- The Metric
- Causal Structure

Refs:

J. Hartle Ch. 7

S. Carroll § 2.5-2.7

R. Wald § 2.4

B. Schutz § 2.4

The metric is an additional structure on a manifold

Provides:

- locally inertial frames (the physics which is easy to understand)
- causality : past, future, max "speed of light"
- geometric, coordinate independent $\begin{cases} \text{path length} \\ \text{proper time of observers} \end{cases}$
- parallel transport, geodesics : $\begin{cases} \text{straigtest (locally) curves} \\ \text{longest " times} \end{cases}$

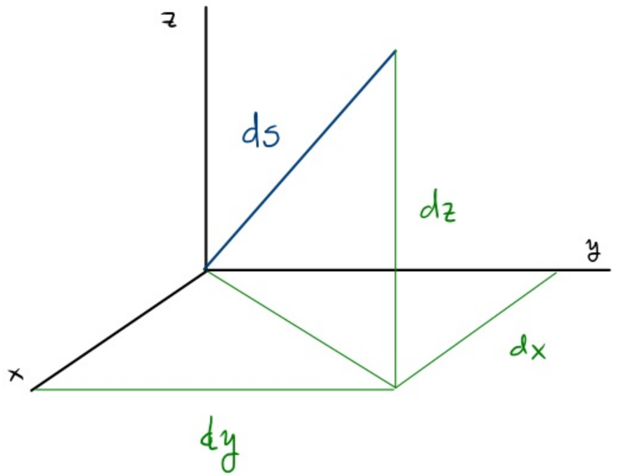
  geodesics: Paths of the free!

- curvature (i.e. gravitation ...)

- Line element: infinitesimal length

  - Euclidean $\mathbb{R}^3$:

$$ds^2 = dx^2 + dy^2 + dz^2$$



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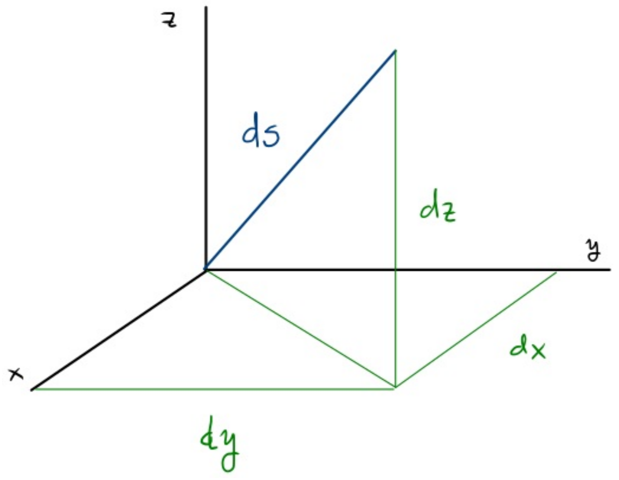

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But also:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

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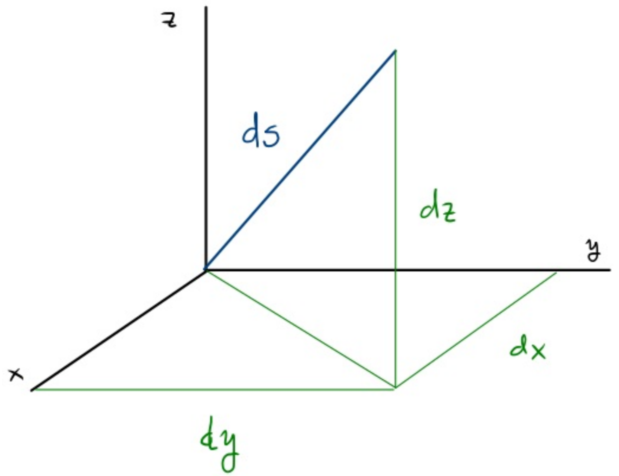

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$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

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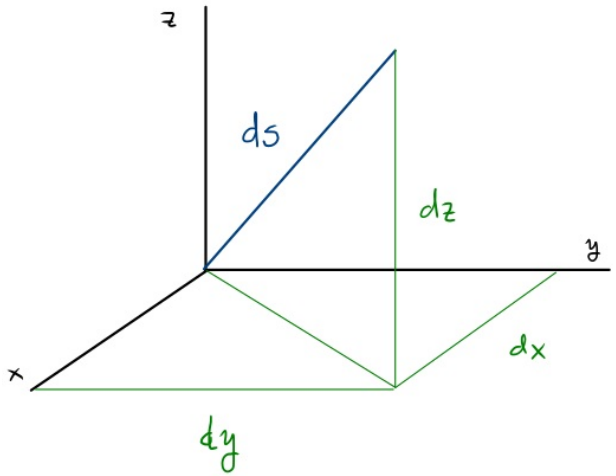

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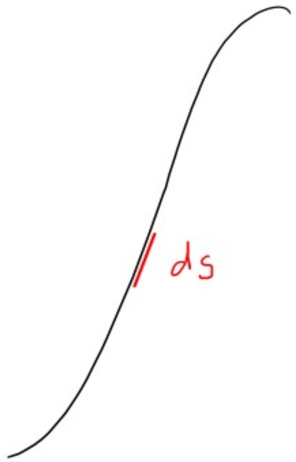

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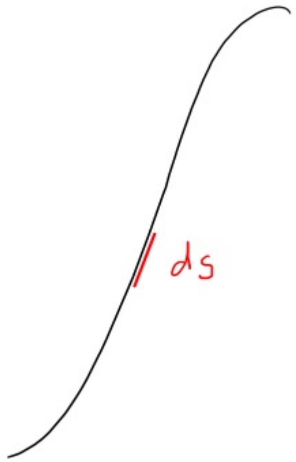

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(... spacetime of events!)

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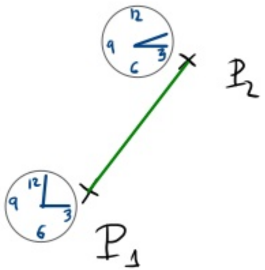
spacetime distance of events



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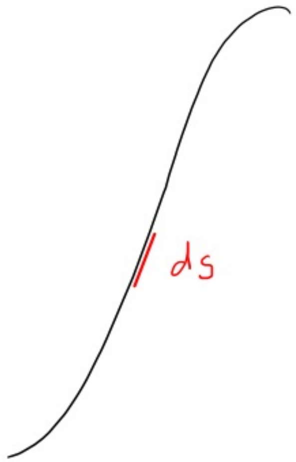

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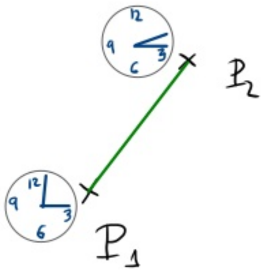

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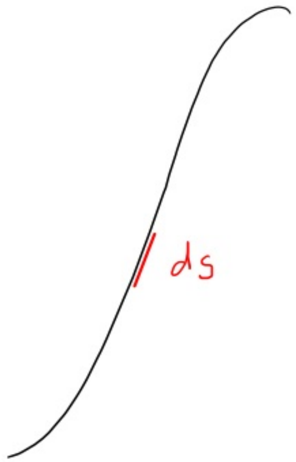

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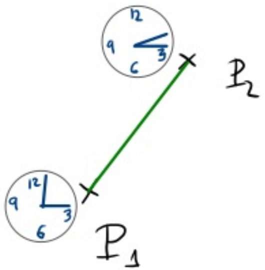

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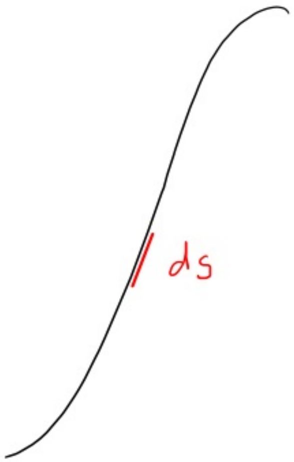

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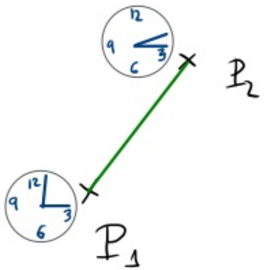

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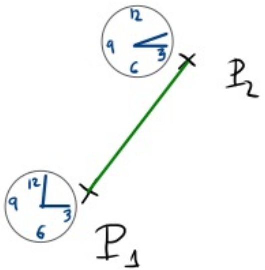

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• $dt = 0$: simultaneous events for an observer, happening @ distance

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, dx^i dx^j$$

$i, j = 1, 2, 3$

Abstract Index Notation

* we have worked with abstract tensor notation:

e.g. $V, V(f), df$

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \quad , \quad \omega = g(V, \ldots)$$

$$\epsilon = \sqrt{|g|}\, \tilde{\epsilon}_{\mu\nu\rho\lambda}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\lambda$$

or with component notation:

$$\omega_\mu = g_{\mu\nu} V^\nu$$

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no reference to a particular base

$$R^a{}_{bad} \equiv R(e^a; \cdot, e_a, \cdot)$$

kind of heavy

(Anti) symmetrization:

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}) \qquad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$$

Abstract Index Notation

We will use both:

Abstract tensor

- when we want to emphasize geometric properties
- better for differential forms
- better for Lie derivatives/brackets

Abstract Index

- all coordinate independent tensorial equations with indices can be thought of as relations between geometric objects. You should play this mental game: very useful/gives insight

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g defines an inner product in T_pM:

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$V^\mu = (1,0,\dots,0)$ gives $g_{0\nu} U^\nu = g_{00} U^0 + g_{01} U^1 + \dots + g_{0n-1} U^{n-1} = 0$

$V^\mu = (0,1,\dots,0)$ " $g_{1\nu} U^\nu = g_{10} U^0 + g_{11} U^1 + \dots + g_{1n-1} U^{n-1} = 0$

⋮

$V^\mu = (0,0,\dots,1)$ " $g_{n-1\nu} U^\nu = g_{n-1\,0} U^0 + g_{n-1\,1} U^1 + \dots + g_{n-1\,n-1} U^{n-1} = 0$

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$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu \quad ,$$

also a symmetric tensor

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* consistent with abstract index notation:

$g^{\mu\nu}$ a $(2,0)$ symmetric tensor s.t.

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⇝ let's see how ...

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$\left(\Leftrightarrow \det g \neq 0 \Leftrightarrow \right.$ non-degenerate volume element from Levi-Civita tensor $\left. \right)$

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$g(U,V) = g_{\mu\nu} U^\mu V^\nu = 0 \quad \forall\, V$, then

$V^\mu = (1,0,\dots,0)$ gives $g_{0\nu} U^\nu = g_{00} U^0 + g_{01} U^1 + \dots + g_{0\,n-1} U^{n-1} = 0$

$V^\mu = (0,1,\dots,0)$ " $g_{1\nu} U^\nu = g_{10} U^0 + g_{11} U^1 + \dots + g_{1\,n-1} U^{n-1} = 0$

$\vdots$

$V^\mu = (0,0,\dots,1)$ " $g_{n-1\,\nu} U^\nu = g_{n-1\,0} U^0 + g_{n-1\,1} U^1 + \;\; + g_{n-1\,n-1} U^{n-1} = 0$

• The Metric

- the $n \times n$ linear system has only trivial solutions $U^\mu = 0$ iff $\det g \neq 0$, g invertible

* we denote g^{-1} by $g^{\mu\nu}$

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu \quad ,$$

also a symmetric tensor

* consistent with abstract index notation:

$g^{\mu\nu}$ a $(2,0)$ symmetric tensor s.t.

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$$

$\rightsquigarrow$ let's see how ...

• Index raising/lowering

A metric gives rise to isomorphism between T_pM and T_p^*M

- If $V \in T_pM$, then $g(V, \cdot) \in T_p^*M$

$$\tilde{V} = g(V, \cdot) \quad \text{or} \quad \tilde{V}_\mu = g_{\mu\nu} V^\nu$$

$\rightarrow$ we write $\tilde{V}_\mu \equiv V_\mu$, call operation index lowering

- If $\omega \in T_p^*\mathcal{M}$, we define $\tilde{\omega} \in T_p\mathcal{M}$ from

$$\omega(V) = g(\tilde{\omega}, V) \qquad \forall\, V \in T_p\mathcal{M}$$

or $\omega_\mu V^\mu = g_{\nu\mu}\tilde{\omega}^\nu V^\mu$

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Index raising/lowering defined for any tensor

e.g. $A_\mu{}^\rho = g_{\mu\lambda} A^{\lambda\rho}$

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(index raising operations)

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Notice that if: $\tilde{g}^{\mu\nu} = g^{\mu\rho} g^{\nu\lambda} g_{\rho\lambda} \Rightarrow$

$$\tilde{g}^{\mu\nu} g_{\nu\sigma} = \underbrace{g^{\mu\rho} g^{\nu\lambda} g_{\rho\lambda} g_{\nu\sigma}}_{\delta^\mu{}_\lambda \quad \delta^\lambda{}_\sigma} = \delta^\mu{}_\lambda \delta^\lambda{}_\sigma = \delta^\mu{}_\sigma$$

Therefore $\tilde{g}^{\mu\nu} = g^{\mu\nu}$, the inverse of $(g_{\mu\nu})$

Finally, notice that if

$$\tilde{V}_\mu = g_{\mu\nu} V^\nu \quad , \text{ then}$$

$$\tilde{\tilde{V}}^\nu = g^{\nu\rho} \tilde{V}_\rho =$$

$$= g^{\nu\rho} g_{\rho\mu} V^\mu$$

$$= \delta^\nu{}_\mu V^\mu$$

$$= V^\nu$$

* Index raising & lowering is 1-1, onto: an isomorphism between $T_p^{(l,k)}M$ for $l+k=$ fixed
* It depends on the choice of metric:

an isomorphism e.g. $V^\mu \leftrightarrow \tilde{V}_\mu$, but

- it does not exist without a metric

- it depends on the metric: if we choose a different metric g', then $V^\mu \leftrightarrow \tilde{V}'_\mu$ s.t.

$$\tilde{V}'_\mu = g'_{\mu\nu} V^\nu$$

* Component Transformation

- coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} g_{\mu\nu}$$

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$$= (\Lambda^T)_{\mu'}{}^{\mu} g_{\mu\nu} \Lambda^{\mu}{}_{\nu'}$$

$$= (\Lambda^T g \Lambda)_{\mu'\nu'}$$

row

$$\Lambda^{\mu}{}_{\mu'} \equiv \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

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$$e_{\mu'} = \Lambda^{\mu}{}_{\mu'} e_{\mu}$$

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- $(g_{\mu\nu})$ a symmetric matrix
  - has real eigenvalues
  - $\det g \neq 0 \Rightarrow$ eigenvalues $\neq 0$
  - $\exists$ orthogonal matrix $O^{-1} = O^T$

$$g_d = O^{-1} g O$$

$g_d = \mathrm{diag}(g_0, \dots, g_{n-1})$ a diagonal matrix

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We can always write Λ as

$$\Lambda = O \cdot D, \quad D \text{ diagonal}$$

* Component Transformation

$$\Rightarrow \Lambda^T = D^T O^T = D O^{-1}$$

Then

$$(g_{\mu'\nu'}) = D O^{-1} (g_{\mu\nu}) O D$$

$$= D \; g_d \; D$$

$$= \begin{pmatrix} d_0 & & \\ & \ddots & \\ & & d_{n-1} \end{pmatrix} \begin{pmatrix} g_0 & & \\ & \ddots & \\ & & g_{n-1} \end{pmatrix} \begin{pmatrix} d_0 & & \\ & \ddots & \\ & & d_{n-1} \end{pmatrix}$$

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- Choose a basis $e_{\mu'} = \Lambda^{\mu}{}_{\mu'} e_\mu = (OD)^{\mu}{}_{\mu'} e_\mu$, where

$$D = \mathrm{diag}\left(\frac{1}{\sqrt{|g_0|}}, \frac{1}{\sqrt{|g_1|}}, \dots, \frac{1}{\sqrt{|g_{n-1}|}} \right)$$

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* Component Transformation

$\Rightarrow (g_{\mu'\nu'}) = \mathrm{diag}(\pm 1, \pm 1, \dots, \pm 1)$

* change the order of columns of O to bring all -1 in front

$\Rightarrow (g_{\mu'\nu'}) = \mathrm{diag}(-1,-1,\dots,-1,+1,+1,\dots,+1)$

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- number of -1 cannot change with any basis xfm

$\hat{=}$ signature of metric

* Orthonormal bases

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$$e_{\mu} = \Lambda^{\mu'}{}_{\mu} e_{\mu'} \quad \text{s.t.}$$

$$\Lambda^{T} \eta \Lambda = \eta$$

is also orthonormal

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If s is the signature of the metric, and

$s=0$: the metric is Euclidean, $\Lambda \in O(n)$

manifold Riemannian ↳ orthogonal group

$$\eta = \mathbb{1}_{n\times n} \qquad \eta = \Lambda^T \eta \Lambda \Leftrightarrow \mathbb{1} = \Lambda^T \Lambda$$

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Lorentz xfm

$s=1$: the metric is Minkowskian, $\Lambda \in O(n-1, n)$
manifold pseudo-Riemannian
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* Orthonormal bases are not coordinate bases (unless $\mathcal{M}$ is flat)

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$\Rightarrow$ coordinate bases can be orthogonal to each other, but cannot make all of them to have unit length

e.g. $(\hat{r}, \hat{\theta}, \hat{\varphi})$ orthonormal

$(\partial_r, \partial_\theta, \partial_\varphi)$ not!

$g(\partial_r, \partial_r) = 1$, $g(\partial_\theta, \partial_\theta) = r^2$, $g(\partial_\varphi, \partial_\varphi) = r^2 \sin^2\theta$

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* We can always choose coordinate system s.t. at one point P the components:

$$g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \quad , \quad \text{and}$$

$$\partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} = 0$$

* Orthonormal bases

* Orthonormal bases are not coordinate bases (unless M is flat)

$\Rightarrow$ coordinate bases can be orthogonal to each other, but cannot make all of them to have unit length

e.g. $(\hat{r}, \hat{\theta}, \hat{\varphi})$ orthonormal

$(\partial_r, \partial_\theta, \partial_\varphi)$ <u>not</u>!

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(cannot make them go away, see Carroll § 2.5)

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– the coordinate basis at P is orthonormal:

locally inertial coordinates

local Lorentz frame

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- effects of curvature go away in a small enough laboratory

(small: compared to "radius of curvature"
"go away": to first order)

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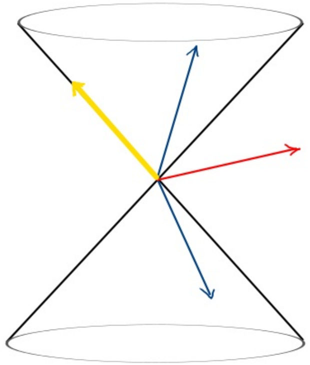
- Minkowski metric at $T_p M$ has causal structure: light cone, past, future



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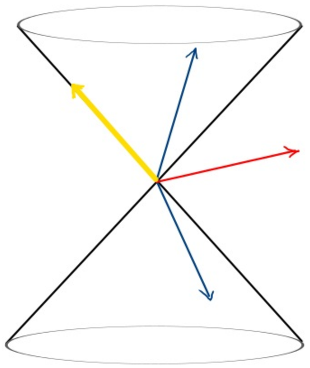
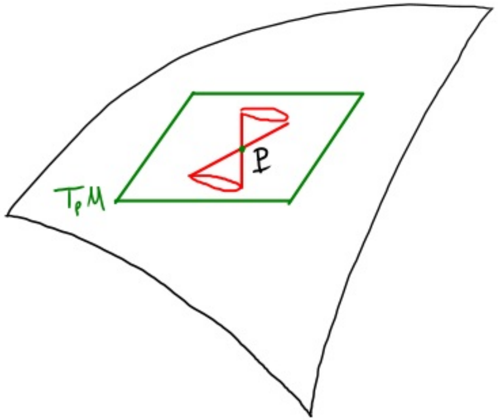


- Manifold inherits causal structure: light always in a direction on the local light cone



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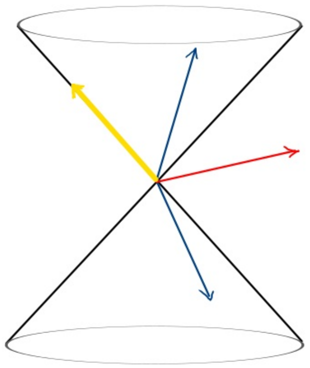

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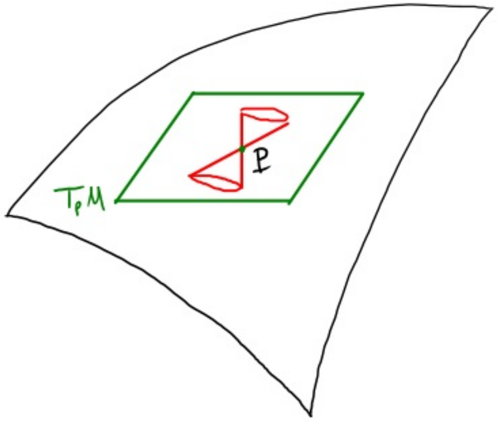


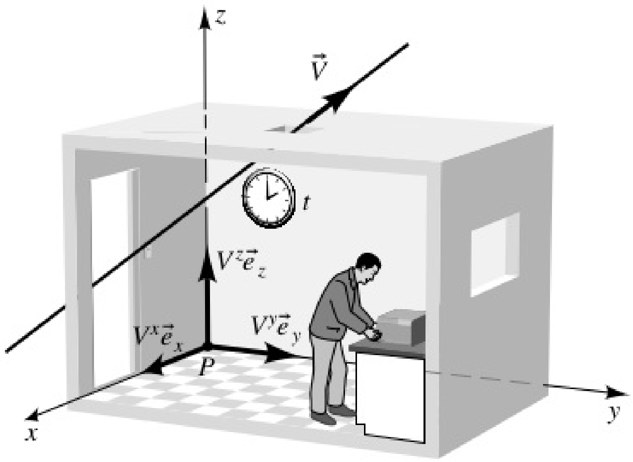
  - massive particles have worldlines with everywhere timelike tangent vectors : timelike curves

  - massless particles have worldlines with everywhere null/lightlike tangent vectors : null curves

  - there are no worldlines with tangent vectors that change category

Local Frames

- Orthonormal bases define an "observer" or "lab"



Hartle, Fig 7.6

Local Frames

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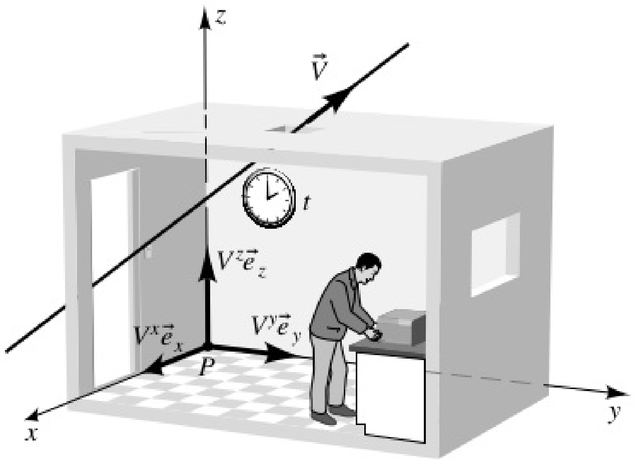
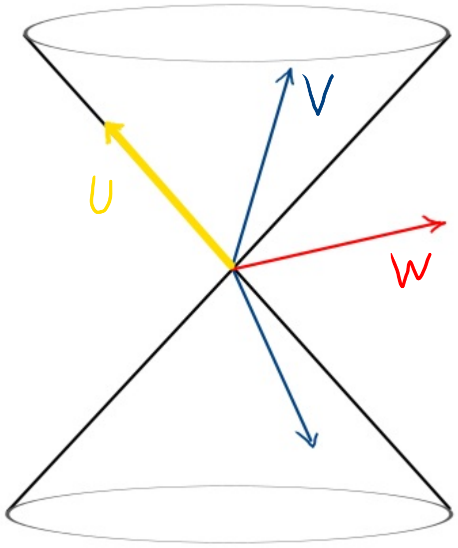
  four-velocity of observer: $u = e_0$

  local cartesian axes: $\{e_1, e_2, e_3\}$



Hartle, Fig 7.6



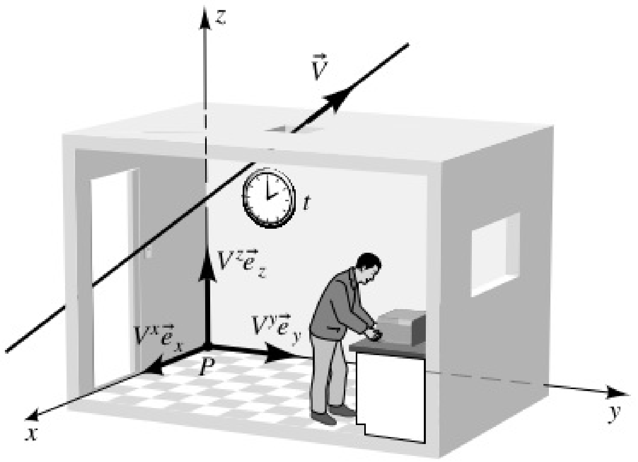
Defines a local light cone:

$g(V,V) < 0$ timelike

$g(U,U) = 0$ null / light like

$g(W,W) > 0$ spacelike

Local Frames



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- Orthonormal bases define an "observer" or "lab"

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velocity of a passing by particle $V = \gamma e_0 + \gamma v e_1$, $\gamma = (1-v^2)^{-1/2}$

$u^\mu = (1,0,0,0)$ $V^\mu = (\gamma, \gamma v, 0, 0)$

$$g_{\mu\nu} u^\mu V^\nu = -\gamma + \gamma v \cdot 0 = -\gamma \Rightarrow v = \left(1 - \frac{1}{\gamma^2}\right)^{1/2} = \underbrace{\left(1 - (u^\mu V_\mu)^{-2}\right)^{1/2}}_{\text{coordinate independent formula}}$$

Local Frames

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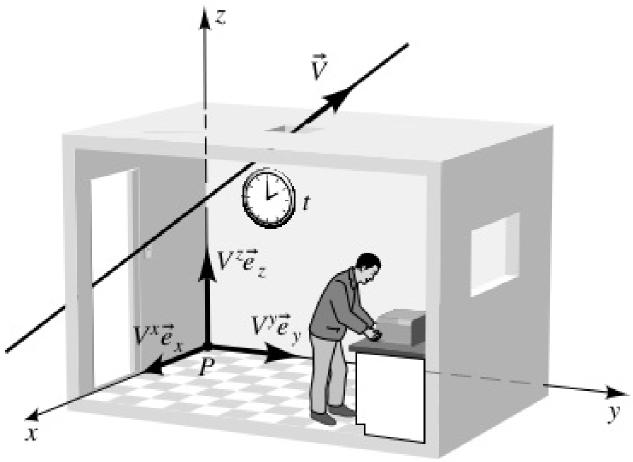


Hartle, Fig 7.6

four-velocity of observer: $u = e_0$

local cartesian axes: $\{e_1, e_2, e_3\}$

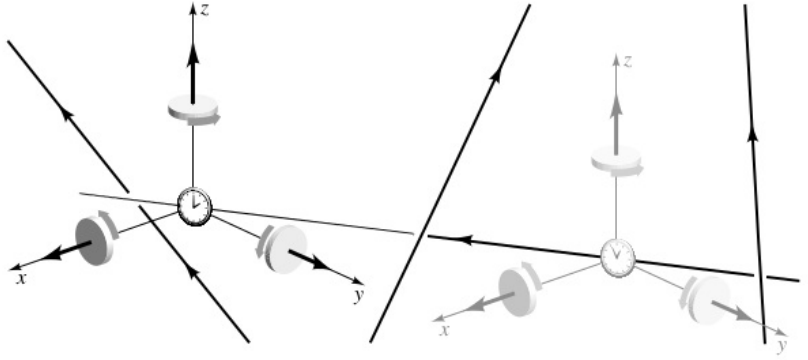
four-momentum of particle:

$$p = p^\mu e_\mu \qquad \left.\begin{aligned} p^\mu &= (E, p^1, p^2, p^3) \\ u^\mu &= (1, 0, 0, 0) \end{aligned}\right\} \Rightarrow p_\mu u^\mu = -E$$

$$\rightsquigarrow \quad E = -p_\mu u^\mu = -\underbrace{g_{\mu\nu} p^\mu u^\nu}_{\text{coordinate independent expression}}$$

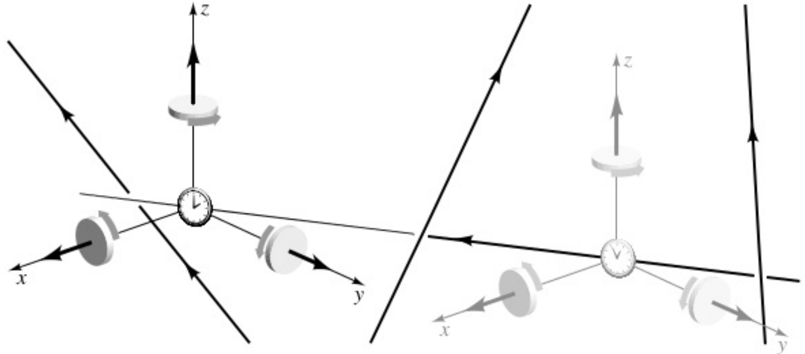
Local Inertial Frames

- Observers that 'observe free particles to move (locally) on straight lines
  - In GR they are the freelly falling observers



Hartle Fig 3.3

Local Inertial Frames



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How to become one:

- follow a free particle and set the origin of axes on it

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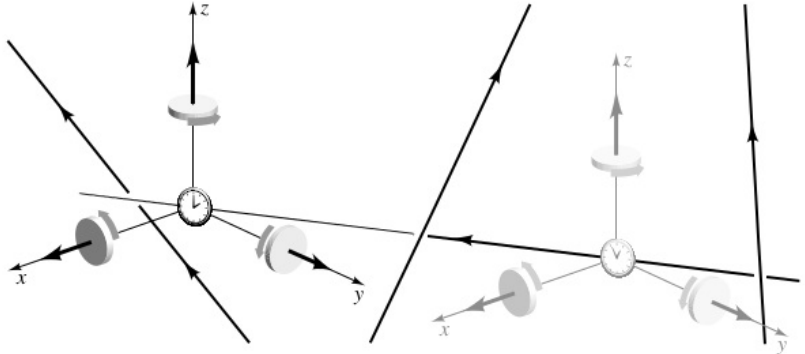
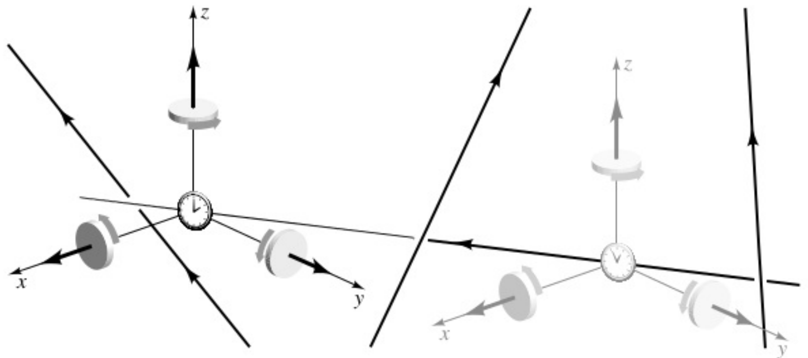


Hartle Fig 3,3

How to become one:

- follow a free particle and set the origin of axes on it
- choose 3 perpendicular axes and set 3 gyroscopes to spin in their direction

Local Inertial Frames



Hartle, Fig 3.3

- Observers that 'observe free particles to move (locally) on straight lines
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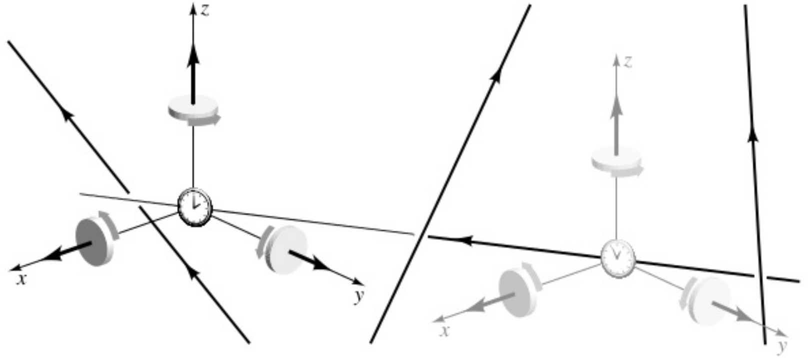
How to become one:

- follow a free particle and set the origin of axes on it
- choose 3 perpendicular axes and set 3 gyroscopes to spin in their direction
- at each instant of time ① let the gyros spin freely and define spatial axes
  ② choose coordinates s.t. $g_{\mu\nu}|_0 = \eta_{\mu\nu}$ and $\partial_\sigma g_{\mu\nu}|_0 = 0$

Local Inertial Frames



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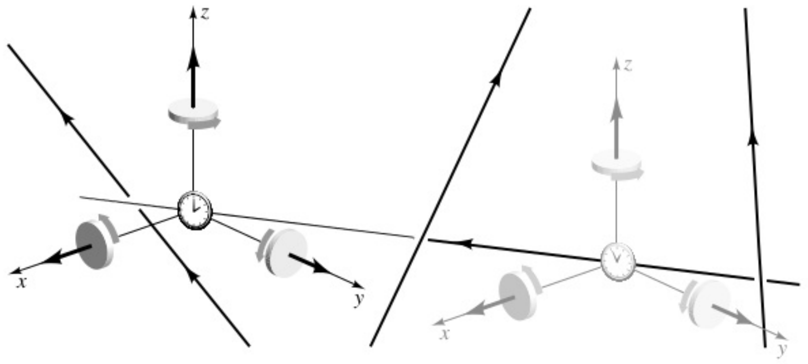
Voila: if your lab is small enough you can do SR-physics!

Local Inertial Frames

- Observers that observe free particles to move (locally) on straight lines
    - In GR they are the freelly falling observers
    - Not all observers are inertial (we are not!)



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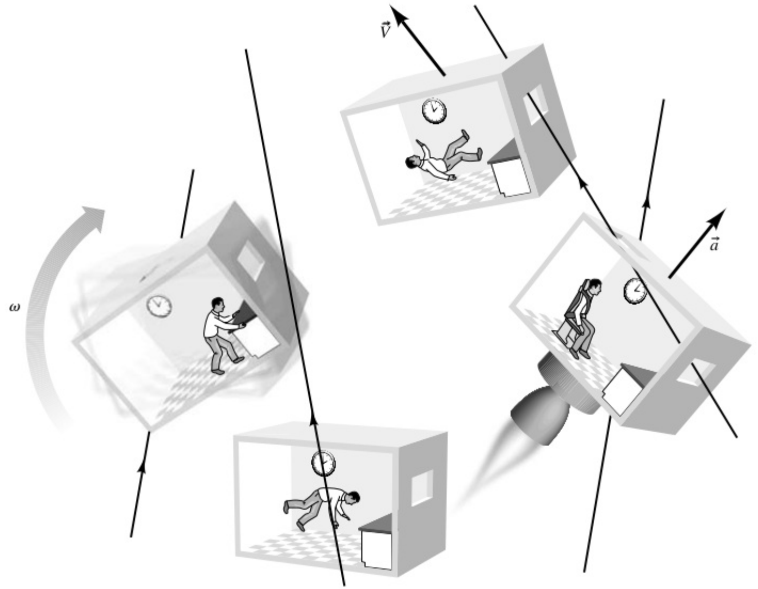

Hartle, Fig 3.2

Dual basis $\{e^\mu\}$ to an orthonormal basis $\{e_\mu\}$

Definition: $e^\mu(e_\nu) = \delta^\mu{}_\nu$

Then: ① $g^{-1}(e^\mu, e^\nu) = \eta^{\mu\nu}$ — also orthonormal

② $e_\mu \cdot e_\nu = \eta_{\mu\nu} \Leftrightarrow g_{\alpha\beta}\, e_\mu{}^\alpha e_\nu{}^\beta = \eta_{\mu\nu} \Rightarrow \eta_{\mu\nu}\, e^\mu{}_\alpha e^\nu{}_\beta = g_{\alpha\beta}$

Dual basis $\{e^\mu\}$ to an orthonormal basis $\{e_\mu\}$

Definition: $e^\mu(e_\nu) = \delta^\mu{}_\nu$

Indeed: $\Rightarrow e^\mu{}_\alpha e_\nu{}^\alpha = \delta^\mu{}_\nu$, and as a matrix equation we have

$$e \cdot \tilde{e}^T = \mathbb{1} \qquad e \equiv (e_\nu{}^\alpha) \quad \tilde{e}^T \equiv (e^T{}_\alpha{}^\mu), \quad e^T{}_a{}^\mu \equiv e^\mu{}_\alpha$$

Dual basis $\{e^\mu\}$ to an orthonormal basis $\{e_\mu\}$

Definition: $e^\mu(e_\nu) = \delta^\mu{}_\nu$

Indeed: $\Rightarrow$ $e^\mu{}_\alpha\, e_\nu{}^\alpha = \delta^\mu{}_\nu$, and as a matrix equation we have

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$$\Rightarrow \quad e = (\tilde{e}^T)^{-1}$$

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$$\Rightarrow \quad e^T_\alpha{}^\mu\, e_\mu{}^\beta = \delta_\alpha{}^\beta$$

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$$\Rightarrow \quad e^\mu{}_\alpha \, e_\mu{}^\beta = \delta_\alpha{}^\beta$$

Dual basis $\{e^\mu\}$ to an orthonormal basic $\{e_\mu\}$

Definition: $e^\mu(e_\nu) = \delta^\mu{}_\nu$

Indeed: $\Rightarrow e^\mu{}_\alpha \, e_\nu{}^\alpha = \delta^\mu{}_\nu \Rightarrow e^\mu{}_\alpha \, e_\mu{}^\beta = \delta_\alpha{}^\beta$

And: $g^{-1}(e^\mu, e^\nu) = g^{\alpha\beta} e^\mu{}_\alpha \, e^\nu{}_\beta$

Dual basis $\{e^\mu\}$ to an orthonormal basis $\{e_\mu\}$

Definition: $e^\mu(e_\nu) = \delta^\mu{}_\nu$

Indeed: $\Rightarrow e^\mu{}_\alpha e_\nu{}^\alpha = \delta^\mu{}_\nu \Rightarrow e^\mu{}_\alpha e_\mu{}^\beta = \delta_\alpha{}^\beta$

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$$= e^\mu{}_\alpha g^{\alpha\beta} e^\nu{}_\beta$$

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$$= \left(e \cdot g^{-1} \cdot \tilde{e}^T\right)^{\mu\nu}$$

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Indeed: ↳ $e^\mu{}_\alpha \, e_\nu{}^\alpha = \delta^\mu{}_\nu \Rightarrow e^\mu{}_\alpha \, e_\mu{}^\beta = \delta_\alpha{}^\beta$

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$$
\begin{aligned}
g^{-1}(e^\mu, e^\nu) &= g^{\alpha\beta} e^\mu{}_\alpha \, e^\nu{}_\beta \\
&= e^\mu{}_\alpha \, g^{\alpha\beta} \, e^\nu{}_\beta \\
&= e^\mu{}_\alpha \, g^{\alpha\beta} \, e^T{}_\beta{}^\nu \\
&= \left(e \cdot g^{-1} \cdot \tilde{e}^T\right)^{\mu\nu} \\
&= \left[(\tilde{e}^T)^{-1} g^{-1} e^{-1}\right]^{\mu\nu}
\end{aligned}
$$

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&= \left[(e \cdot g \cdot \tilde{e}^T)^{-1}\right]^{\mu\nu}
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And: $$g^{-1}(e^\mu, e^\nu) = g^{\alpha\beta} e^\mu{}_\alpha e^\nu{}_\beta$$
$$= e^\mu{}_\alpha g^{\alpha\beta} e^\nu{}_\beta$$
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$$= \left(e \cdot g^{-1} \cdot \tilde{e}^T\right)^{\mu\nu}$$
$$= \left[(\tilde{e}^T)^{-1} g^{-1} e^{-1}\right]^{\mu\nu}$$
$$= \left[(e \cdot g \cdot \tilde{e}^T)^{-1}\right]^{\mu\nu} = (\eta^{\mu\nu})^{-1} = \eta^{\mu\nu}$$

Dual basis $\{e^\mu\}$ to an orthonormal basis $\{e_\mu\}$

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we have that: $\quad \eta_{\mu\nu} = g_{\alpha\beta} \, e_\mu{}^\alpha \, e_\nu{}^\beta \quad \Rightarrow$

$$\eta_{\mu\nu} \, e^\mu{}_\gamma \, e^\nu{}_\delta = g_{\alpha\beta} \, e_\mu{}^\alpha \, e_\nu{}^\beta \, e^\mu{}_\gamma \, e^\nu{}_\delta$$

$\delta^\beta{}_\delta$

$\delta^\alpha{}_\gamma$

$$= g_{\gamma\delta}$$

* Orthogonal coordinate bases:

$\{\partial_\mu\}$ orthogonal $\Leftrightarrow$ $(g_{\mu\nu}) = \mathrm{diag}(g_{00}, g_{11}, g_{22}, g_{33})$

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orthonormal

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orthonormal

* components of $\{e_\mu\}$ in $\{\partial_\mu\}$

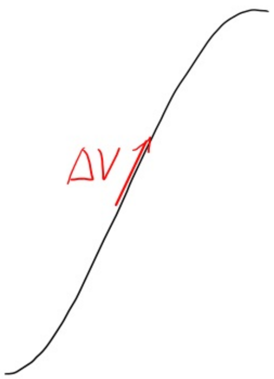
$$\left.\begin{aligned} e_0 &= \left(\frac{1}{\sqrt{|g_{00}|}}, 0, 0, 0\right) & e_1 &= \left(0, \frac{1}{\sqrt{|g_{11}|}}, 0, 0\right) \\ e_2 &= \left(0, 0, \frac{1}{\sqrt{|g_{22}|}}, 0\right) & e_3 &= \left(0, 0, 0, \frac{1}{\sqrt{|g_{33}|}}\right) \end{aligned}\right\}$$

useful formulas

Line Element

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$\Delta V = \Delta x^\mu \partial_\mu$$



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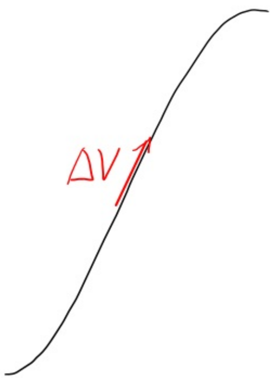

$$g(\Delta V, \Delta V) = g(\Delta x^\mu \partial_\mu, \Delta x^\nu \partial_\nu)$$

Line Element

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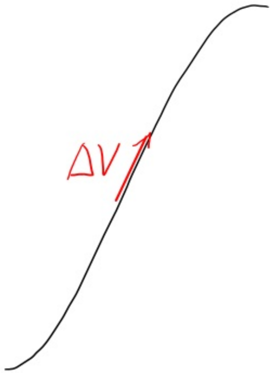

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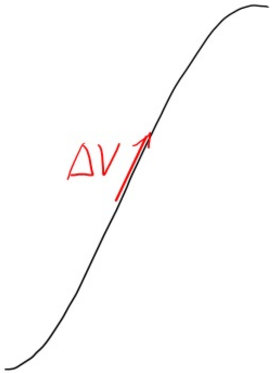
Line Element

$$g = g_{\mu\nu} \, dx^\mu \otimes dx^\nu$$



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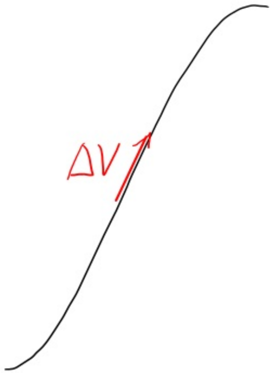

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we write $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, and use it as infinitesimal line element,

e.g. $$s_{AB} = \int_A^B ds = \int_A^B \left\{ \left| g_{\mu\nu} dx^\mu dx^\nu \right| \right\}^{1/2}$$

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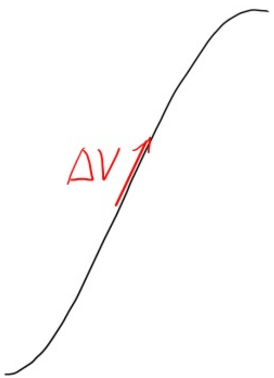

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Line Element

In the bibliography, sometimes

$$ds^2 \equiv g = g_{\mu\nu}\, dx^\mu dx^\nu \quad \text{means} \quad \underbrace{g_{\mu\nu}}_{\substack{\text{component} \\ \text{(scalar)}}} \overbrace{dx^\mu \otimes dx^\nu}^{\text{tensor}} \quad (\text{e.g. } dx^\mu dx^\nu \neq dx^\nu dx^\mu)$$

$\rightsquigarrow$ you should understand from the context (if it matters at all)

---

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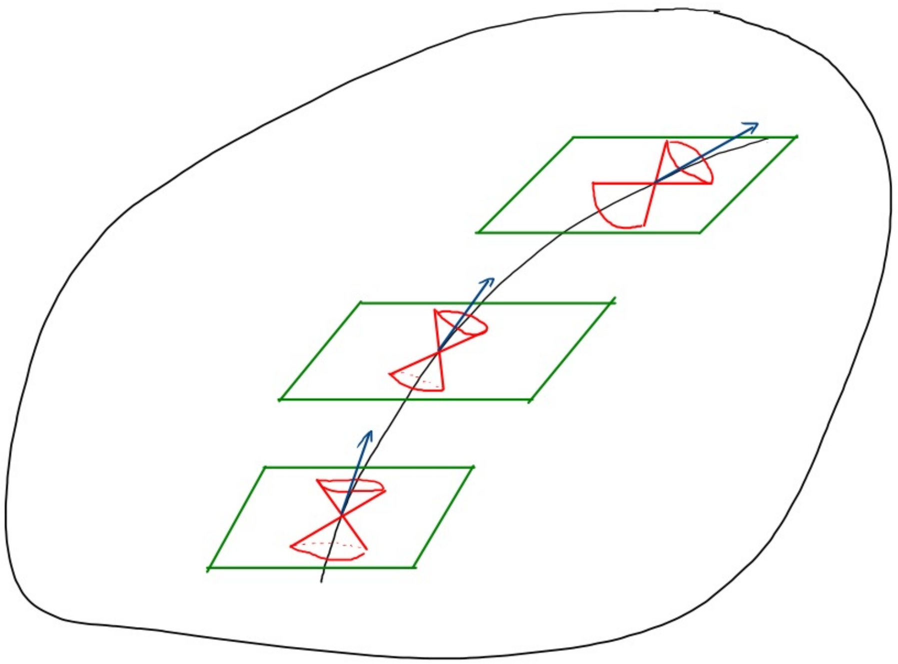
We consider curves of 3 types:

$ds^2 < 0$ everywhere $\rightarrow$ timelike curves

$ds^2 = 0$ " null/lightlike

$ds^2 > 0$ " spacelike



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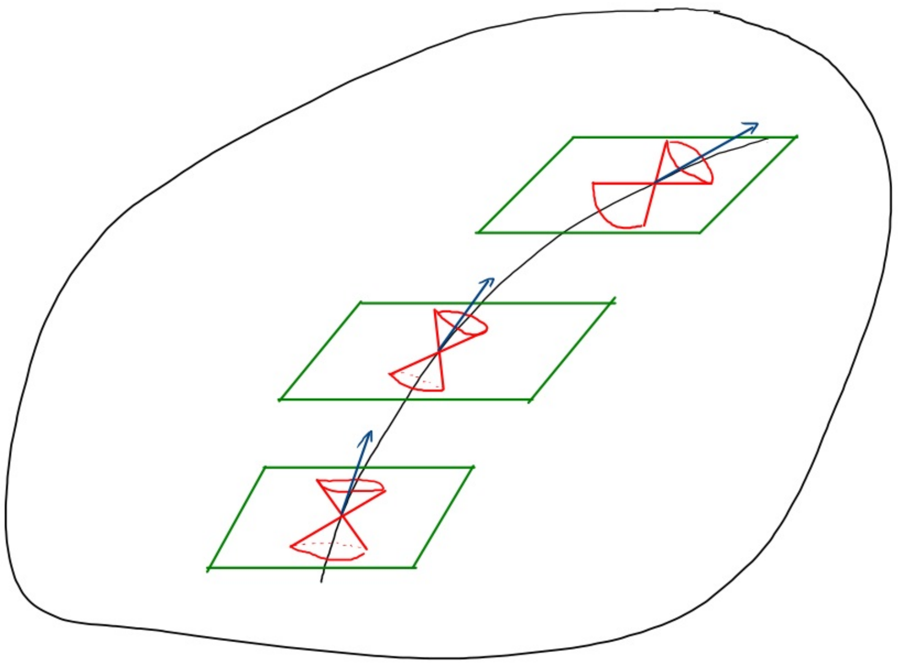

$\Rightarrow$ tangent vector V is of the same type at each point ($g(V,V)$ has the same sign)

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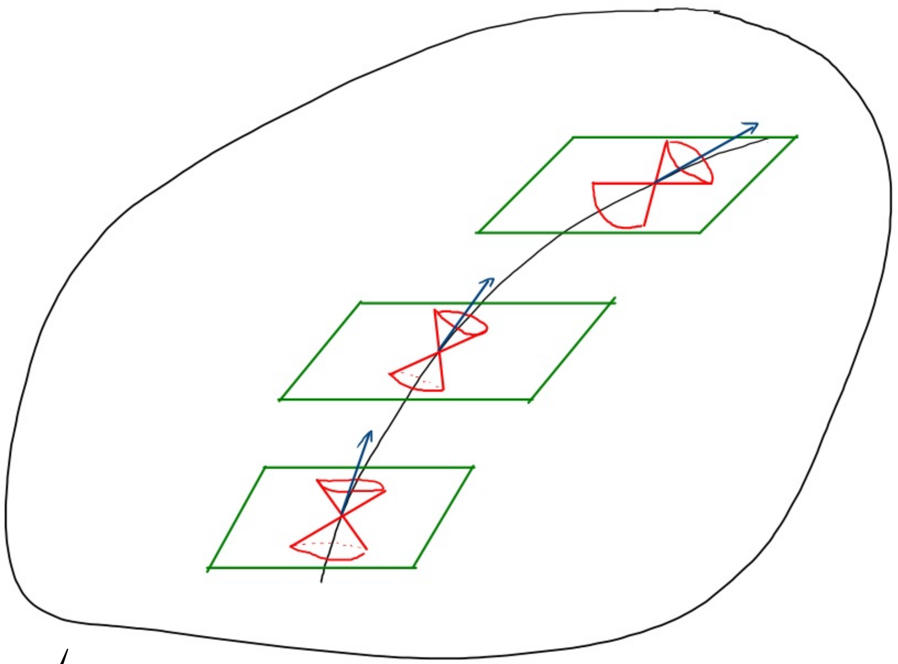

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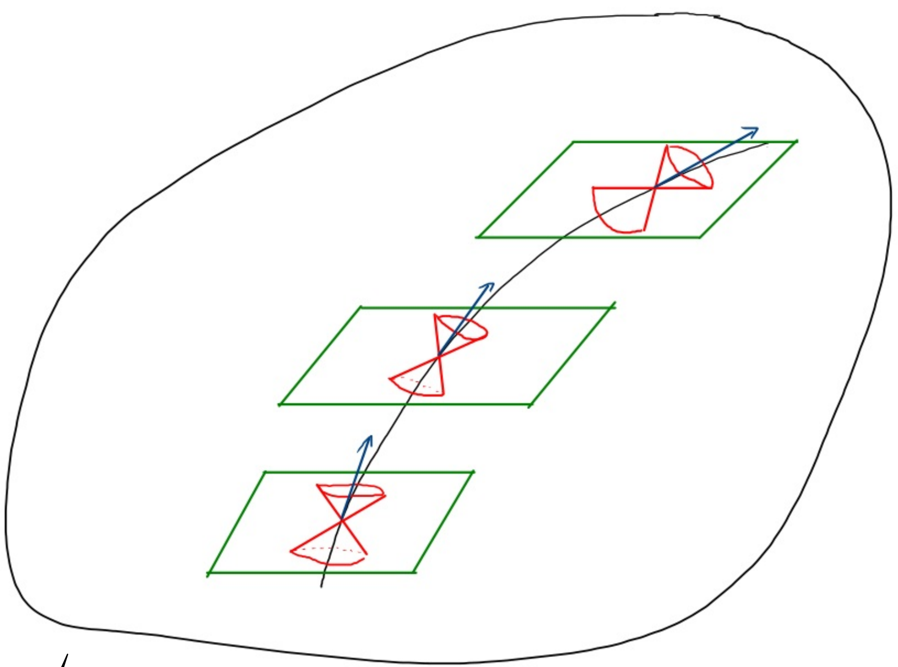

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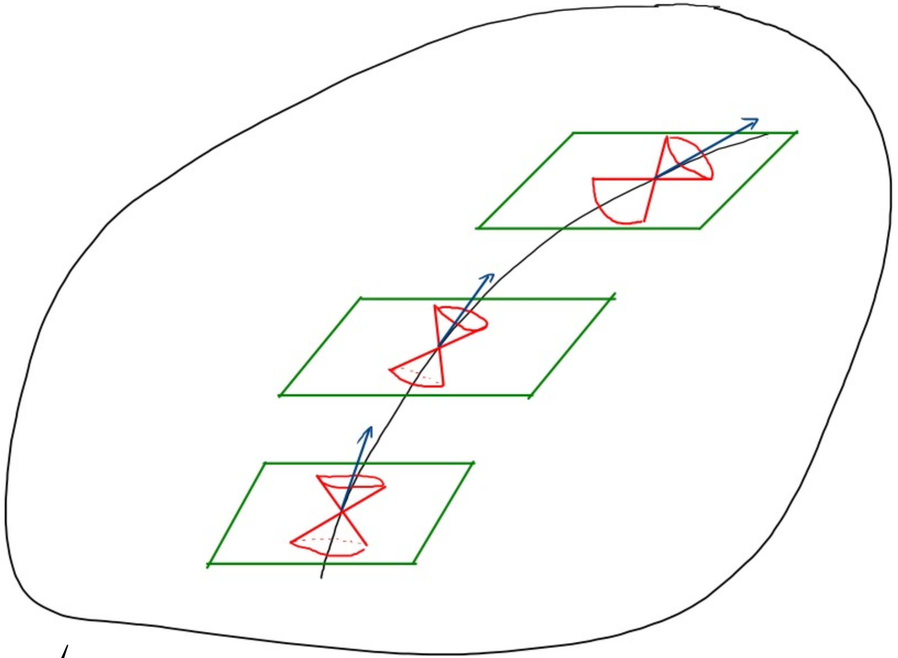

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$\rightarrow$ tangent vector the 4-velocity of particle

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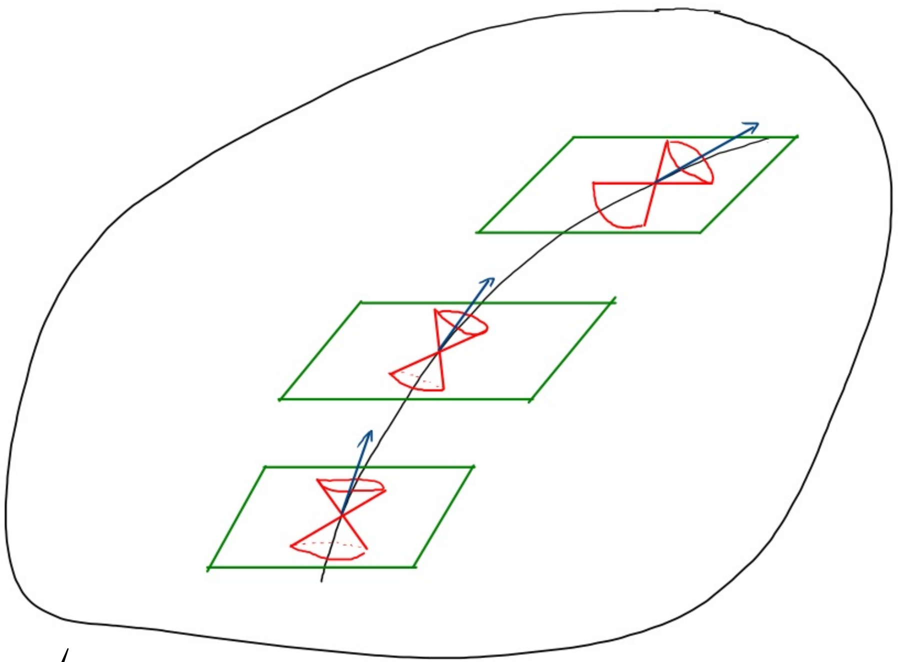
* Observer chooses orthonormal frame $u = e_0$ and stays at fixed position: $dx^1 = dx^2 = dx^3 = 0$
  $\Rightarrow$ $ds^2 = -d\tau^2$, $d\tau$ her/his proper time



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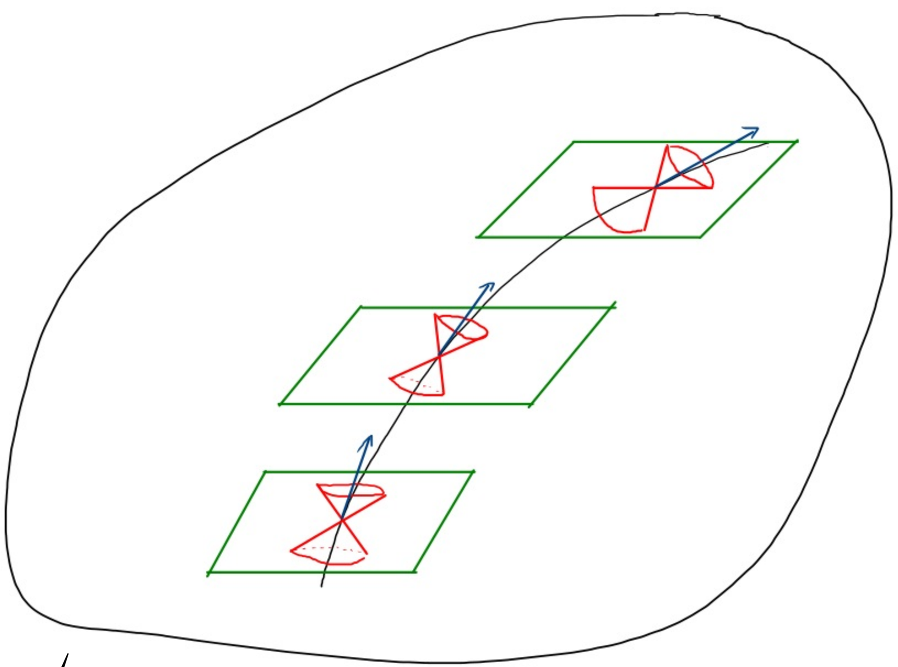
* The tangent of the observer's curve within lightcone:
  - moves "slower than light" at each point
  - a local concept: no "relative speed" for particles far from each other



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Examples:

Hartle, ex. 7.3

$$ds^2 = -x^2 dt^2 + dx^2 \qquad x > 0$$

(not cartesian coordinates ...)

$\partial_t \cdot \partial_t = -x^2 < 0$ everywhere timelike

$\partial_x \cdot \partial_x = +1 > 0$

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Hartle, ex. 7.3

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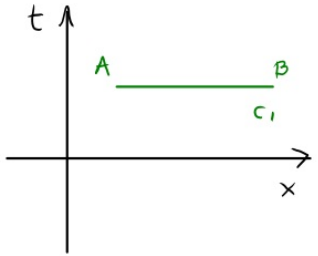
$$C_1: \quad ds^2 = dx^2 \quad \Rightarrow \quad S_{AB} = \int_A^B ds = \int_{x_A}^{x_B} dx = x_B - x_A$$



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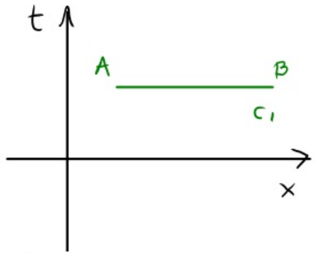

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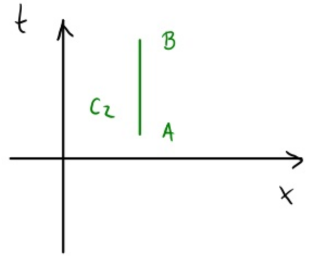

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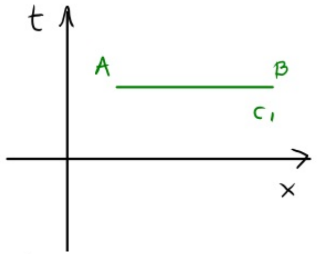

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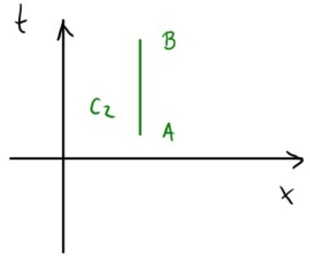

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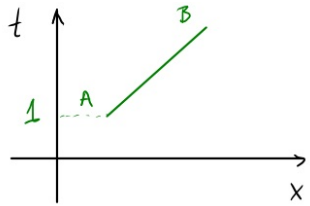

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Notice that this curve is timelike only for $t > 1$!

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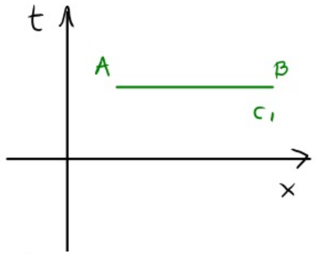

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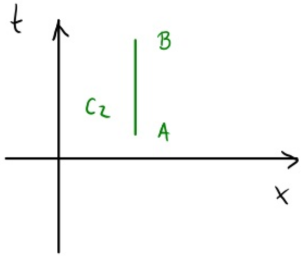

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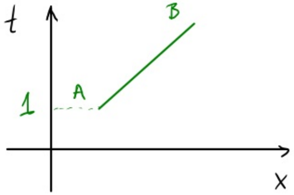

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Not $\sqrt{1-v^2}\,\Delta t$!

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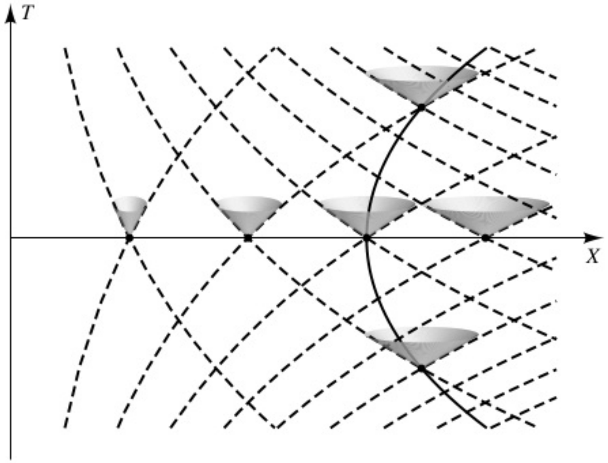

Null curves: $\quad ds^2 = 0 \Leftrightarrow -x^2 dt^2 + dx^2 = 0$

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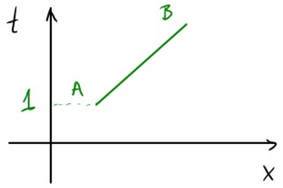
c_s: $\quad x = vt$



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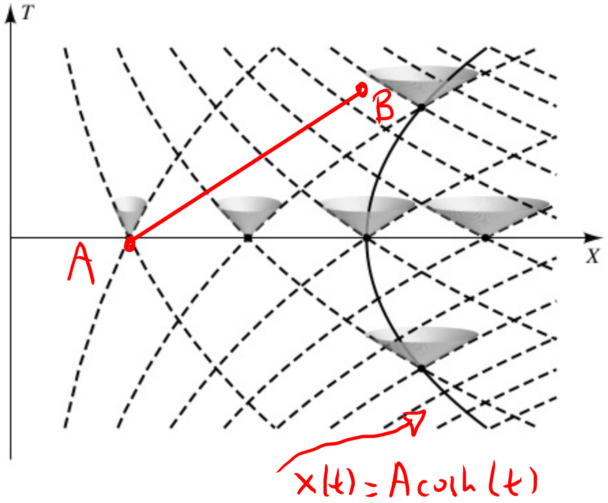

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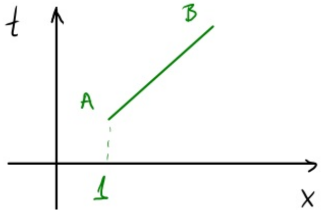
$$\Rightarrow \quad \frac{dt}{dx} = \pm \frac{1}{x} \iff t = \pm \ln\left(\frac{x}{x_0}\right)$$

- this is why our $x = vt$ curve spacelike $\to$ timelike

inside lightcone

- $x(t) = A\cosh(t)$ a timelike curve: $\frac{dx}{dt} = A\sinh(t) < A\cosh(t) = x \Rightarrow \underbrace{\frac{dt}{dx} > \frac{1}{x}}$



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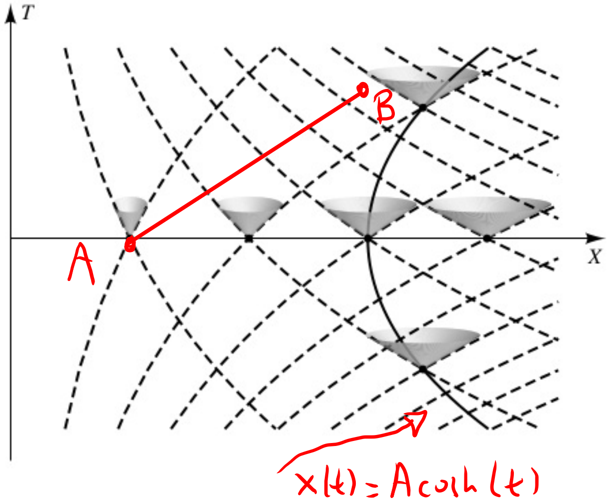

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Indeed: $\quad dx = A\sinh(t)\,dt \Rightarrow$

$$ds^2 = -A^2\cosh^2 t\, dt^2 + A^2 \sinh^2 t\, dt^2 = -A^2 dt^2 < 0$$

and $d\tau = A\,dt \Rightarrow \Delta\tau = A\,\Delta t$

Examples: a wormhole (Hartle, ex 7.7)

$$ds^2 = -dt^2 + dr^2 + (r^2+b^2)(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad , \quad b>0$$

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– consider cylindrical coords in 3d: $\quad d\sigma^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2$

$\hookrightarrow$ Euclidean flat metric in 3d

Examples: a wormhole (Hartle, ex 7.7)

The surface $\rho(z,\varphi) = b\cosh\left(\frac{z}{b}\right)$ will have $d\Sigma^2$ if we choose

$\rho^2 = r^2 + b^2$ $\quad z(r) = b\sinh^{-1}\left(\frac{r}{b}\right)$ and φ being the same

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$dz = \frac{b}{(r^2+b^2)^{1/2}} dr$, $\quad d\rho = \frac{r}{(r^2+b^2)^{1/2}} dr$, $\quad \rho^2 = r^2+b^2$

$\Rightarrow d\sigma^2 = d\Sigma^2$

$$d\Sigma^2 = dr^2 + (r^2+b^2)\, d\varphi^2$$

$r=0 \rightsquigarrow d\Sigma^2 = b^2 d\varphi^2$ circle of $R=b$ (corresponds to sphere: $ds^2 = b^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$)

$\hookrightarrow$ not a point

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$$\rho^2 = r^2 + b^2 \qquad z(r) = b\sinh^{-1}\left(\frac{r}{b}\right)$$ and φ being the same

Indeed, on the surface: $d\sigma^2 = \left[\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial \rho}{\partial r}\right)^2\right] dr^2 + \rho^2 d\varphi^2$

$$dz = \frac{b}{(r^2+b^2)^{1/2}}\,dr \,, \quad d\rho = \frac{r}{(r^2+b^2)^{1/2}}\,dr, \quad \rho^2 = r^2 + b^2$$

$$\Rightarrow d\sigma^2 = d\Sigma^2$$

$$d\Sigma^2 = dr^2 + (r^2+b^2)\,d\varphi^2$$

$r = 0 \rightsquigarrow d\Sigma^2 = b^2 d\varphi^2$ circle of $R = b$ (corresponds to sphere: $ds^2 = b^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$)

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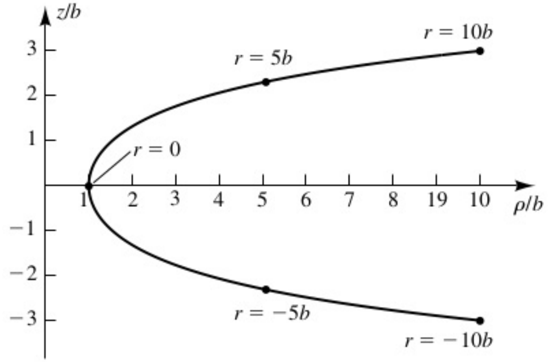
Examples: a wormhole (Hartle, ex 7.7)



Hartle, Fig 7.4

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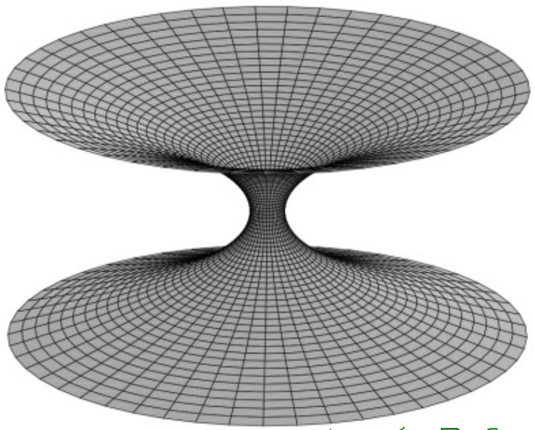
Examples:

a wormhole (Hartle, ex 7.7)

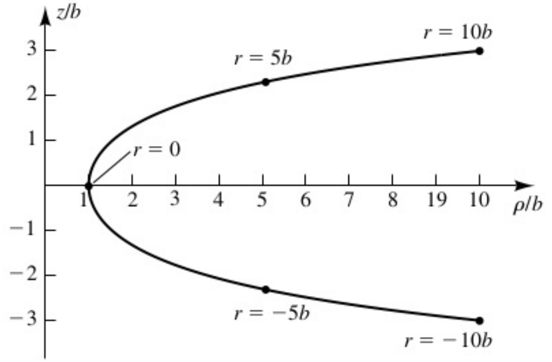


Hartle, Fig 7.5

* at large ρ we obtain two different asymptotically flat spaces

* they are connected via the "throat" of the wormhole at $r=0$ (a sphere)

* The topology of space is not $\simeq \mathbb{R}^3$



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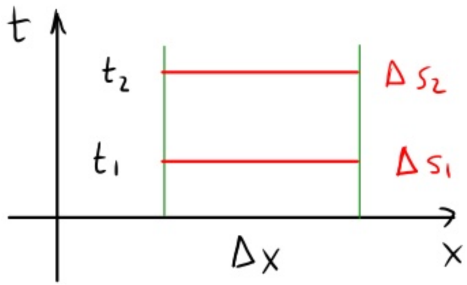
  $dx=dy=dz=0$
  $\left\{\begin{array}{l}\text{comoving frame/observers} \\ t\text{: their proper time}\end{array}\right.$
- $a(t)$: scale factor, determines expansion of $dt=0$ space (the "universe")



$dt=dy=dz=0 \Rightarrow ds^2 = a^2(t)\,dx^2$

$$\left.\begin{array}{l}\Delta s_1 = a_1\,\Delta x \\ \Delta s_2 = a_2\,\Delta x\end{array}\right\} \Rightarrow \frac{\Delta s_1}{\Delta s_2} = \frac{a_1}{a_2}$$

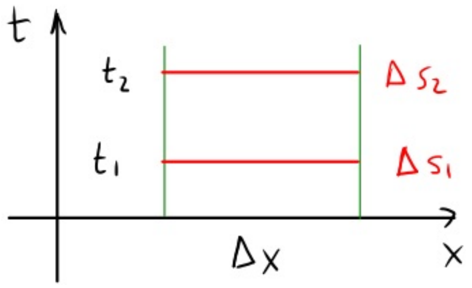
Examples: flat space cosmology (Carroll §2.6)

* Typically $a(t) = t^q \quad 0 < q < 1$

q is determined by matter: e.g. $q = \frac{2}{3}$ matter dominated (today)

$q = \frac{1}{2}$ radiation dominated ($t \lesssim 10$ kyrs)

$dt = dy = dz = 0 \Rightarrow ds^2 = a^2(t)\, dx^2$



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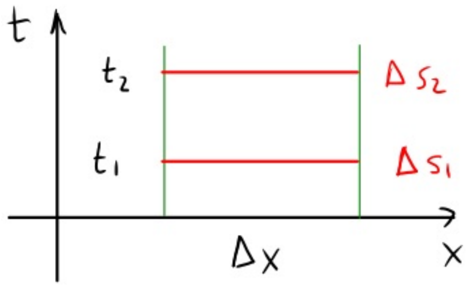

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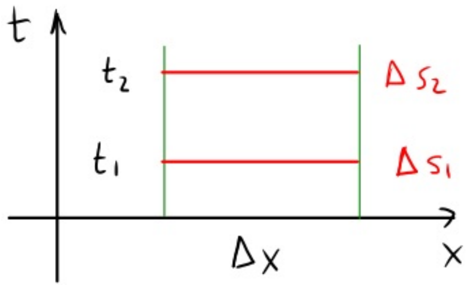

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$r = 3$ radiation dom

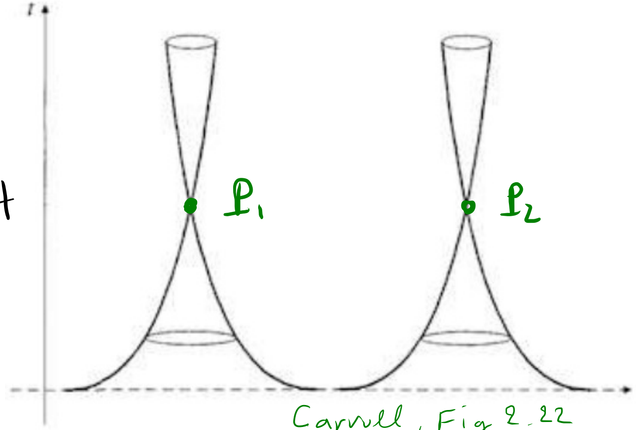
$r = 2$ matter dom

Examples: flat space cosmology (Carroll §2.6)

* the past of P_1, P_2 is non overlapping

  events define horizons: events outside horizons have no causal contact

* light cones tangent to $t=0$ (singularity)



Carroll, Fig 2.22

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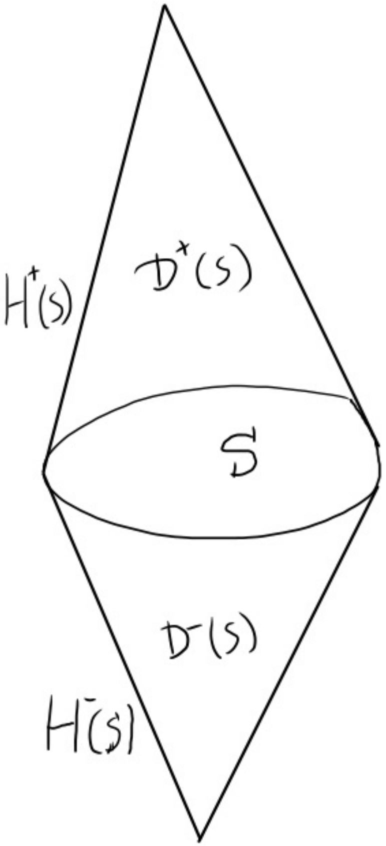
* initial value problem depends on global causal structure:
  - give condition on spacelike surface S (initial "time")
  - the future is determined for future events which are causally connected only with events on S (similar with past...)
  - only globally hyperbolic manifolds have a well defined initial value problem

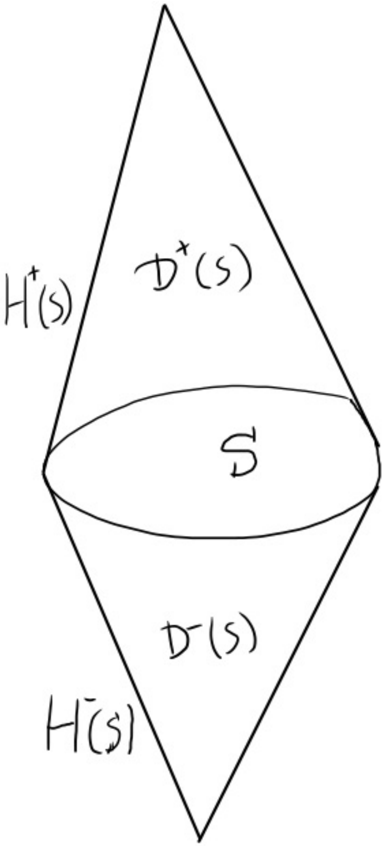


$S \subset \mathcal{M}$

$J^+(S)$ causal future of S: events connected to S by future causal curve

$J^-(S)$ " past " past "

causal curve: timelike or null



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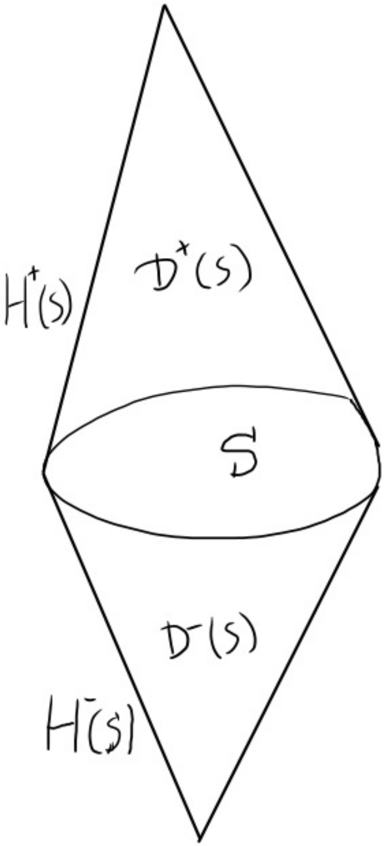

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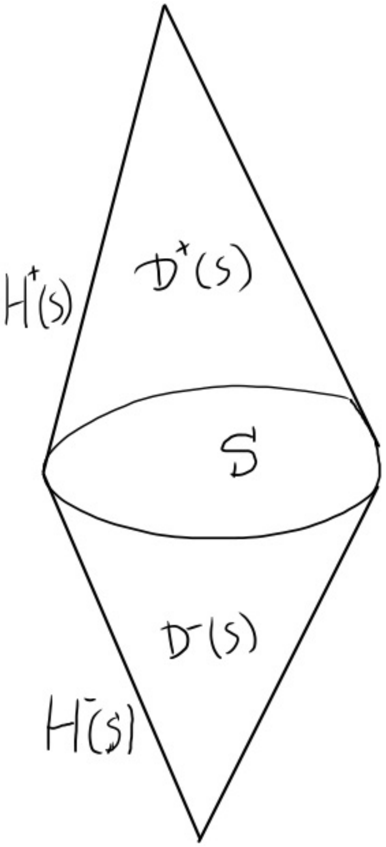
S is achronal if no two of its points connected by timelike curve
Consider $S \subset \mathcal{M}$ closed + achronal



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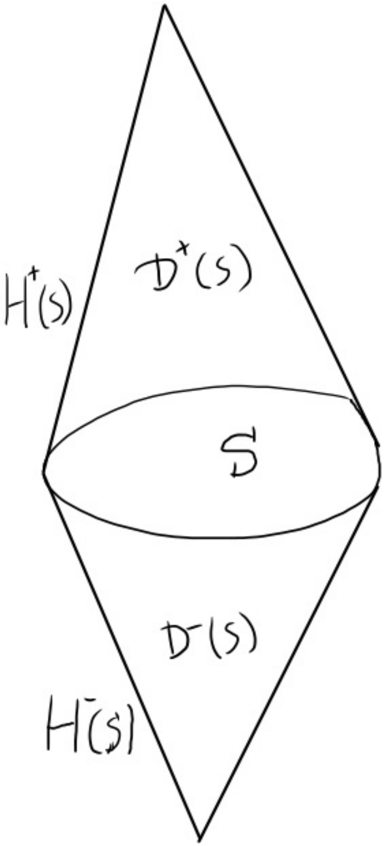

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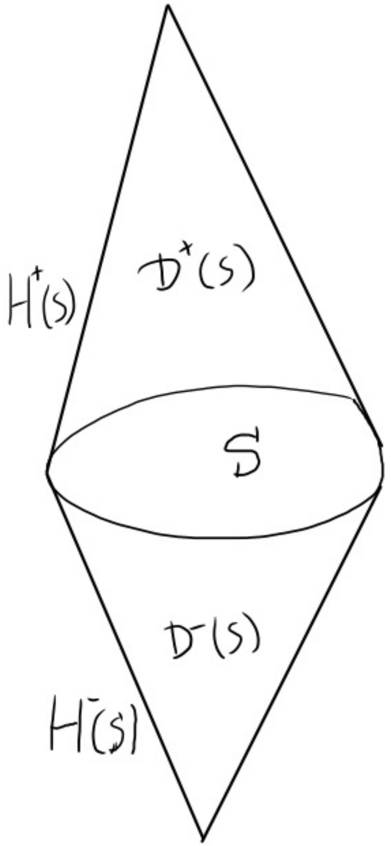

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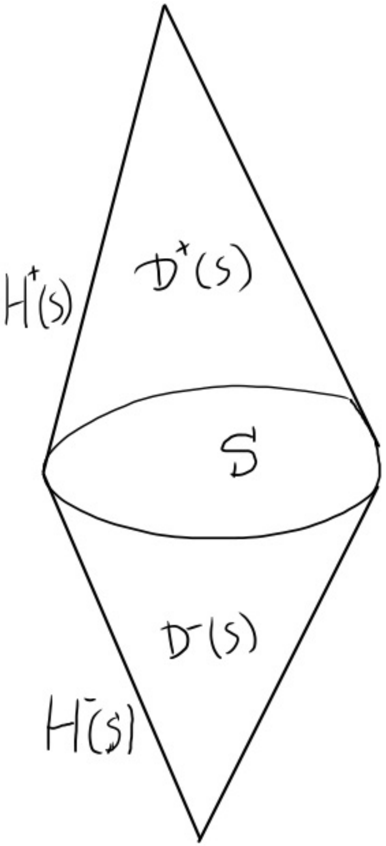

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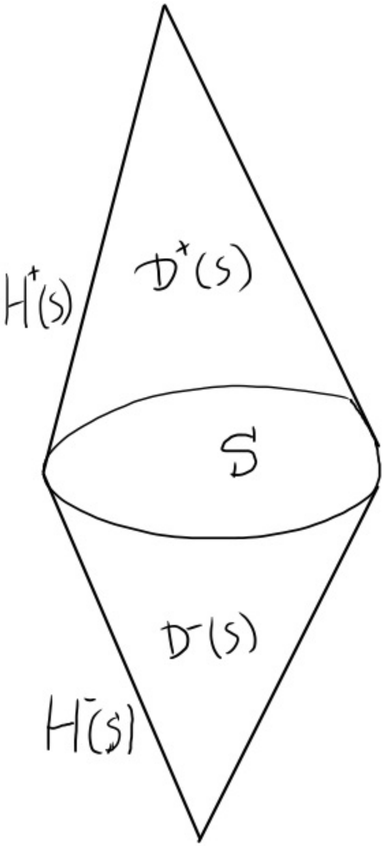
$\Rightarrow$ Initial data on S determines what happens in $D(S) = D^+(S) \cup D^-(S)$



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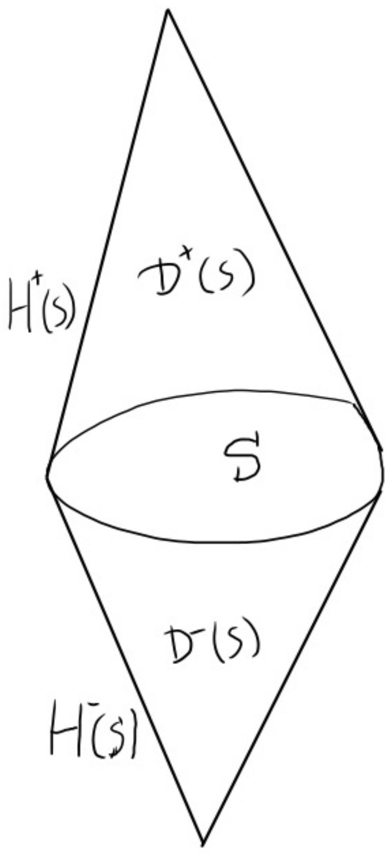

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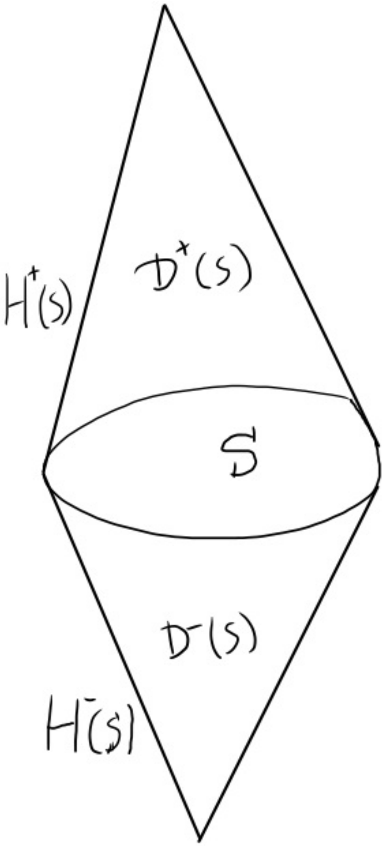

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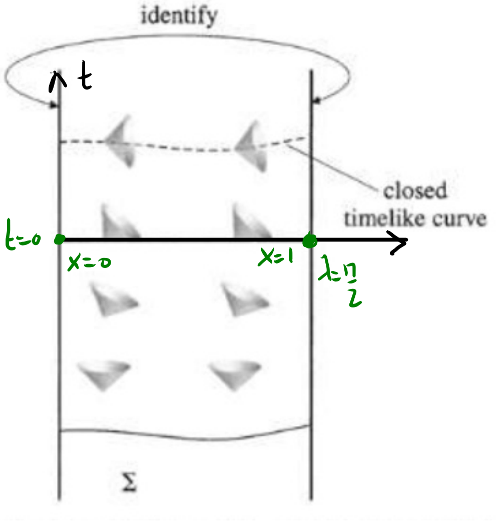

Carrol, Fig 2.25

e.g. (Carroll § 2.7) Misner space

$$ds^2 = -\cos\lambda\, dt^2 - \sin\lambda\,[dt\,dx + dx\,dt] + \cos\lambda\, dx^2 \qquad -\infty < t < +\infty$$

$$0 < \lambda < \pi$$

$$t = \cot\lambda \qquad \mathbb{R} \times S^1 : (t, x) \sim (t, x+1)$$

non-degenerate: $\det g = \begin{vmatrix} -\cos\lambda & -\sin\lambda \\ -\sin\lambda & \cos\lambda \end{vmatrix} = -1$

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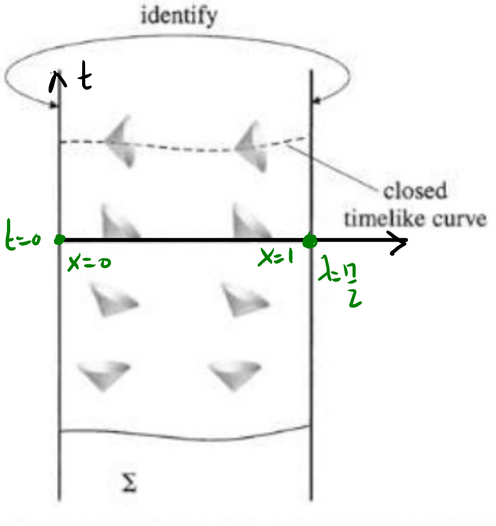

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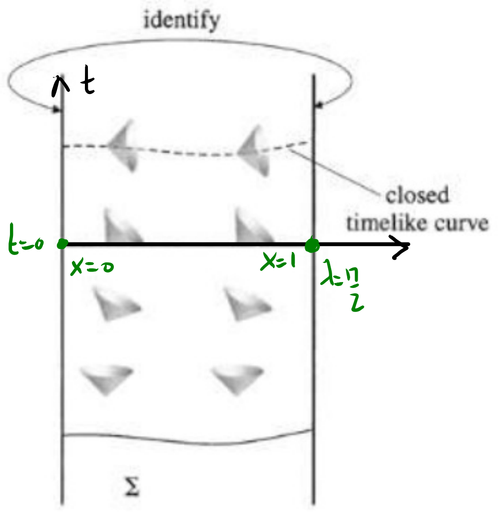

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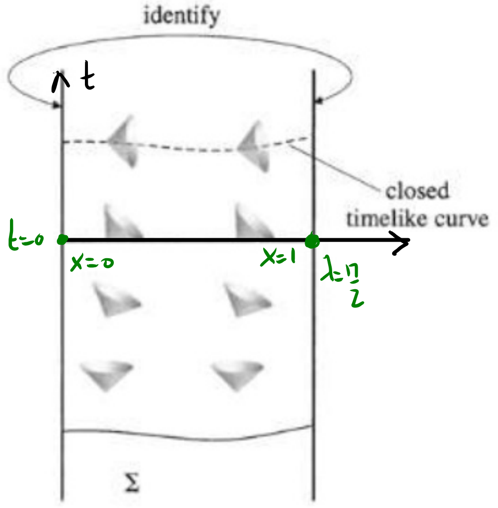

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closed timelike curve
$\Uparrow$

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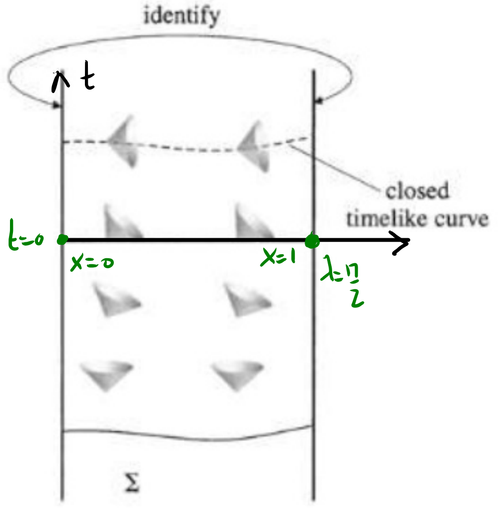

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Not a solution to Einstein eqs, don't worry!

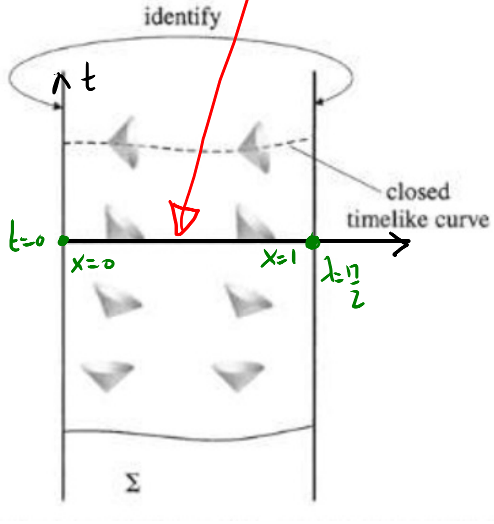
$$\partial_x \cdot \partial_x = g(\partial_x, \partial_x) = \cos\lambda = \begin{cases} (+) & 0 < \lambda < \frac{\pi}{2} \quad \text{spacelike} \\ (-) & \frac{\pi}{2} < \lambda < \pi \quad \text{timelike} \end{cases}$$

closed timelike curve
⇑
timelike → time direction!

* If $D(\Sigma) = M$, Σ is a Cauchy surface
* If M has a Σ, it is globally hyperbolic

Cauchy horizon at $t=0$, no well defined initial problem for Σ

* The existence of closed timelike curves is a reason not to have global hyperbolicity (spacetime has Cauchy horizon)



Carrol, Fig 2.25

e.g. (Carroll § 2.7) Misner space

$$ds^2 = -\cos\lambda\, dt^2 - \sin\lambda\,[dt\,dx + dx\,dt] + \cos\lambda\, dx^2 \qquad -\infty < t < +\infty$$

$$0 < \lambda < \pi$$

$$t = \cot\lambda \qquad \mathbb{R}\times S^1 : (t,x) \sim (t, x+1)$$

$$\partial_t \cdot \partial_t \equiv g(\partial_t, \partial_t) = -\cos\lambda = \begin{cases} (-) & 0 < \lambda < \frac{\pi}{2} \\ (+) & \frac{\pi}{2} < \lambda < \pi \end{cases} \qquad \begin{matrix} \text{timelike} \\ \text{spacelike} \end{matrix}$$

closed timelike curve

$$\partial_x \cdot \partial_x = g(\partial_x, \partial_x) = \cos\lambda = \begin{cases} (+) & 0 < \lambda < \frac{\pi}{2} \\ (-) & \frac{\pi}{2} < \lambda < \pi \end{cases} \qquad \begin{matrix} \text{spacelike} \\ \text{timelike} \to \text{time direction!} \end{matrix}$$

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