Affine Connection & Curvature Calculations Using Maxima

* use package ctensor (included...)
* Read ctensor documentation and arXiv:cs/0503073
  V. Toth, Tensor manipulation with GPL Maxima
* Basic Usage: Input: metric
  Output: $\Gamma^{\mu}{}_{\nu\rho}$, $R^{\mu}{}_{\nu\rho\sigma}$, $R_{\mu\nu}$, R, $G_{\mu\nu}$, R^2, ...
  (there is more than that in ctensor)

* Assume that you know the very basic:
  - entering input in notebooks + evaluation
  - assignment of variables
  - representation of matrices

* The wxmaxima Help menu has links to documentation ctensor documentation (online + local):

https://maxima.sourceforge.io/docs/manual/maxima_128.html

and the math:

https://arxiv.org/abs/cs/0503073

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Ubuntu install:

sudo apt install wxmaxima

Otherwise:
https://wxmaxima-developers.github.io/wxmaxima/
https://maxima.sourceforge.io/

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Introductory videos by me: https://youtu.be/RmF_MECumyI
(in Greek) https://youtu.be/kvtrETJotx8

* Basic Usage:

Documentation:
https://maxima.sourceforge.io/docs/manual/maxima_128.html

- load package
- enter dimension of M
- enter coordinate system: names of coordinates
- enter metric $g_{\mu\nu}$
- Compute:
  - cmetric() $\rightarrow$ $g^{\mu\nu}$, $\det g$
  - christof(mcs) $\rightarrow$ $\Gamma^{\mu}{}_{\nu\rho}$
  - riemann(true) $\rightarrow$ $R^{\mu}{}_{\nu\rho\sigma}$
  - ricci(true) $\rightarrow$ $R_{\mu\nu}$

cgeodesic(true) $\rightarrow$ geodesic equations

leinstein(true) $\rightarrow$ $G_{\mu\nu}$

scurvature() $\rightarrow$ R

- Variables:

lg[i,j]	g_{ij}	dim	n (dimension of M)
ug[i,j]	g^{ij}	gdet	g
mcs[i,j,k]	$\Gamma^k{}_{ij}$	ct_coords[]	coordinates
riem[i,j,k,m]	$-R^m{}_{ijk}$		
ric[i,j]	R_{ij}		
lein[i,j]	G_{ij}		
geod[i]	geodesic equation for x^i		

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* Careful with Γ + R indices!

$$\mathrm{mcs}[i,j,k] = \Gamma^{k}{}_{ij} = \frac{1}{2} g^{km}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right)$$

$$\mathrm{riem}[i,j,k,m] = -R^{m}{}_{ijk}$$

$$= -\left\{ \partial_j \Gamma^{m}_{ki} - \partial_k \Gamma^{m}_{ji} + \Gamma^{m}_{jn}\Gamma^{n}_{ki} - \Gamma^{m}_{kn}\Gamma^{n}_{ji} \right\}$$

$$\mathrm{lriem}[i,j,k,m] = -R_{mijk} = -g_{mn} R^{n}{}_{ijk}$$

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$$\mathrm{lriem}[i,j,k,m] = -R_{mijk} = -g_{mn} R^n{}_{ijk}$$

* $R_{\mu\nu}$, $G_{\mu\nu}$ same as the conventions in class:

$$\mathrm{ric}[i,j] = R_{ij} = R^m{}_{imj}$$

$$\mathrm{lein}[i,j] = G_{ij}$$

Results:

Documentation:
https://maxima.sourceforge.io/docs/manual/maxima_128.html

- **Friedman Metric:**

$$ds^2 = -dt^2 + a^2(t)\,d\chi^2 + a^2(t)\sin^2\chi\,d\theta^2 + a^2(t)\sin^2\chi\sin^2\theta\,d\varphi^2$$

$$[t, \chi, \theta, \varphi]$$

$$(g_{\mu\nu}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2\sin^2\chi & 0 \\ 0 & 0 & 0 & a(t)^2\sin^2\chi\sin^2\theta \end{bmatrix}$$

Connection

$$\Gamma^1{}_{22} = a a'$$

$$\Gamma^2{}_{12} = \frac{a'}{a}$$

$$\Gamma^3{}_{13} = \frac{a'}{a}$$

$$\Gamma^4{}_{14} = \frac{a'}{a}$$

$$\Gamma^1{}_{33} = a a' \sin^2\chi$$

$$\Gamma^2{}_{33} = -\cos\chi \sin\chi$$

$$\Gamma^3{}_{23} = \frac{\cos\chi}{\sin\chi}$$

$$\Gamma^4{}_{24} = \frac{\cos\chi}{\sin\chi}$$

$$\Gamma^1{}_{44} = a a' \sin^2\chi \sin^2\theta$$

$$\Gamma^2{}_{44} = -\cos\chi \sin\chi \sin^2\theta$$

$$\Gamma^3{}_{44} = -\cos\theta \sin\theta$$

$$\Gamma^4{}_{34} = \frac{\cos\theta}{\sin\theta}$$

$1 \to t \qquad 2 \to \chi \qquad 3 \to \theta \qquad 4 \to \phi$

Riemann Tensor

$$R^1{}_{221} = -aa''$$

$$R^2{}_{121} = -\frac{a''}{a}$$

$$R^3{}_{131} = -\frac{a''}{a}$$

$$R^4{}_{141} = -\frac{a''}{a}$$

$$R^1{}_{331} = -aa'' \sin^2\chi$$

$$R^2{}_{332} = -\left[(a')^2+1\right] \sin^2\chi$$

$$R^3{}_{232} = \left[(a')^2+1\right]$$

$$R^4{}_{242} = \left[(a')^2+1\right]$$

$$R^1{}_{441} = -aa'' \sin^2\chi \sin^2\theta$$

$$R^2{}_{442} = -\left[(a')^2+1\right] \sin^2\chi \sin^2\theta$$

$$R^3{}_{443} = -\left[(a')^2 \sin^2\chi + 1 - \cos^2\chi\right] \sin^2\theta$$

$$R^4{}_{343} = \underbrace{\left[(a')^2 \sin^2\chi + 1 - \cos^2\chi\right]}_{\left[(a')^2+1\right]\sin^2\chi}$$

Ricci Tensor

$$R_{11} = -\frac{3a''}{a} \qquad R_{22} = a\,a'' + 2(a')^2 + 2$$

$$R_{33} = \left[a\,a'' + 2\,(a')^2 + 2\right]\sin^2\chi$$

$$R_{44} = \left[a a'' + 2(a')^2 + 2\right]\sin^2\chi\,\sin^2\theta$$

$$R = \frac{1}{a^2}\Big[6aa'' + 6(a')^2 + \underbrace{4\sin^2\chi - 2\cos^2\chi}_{\substack{\shortparallel \\ 6\sin^2\chi - 2\sin^2\chi}} + 2\Big] = \frac{6}{a^2}\left[aa'' + (a')^2 + 1\right]$$

Einstein Tensor

$$G_{11} = \frac{1}{a^2}\left[3(a')^2 + \underset{\substack{\| \\ 3\sin^2\chi - \sin^2\chi}}{2\sin^2\chi} - \cos^2\chi + 1\right] = \frac{3}{a^2}\left[(a')^2 + 1\right]$$

$$G_{22} = -\left[2aa'' + (a')^2 + 1\right]$$

$$G_{33} = -\left[2aa'' + (a')^2 + 1\right]\sin^2\chi$$

$$G_{44} = -\left[2aa'' + (a')^2 + 1\right]\sin^2\chi\,\sin^2\theta$$

Geodesic equations (affine parameter s)

$$t_{ss} + a a' \chi_s^2 + a a' \sin^2\chi \; \theta_s^2 + a a' \sin^2\chi \sin^2\theta \; \phi_s^2 = 0$$

$$\chi_{ss} + 2\frac{a'}{a} t_s \chi_s - \cos\chi \sin\chi \; \theta_s^2 - \frac{1}{a} \cos\chi \sin\chi \; \phi_s^2 = 0$$

$$\theta_{ss} + 2\frac{a'}{a} t_s \theta_s + 2\cot\chi \; \chi_s \theta_s - \cos\theta \sin\theta \; \phi_s^2 = 0$$

$$\phi_{ss} + 2\frac{a'}{a} t_s \phi_s + 2\cot\chi \; \chi_s \phi_s + 2\cot\theta \; \theta_s \phi_s = 0$$

$$R^2 \equiv R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = 12\left[\left(\frac{a''}{a}\right)^2 + \left[\left(\frac{a'}{a}\right)^2 + \frac{1}{a^2} \right]^2 \right]$$

Schwarzschild

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$(g_{\mu\nu}) = \begin{bmatrix} -\left(1 - \frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \dfrac{1}{1 - \frac{2m}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

Schwarzschild

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

→ ct_coordsys(exteriorschwarzschild)

Connection

$$\Gamma^1{}_{12} = \Gamma^t{}_{tr} = \frac{m}{r(r-2m)} = \frac{m}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\Gamma^2{}_{11} = \Gamma^r{}_{tt} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)$$

$$\Gamma^2{}_{22} = \Gamma^r{}_{rr} = -\frac{m}{r^2\left(1-\frac{2m}{r}\right)}$$

$$\Gamma^2{}_{33} = \Gamma^r{}_{\theta\theta} = \frac{1}{r}$$

$$\Gamma^3{}_{44} = \Gamma^\theta{}_{\varphi\varphi} = -\cos\theta\sin\theta$$

$$\Gamma^4{}_{42} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}$$

$$\Gamma^4{}_{34} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$$

Carrol: (5.52) p206

Riemann:

$$R^1{}_{221} = R^t{}_{rtt} = -\frac{2m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^2{}_{121} = R^r{}_{trt} = -\frac{2m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^3{}_{131} = R^\theta{}_{t\theta t} = \frac{m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^4{}_{141} = R^\phi{}_{t\phi t} = \frac{m}{r^3}\left(1-\frac{2m}{r}\right)$$

$$R^1{}_{331} = R^t{}_{\theta\theta t} = \frac{m}{r}$$

$$R^2{}_{332} = R^r{}_{\theta\theta r} = \frac{m}{r}$$

$$R^3{}_{232} = R^\theta{}_{r\theta r} = -\frac{m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^4{}_{242} = R^\phi{}_{r\phi r} = -\frac{m}{r^3\left(1-\frac{2m}{r}\right)}$$

$$R^1{}_{441} = R^t{}_{\phi\phi t} = \frac{m}{r}\sin^2\theta$$

$$R^2{}_{442} = R^r{}_{\phi\phi r} = \frac{m}{r}\sin^2\theta$$

$$R^3{}_{443} = R^\theta{}_{\phi\phi\theta} = -\frac{2m}{r}\sin^2\theta$$

$$R^4{}_{343} = R^\phi{}_{\theta\phi\theta} = \frac{2m}{r}$$

Ricci

$R_{11} = \cdots = 0$

$R_{22} = \cdots = 0$

Scalar Curvature

$R = 0$

R^2

$R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48m^2}{r^6}$ (Carroll (5.50))

Ricci

$R_{11} = \dots = 0$

$R_{22} = \dots = 0$

Scalar Curvature

$R = 0$

R^2

$R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 m^2}{r^6}$

Blows up as $r \to 0$

(Carroll (5.50))

$R_{\mu\nu\rho\sigma}$

$$R_{1221} = \frac{2m}{r^3} \qquad R_{1331} = -\frac{m}{r}\left(1-\frac{2m}{r}\right)$$

$$R_{2332} = \frac{m}{r}\frac{1}{\left(1-\frac{2m}{r}\right)} \qquad R_{2442} = \frac{m}{r}\frac{1}{\left(1-\frac{2m}{r}\right)}\sin^2\theta$$

$$R_{3443} = -2mr\sin^2\theta$$

Geodesics

$$g_1 \Rightarrow \quad t_{ss} + \frac{2m}{r(r-2m)} t_s r_s = 0$$

$$g_2 \Rightarrow \quad r_{ss} + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right) t_s^2 - \frac{m}{r^2} \frac{1}{1-\frac{2m}{r}} r_s^2 - r\left(1-\frac{2m}{r}\right)\left[\theta_s^2 + \sin^2\theta \, \phi_s^2 \right] = 0$$

$$g_3 \Rightarrow \quad \theta_{ss} + \frac{2}{r} r_s \theta_s - (\phi_s)^2 \cos\theta \sin\theta = 0$$

$$g_4 \Rightarrow \quad \phi_{ss} + \frac{2}{r} r_s \phi_s + 2 \theta_s \phi_s \cot\theta = 0$$

(Carroll (5.53))