\vec{B} = \vec{\nabla} \times \vec{A} \iff B_i = \epsilon_{ijk} \partial_j A_k$$

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$\ast$ Examples:

(α)

$$A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(+A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \Big)$$

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The $k > 3$ cases

* How to construct all permutations of objects $(1\ 2 \ldots k)$

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  - I have k slots to fill: $\underbrace{\_\ \_\ \_\ \cdots\ \_\ \_}_{k}$

The $k > 3$ cases

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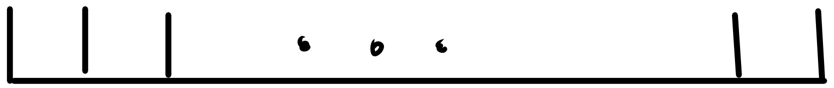
  - I have k slots to fill:



  - I fill the 1st one in k ways choosing 1 out of k available

k ways

The $k > 3$ cases

* How to construct all permutations of objects $(1\ 2 \dots k)$

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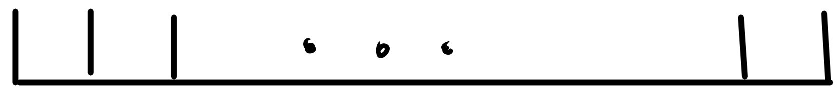

- I fill the 1^{st} one in k ways choosing 1 out of k available

k ways

- I fill the 2^{nd} one in $(k-1)$ way choosing 1 out of the remaining $(k-1)$

$k(k-1)$ ways

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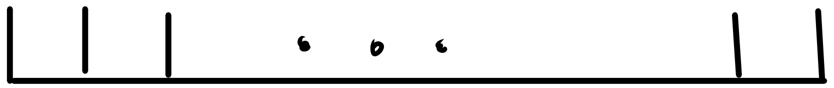

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$k(k-1)$ ways

  - Then fill the other slots in $k \times (k-1) \times (k-2) \times (k-3) \times \cdots$ ways

The $k > 3$ cases

* How to construct all permutations of objects $(1\ 2 \ldots k)$

- I have k slots to fill:

- I fill the 1st one in k ways choosing 1 out of k available

- I fill the 2nd one in $(k-1)$ way choosing 1 out of the remaining $(k-1)$

- Then fill the other slots in $k \times (k-1) \times (k-2) \times (k-3) \times \cdots$ ways

- For the last slot, I have no choice: only one element left!

Total: $k(k-1) \cdots 2 \cdot 1$

$k!$

1				
2				
3				
4				

1	1 2			
	1 3			
	1 4			
2	2 1			
	2 3			
	2 4			
3	3 1			
	3 2			
	3 4			
4	4 1			
	4 2			
	4 3			

1	12	12 3		
		12 4		
	13	13 2		
		13 4		
	14	14 2		
		14 3		
2	21	21 3		
		21 4		
	23	23 1		
		23 4		
	24	24 1		
		24 3		
3	31	31 2		
		31 4		
	32	32 1		
		32 4		
	34	34 1		
		34 2		
4	41	41 2		
		41 3		
	42	42 1		
		42 3		
	43	43 1		
		43 2		

1	12	123	1234	
		124	1243	
	13	132	1324	
		134	1342	
	14	142	1423	
		143	1432	
2	21	213	2134	
		214	2143	
	23	231	2314	
		234	2341	
	24	241	2413	
		243	2431	
3	31	312	3124	
		314	3142	
	32	321	3214	
		324	3241	
	34	341	3412	
		342	3421	
4	41	412	4123	
		413	4132	
	42	421	4213	
		423	4231	
	43	431	4312	
		432	4321	

Sign of Permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{pmatrix}$$

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$$\text{sign}(\sigma) = \prod_{0 \leq i < j \leq k} \text{sign}(\sigma(j) - \sigma(i))$$

$$= \text{singn}(\sigma(2) - \sigma(1)) \cdot \text{sign}(\sigma(3) - \sigma(1)) \cdot \text{sign}(\sigma(3) - \sigma(2)) \cdots$$

Sign of Permutation

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---

* need $\Theta(k^2)$ operations

Sign of Permutation

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---

* need $\Theta(k^2)$ operations

* better way: $\Theta(k \ln k)$ operations : compute parity of disjoint cycles

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Cycles of permutation

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$$(1\ 3\ 5)$$

cycle

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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

$(2\ 4\ 7)$
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$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

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Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\,3\,5) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \end{array}$$

$(1\ 3\ 5)$ cycle

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

$(2\ 4\ 7)$ cycle

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

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$(1\ 3\ 5)$ cycle

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$(2\ 4\ 7)$ cycle

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But $(1\ 3\ 5) = (1\ 3)(3\,5)$

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But $(1\;3\;5) = (1\;3)(3\;5)$

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\;3) & \bigcirc & 2 & \bigcirc & 4 & 5 & 6 & 7 \end{array}$$

Cycles of permutation

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But $(1\ 3\ 5) = (1\ 3)(3\ 5)$

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\,3) & 3 & 2 & 1 & 4 & 5 & 6 & 7 \\ (3\,5) & 3 & 2 & \bigcirc & 4 & \bigcirc & 6 & 7 \end{array}$$

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

	1	2	3	4	5	6	7
$(1\,3\,5)$	3	2	5	4	1	6	7
$(2\,4\,7)$	3	4	5	7	1	6	2

But $(1\,3\,5) = (1\,3)(3\,5)$

	1	2	3	4	5	6	7
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Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

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$$\begin{array}{l|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(1\,3\,5) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \\
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\end{array}$$

But $(1\ 3\ 5) = (1\ 3)(3\ 5)$

$$\begin{array}{l|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
(1\ 3) & 3 & 2 & 1 & 4 & 5 & 6 & 7 \\
(3\ 5) & 3 & 2 & 5 & 4 & 1 & 6 & 7
\end{array}$$

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(135)(247)$ product

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (135) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \\ (247) & 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{array}$$

But $(1\ 3\ 5) = (1\ 3)(3\ 5)$

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\ 3) & 3 & 2 & 1 & 4 & 5 & 6 & 7 \\ (3\ 5) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \end{array}$$

and $(2\ 4\ 7) = (2\ 4)(4\ 7)$

$$\begin{array}{c|ccccccc} \text{original} & 3 & 2 & 5 & 4 & 1 & 6 & 7 \\ (2,4) & 3 & & 5 & & 1 & 6 & 7 \end{array}$$

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

	1	2	3	4	5	6	7
$(1\,3\,5)$	3	2	5	4	1	6	7
$(2\,4\,7)$	3	4	5	7	1	6	2

But $(\overline{1\,3}\,\underline{\,5})$... $(1\,3\,5) = (1\,3)(3\,5)$

	1	2	3	4	5	6	7
$(1\ 3)$	3	2	1	4	5	6	7
$(3\ 5)$	3	2	5	4	1	6	7

and $(2\,4\,7) = (2\,4)(4\,7)$

original	3	2	5	4	1	6	7
$(2,4)$	3	4	5	2	1	6	7

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1 3 5)(2 4 7)$ product

	1	2	3	4	5	6	7
$(1\,3\,5)$	3	2	5	4	1	6	7
$(2\,4\,7)$	3	4	5	7	1	6	2

But $(1\,3\,5) = (1\,3)(3\,5)$

	1	2	3	4	5	6	7
$(1\,3)$	3	2	1	4	5	6	7
$(3\,5)$	3	2	5	4	1	6	7

and $(2\,4\,7) = (2\,4)(4\,7)$

original	3	2	5	4	1	6	7
$(2,4)$	3	4	5	2	1	6	7
$(4,7)$	3	4	5		1	6	

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

	1	2	3	4	5	6	7
(1 3 5)	3	2	5	4	1	6	7
(2 4 7)	3	4	5	7	1	6	2

But $(1\,3\,5) = (1\,3)(3\,5)$

	1	2	3	4	5	6	7
(1 3)	3	2	1	4	5	6	7
(3 5)	3	2	5	4	1	6	7

and $(2\,4\,7) = (2\,4)(4\,7)$

original	3	2	5	4	1	6	7
(2, 4)	3	4	5	2	1	6	7
(4, 7)	3	4	5	7	1	6	2

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(135)(247)$ product

	1	2	3	4	5	6	7
$(1\,3\,5)$	3	2	5	4	1	6	7
$(2\,4\,7)$	3	4	5	7	1	6	2

But $(1\ 3\ 5) = (1\ 3)(3\ 5)$

	1	2	3	4	5	6	7
$(1\ 3)$	3	2	1	4	5	6	7
$(3\ 5)$	3	2	5	4	1	6	7

and $(2\ 4\ 7) = (2\ 4)(4\ 7)$

original	3	2	5	4	1	6	7
$(2,4)$	3	4	5	2	1	6	7
$(4,7)$	3	4	5	7	1	6	2

Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

	1	2	3	4	5	6	7
$(1\,3\,5)$	3	2	5	4	1	6	7
$(2\,4\,7)$	3	4	5	7	1	6	2

Therefore:

$$\sigma = (1\,3)(3\,5)(2\,4)(4\,7)$$

But $(1\,3\,5) = (1\,3)(3\,5)$

	1	2	3	4	5	6	7
$(1\,3)$	3	2	1	4	5	6	7
$(3\,5)$	3	2	5	4	1	6	7

and $(2\,4\,7) = (2\,4)(4\,7)$

original	3	2	5	4	1	6	7
$(2,4)$	3	4	5	2	1	6	7
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Cycles of permutation

Example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Consistent with $(1\,3\,5)(2\,4\,7)$ product

$$\begin{array}{l|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\,3\,5) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \\ (2\,4\,7) & 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{array}$$

Therefore: Even!

$\sigma = (1\,3)(3\,5)(2\,4)(4\,7)$

$\Rightarrow \operatorname{sign}(\sigma) = +1$

But $(1\;3\;5) = (1\;3)(3\;5)$

$$\begin{array}{l|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ (1\;3) & 3 & 2 & 1 & 4 & 5 & 6 & 7 \\ (3\;5) & 3 & 2 & 5 & 4 & 1 & 6 & 7 \end{array}$$

and $(2\;4\;7) = (2\;4)(4\;7)$

$$\begin{array}{l|ccccccc} \text{original} & 3 & 2 & 5 & 4 & 1 & 6 & 7 \\ (2,4) & 3 & 4 & 5 & 2 & 1 & 6 & 7 \\ (4,7) & 3 & 4 & 5 & 7 & 1 & 6 & 2 \end{array}$$

Exercise:

Construct the k=7 permutations:

$(1\ 3\ 5\ 7) = (1\ 3)(3\ 5)(5\ 7)$ odd

$(1\ 3\ 5\ 7\ 4) = (1\ 3)\ (3\ 5)\ (5\ 7)\ (7\ 4)$ even

$(2\ 4\ 6)(3\ 5\ 7) = (2\ 4)(4\ 6)\ (3\ 5)\ (5\ 7)$ even

sign(σ) computation algorithm:

- determine k-cycles of permutation
- each cycle contributes $(-1)^{k-1}$ to the sign

A LOT OF WORK FOR LARGE k

$\rightarrow$ computer aid ...

$\mathrm{sign}(\sigma)$ computation algorithm:

- determine k-cycles of permutation
- each cycle contributes $(-1)^{k-1}$ to the sign

1	12	12 3	123 4	+	
		12 4	124 3	−	(43)
	13	13 2	132 4		
		13 4	134 2		
	14	14 2	142 3		
		14 3	143 2		
2	21	21 3	213 4		
		21 4	214 3		
	23	23 1	231 4		
		23 4	234 1		
	24	24 1	241 3		
		24 3	243 1		
3	31	31 2	312 4		
		31 4	314 2		
	32	32 1	321 4		
		32 4	324 1		
	34	34 1	341 2		
		34 2	342 1		
4	41	41 2	412 3		
		41 3	413 2		
	42	42 1	421 3		
		42 3	423 1		
	43	43 1	431 2		
		43 2	432 1		

1 2 3 4

1 2 4 3

1	12	123	1234	+	
		124	1243	−	(43)
	13	132	1324	−	(32)
		134	1342		
	14	142	1423		
		143	1432		
2	21	213	2134		
		214	2143		
	23	231	2314		
		234	2341		
	24	241	2413		
		243	2431		
3	31	312	3124		
		314	3142		
	32	321	3214		
		324	3241		
	34	341	3412		
		342	3421		
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	42	421	4213		
		423	4231		
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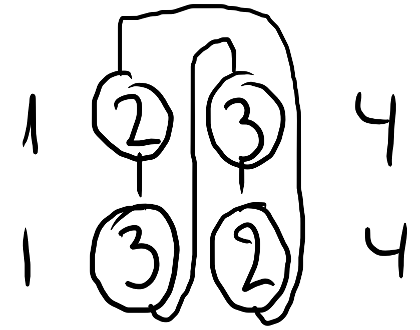

1	12	123	1234	+	
		124	1243	−	(43)
	13	132	1324	−	(32)
		134	1342	+	(234) = (23)(34)
	14	142	1423		
		143	1432		
2	21	213	2134		
		214	2143		
	23	231	2314		
		234	2341		
	24	241	2413		
		243	2431		
3	31	312	3124		
		314	3142		
	32	321	3214		
		324	3241		
	34	341	3412		
		342	3421		
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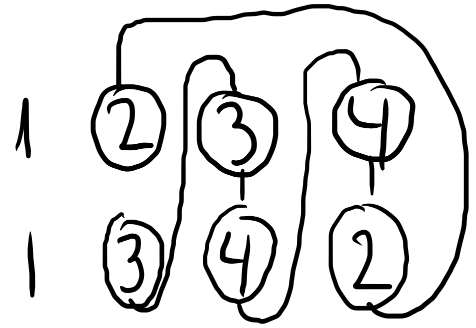

1	12	123	1234	+	
		124	1243	−	(43)
	13	132	1324	−	(32)
		134	1342	+	(234) = (23)(34)
	14	142	1423	+	(243) = (24) (43)
		143	1432		
2	21	213	2134		
		214	2143		
	23	231	2314		
		234	2341		
	24	241	2413		
		243	2431		
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		314	3142		
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		342	3421		
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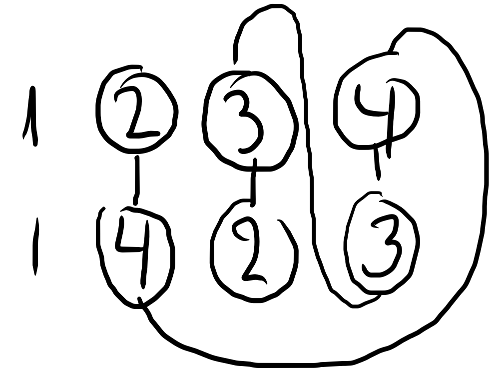

1	12	123	1234	+	
		124	1243	−	(43)
	13	132	1324	−	(32)
		134	1342	+	(234) = (23)(34)
	14	142	1423	+	(243) = (24) (43)
		143	1432	−	(24)
2	21	213	2134	−	(12)
		214	2143	+	(12)(34)
	23	231	2314	+	(123) = (12)(23)
		234	2341	−	(1234) = (12) (23)(34)
	24	241	2413	−	(2431) = (24) (43)(31)
		243	2431	+	(124) = (12) (24)
3	31	312	3124	+	(132) = (13) (32)
		314	3142	−	(1342) = (13) (34) (42)
	32	321	3214	−	(13)
		324	3241	+	(134) = (13)(34)
	34	341	3412	+	(13)(24)
		342	3421	−	(1324) = (13)(32)(24)
4	41	412	4123	−	(1432) = (14)(43) (32)
		413	4132	+	(142) = (14) (42)
	42	421	4213	+	(143) = (14) (43)
		423	4231	−	(14)
	43	431	4312	−	(1423) = (14) (42) (23)
		432	4321	+	(14) (23)