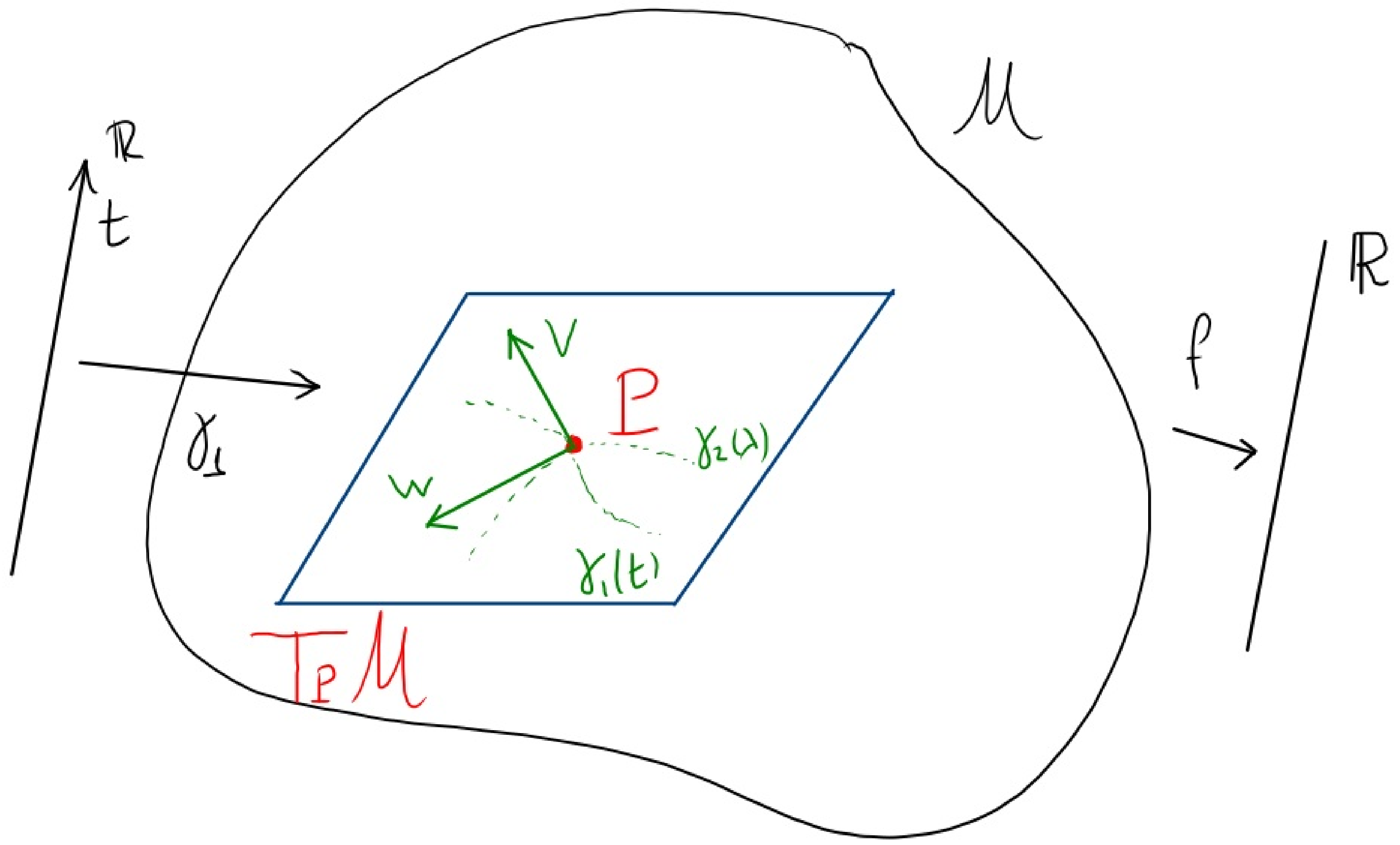
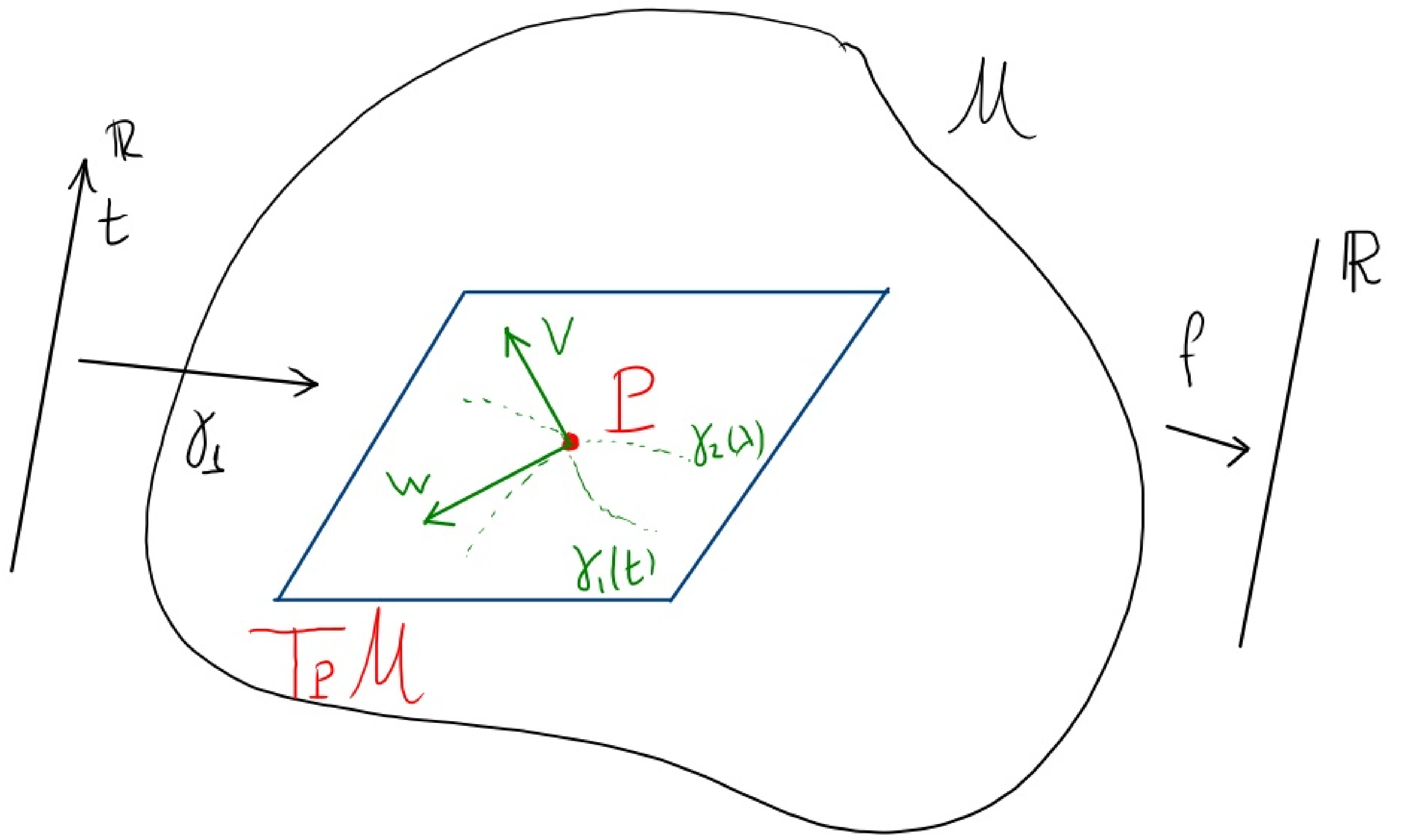
* Vectors : tangent to curves



* Vectors: tangent to curves
* Measure rate of change of functions on M along the curve they are tangent to:

$$V = \frac{d}{dt}\Big|_p : f \mapsto \frac{df}{dt}\Big|_p$$



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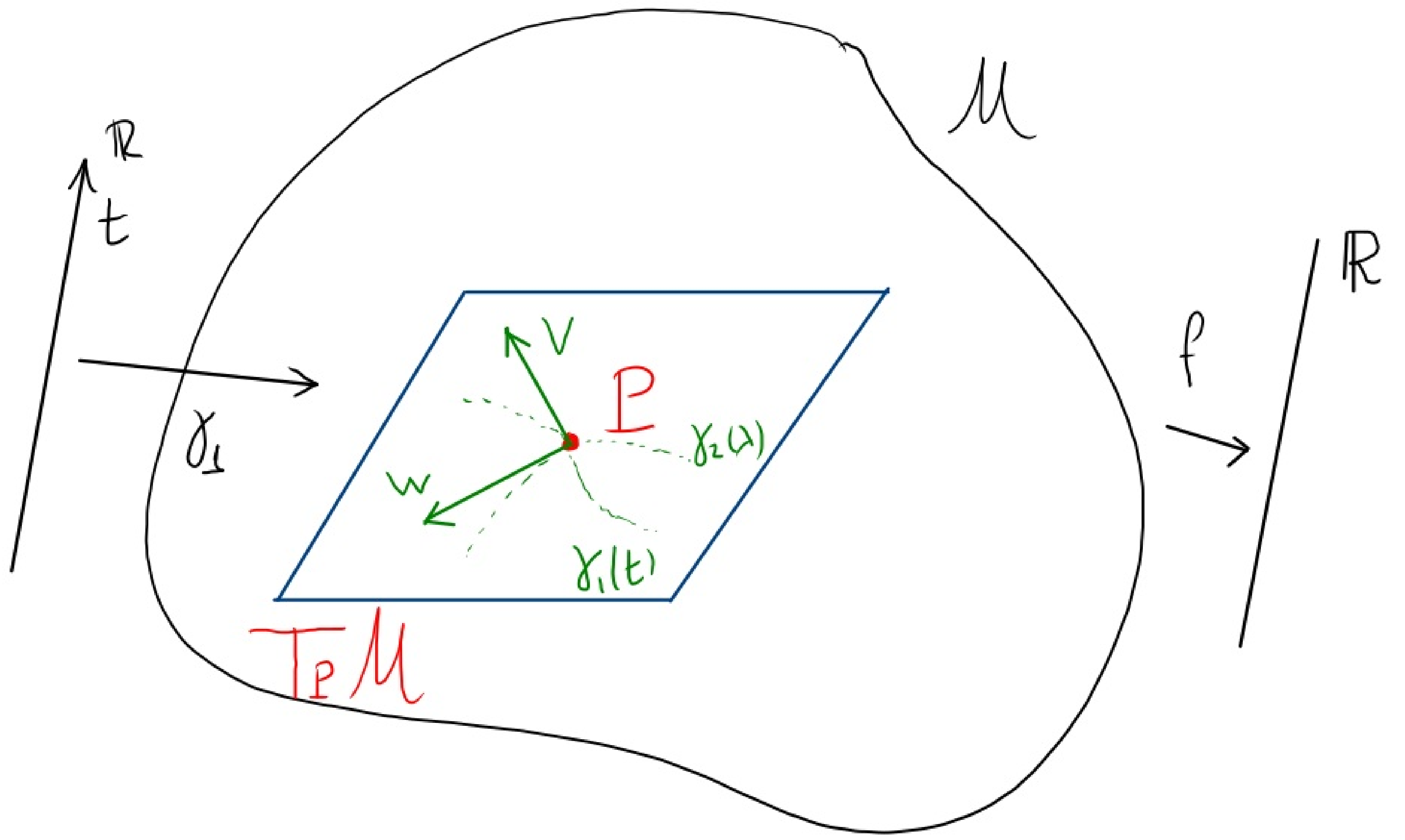

* They are derivations: $V(\alpha f + \beta g) = \alpha V(f) + \beta V(g)$

$$V(f \cdot g) = V(f) \cdot g + f \cdot V(g)$$

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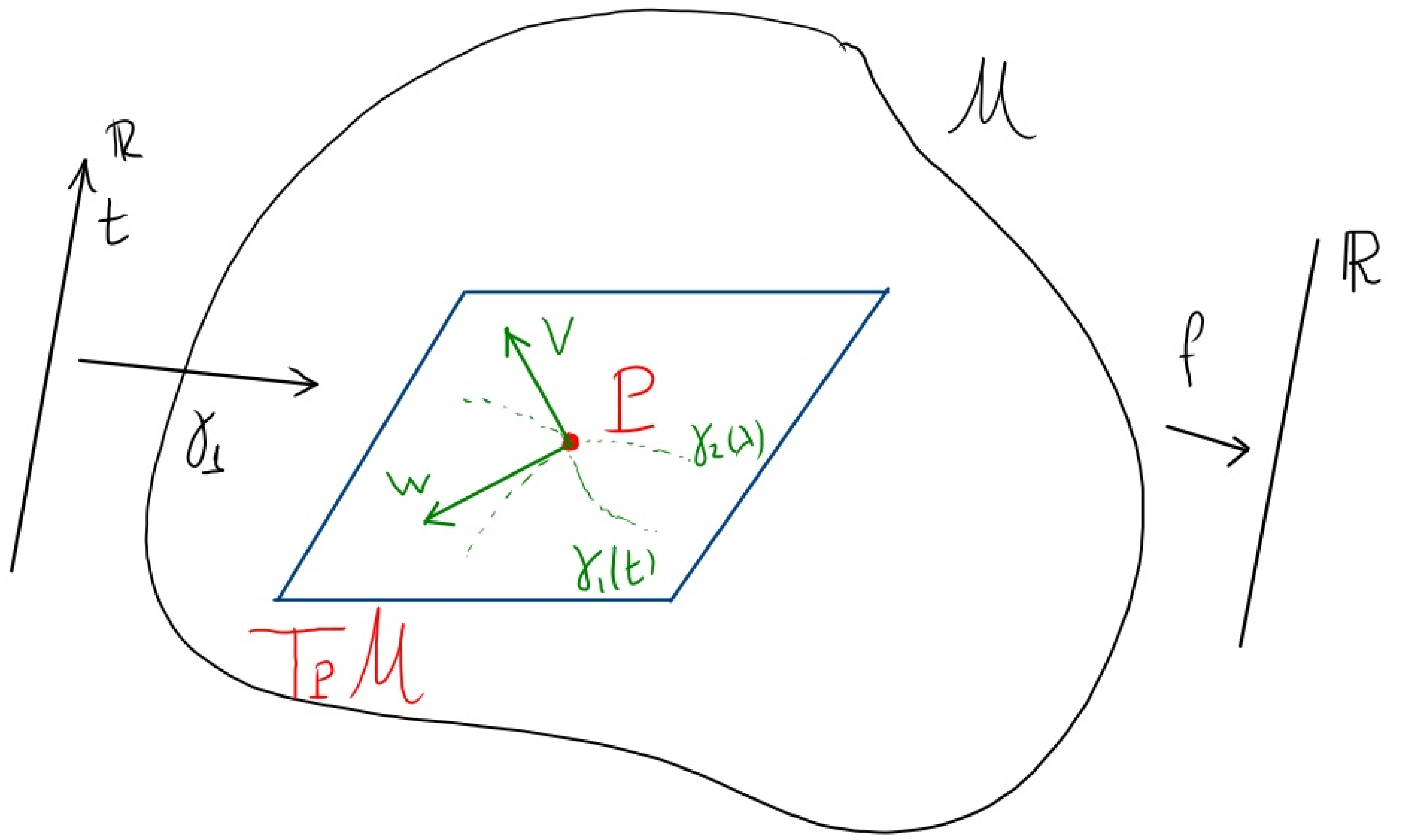

* They are derivations
* They form an n-dim vector space: $T_{\underline{p}}\mathcal{M}$

↳ • associated w/ $\underline{P}$
 • different at each $\underline{P}$

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$$V = \frac{d}{dt}\Big|_p : f \mapsto \frac{df}{dt}\Big|_p$$

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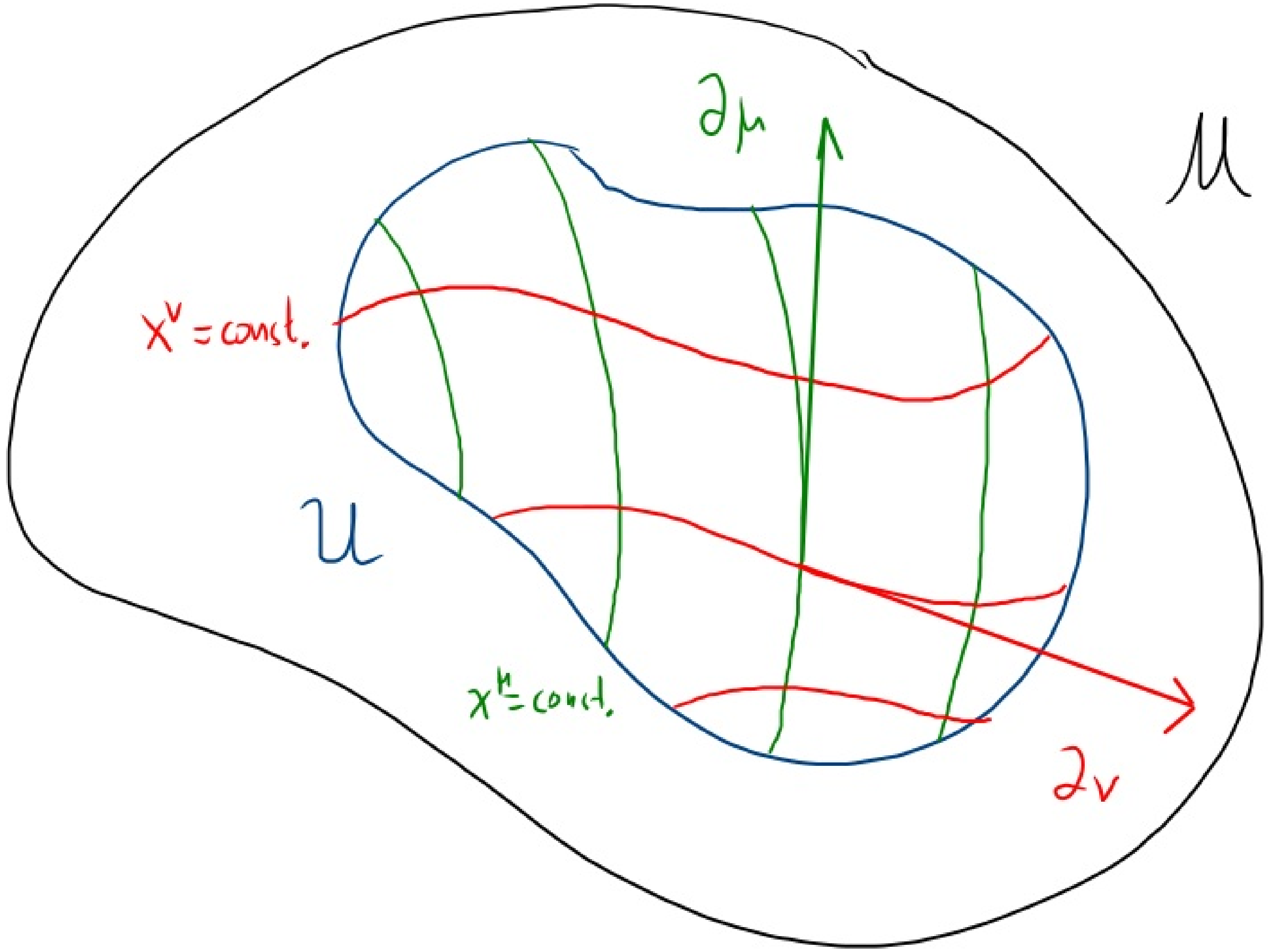
* They form an n-dim vector space: $T_p\mathcal{M}$

  _ choice of coord. system x^μ selects a special base $\{\partial_\mu\}$

  ↳ coordinate basis



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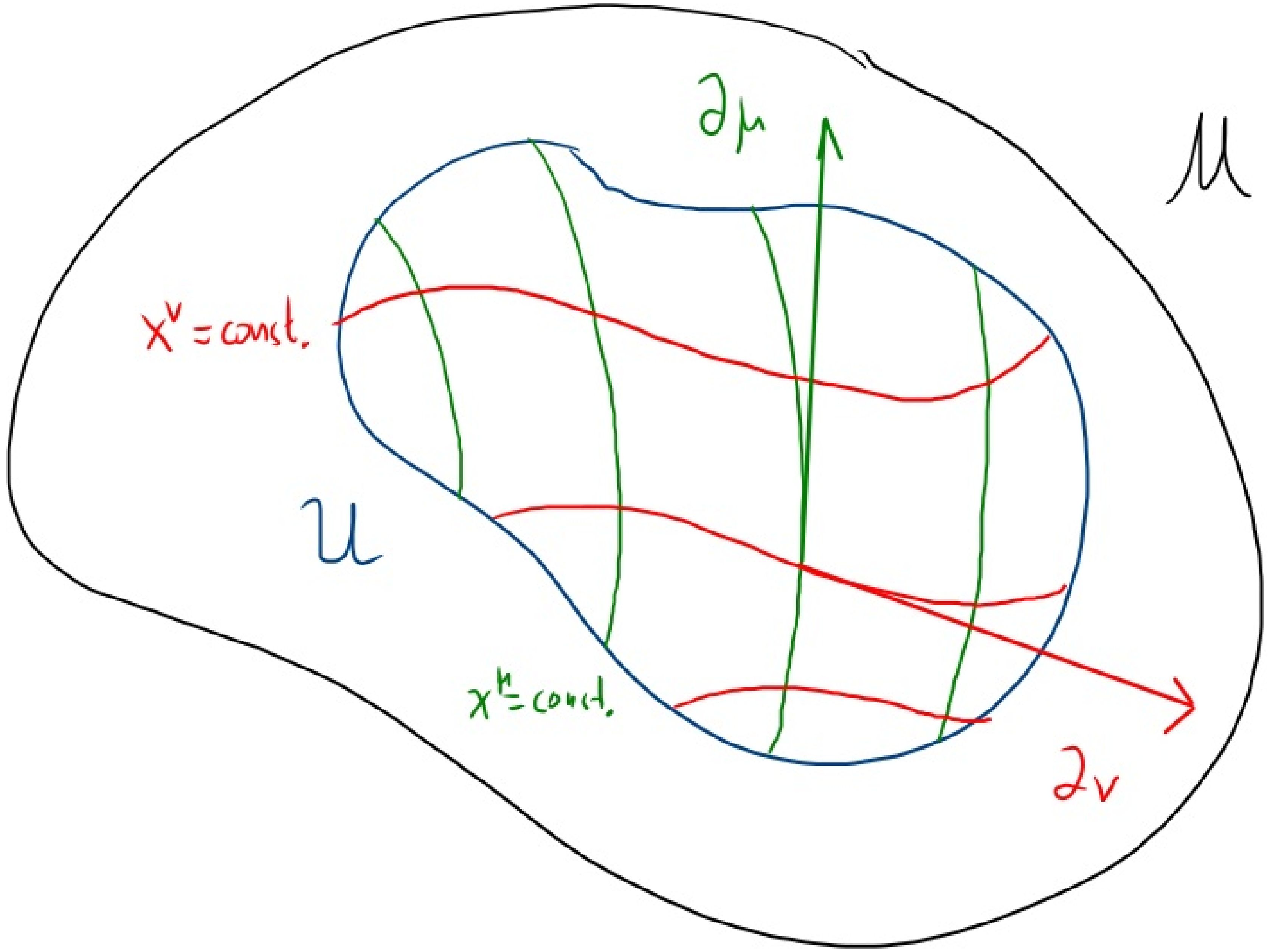


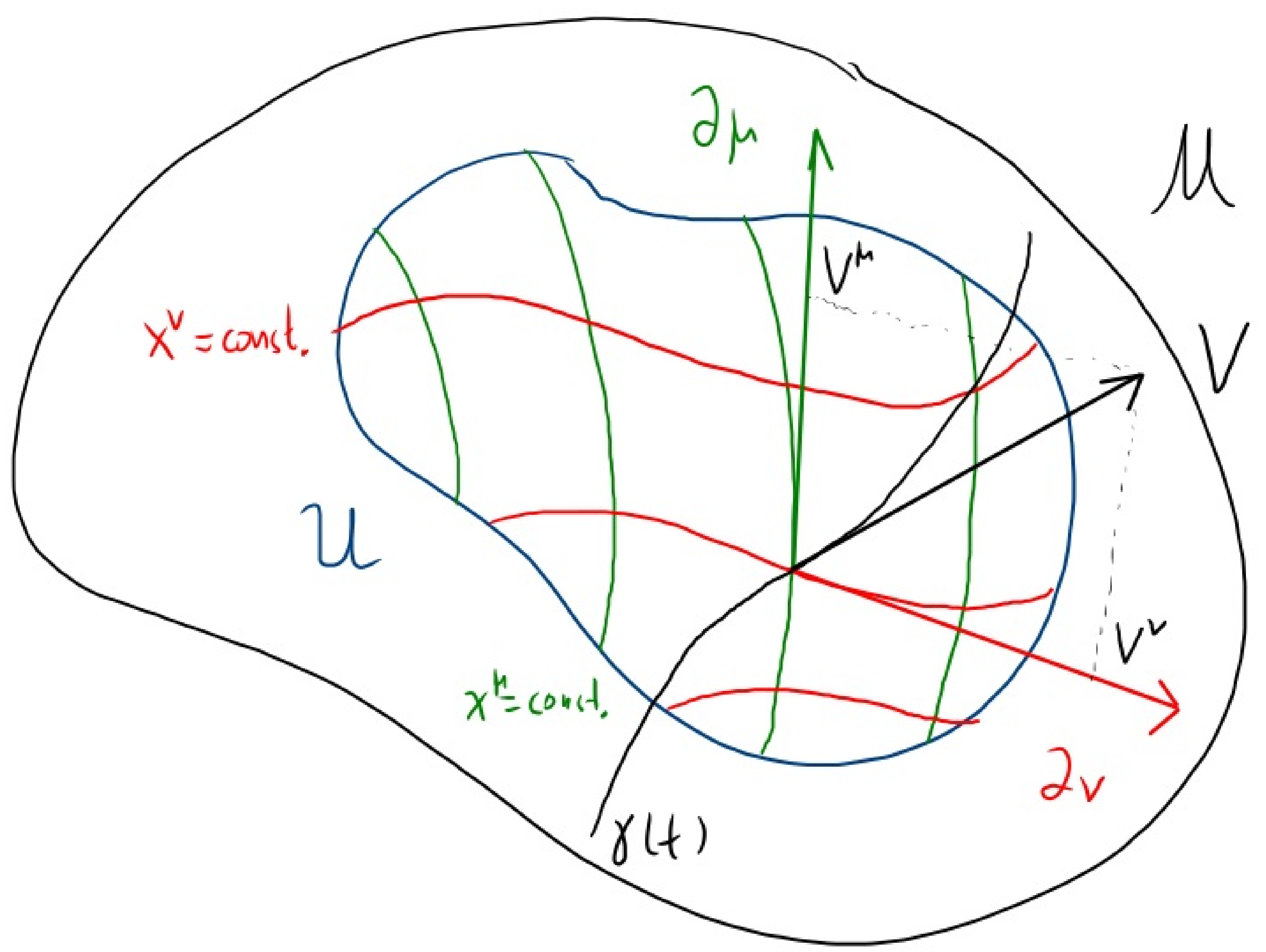
* They are derivations

* They form an n-dim vector space: $T_p\mathcal{M}$
  - choice of coord. system x^μ selects a coordinate basis, s.t.

$$V = V^\mu \partial_\mu \qquad V^\mu \text{ components of } V \text{ in } \{\partial_\mu\}$$

$$V^\mu = \frac{dx^\mu}{dt} = \left(\begin{array}{l} \text{rate of change} \\ \text{of } x^\mu \text{ along } \gamma \end{array} \right)$$

$$V = \frac{dx^\mu}{dt} \cdot \partial_\mu$$



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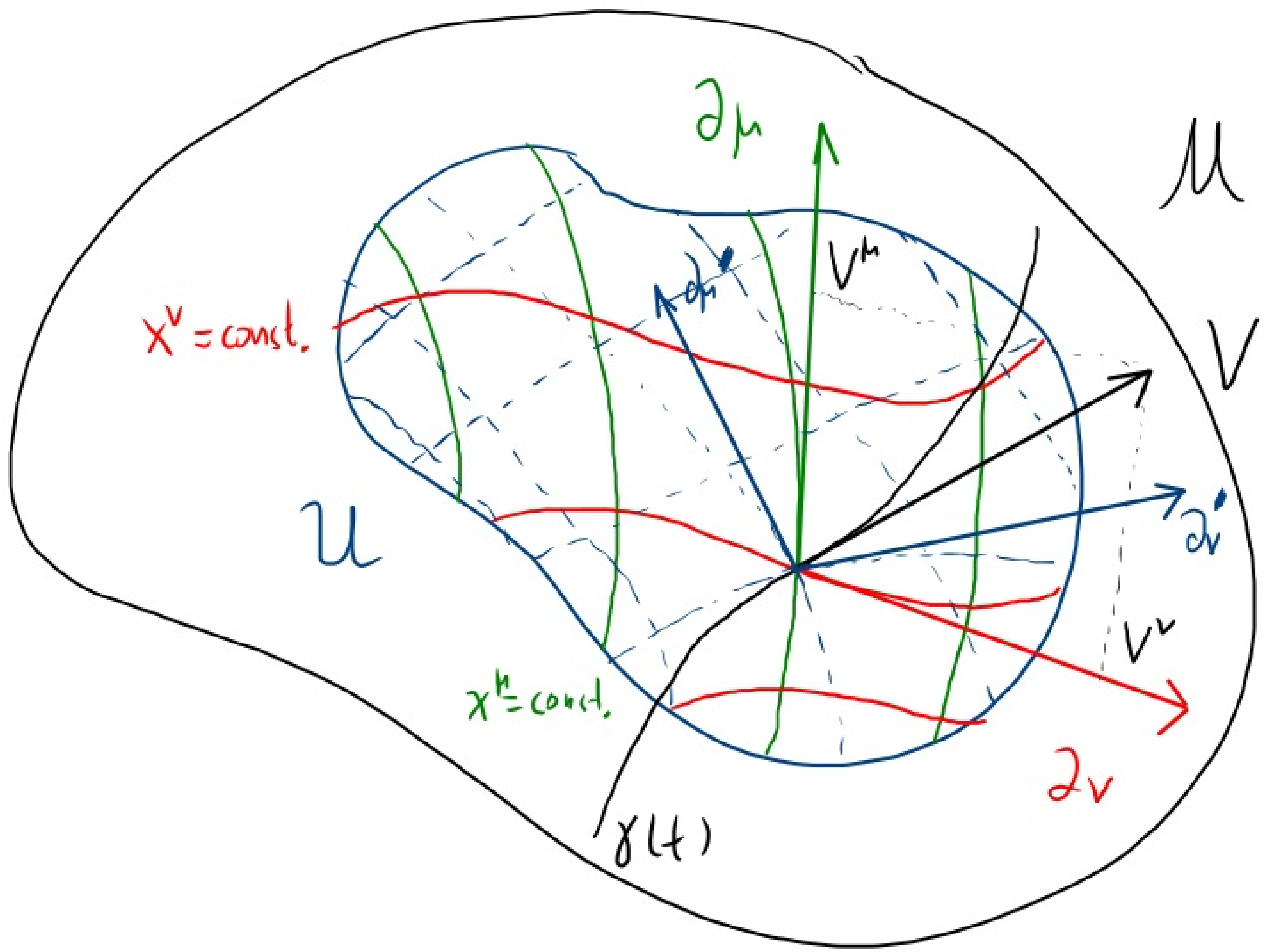
* Coordinate xfm: $x^\mu \to x^{\mu'}$

$$V^{\mu'} = \frac{dx^{\mu'}}{dt} \qquad V = V^{\mu'} \partial_{\mu'}$$

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$



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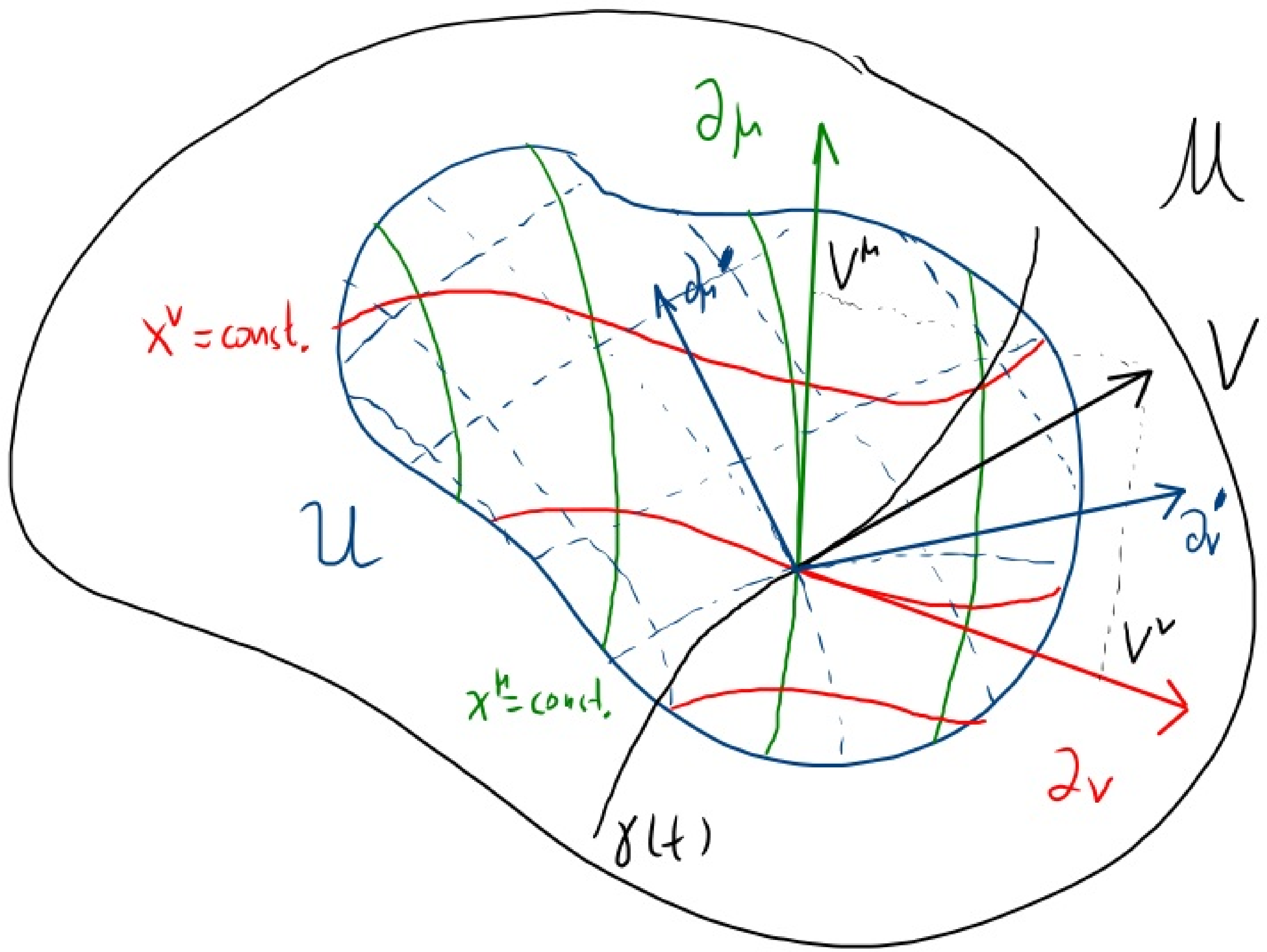

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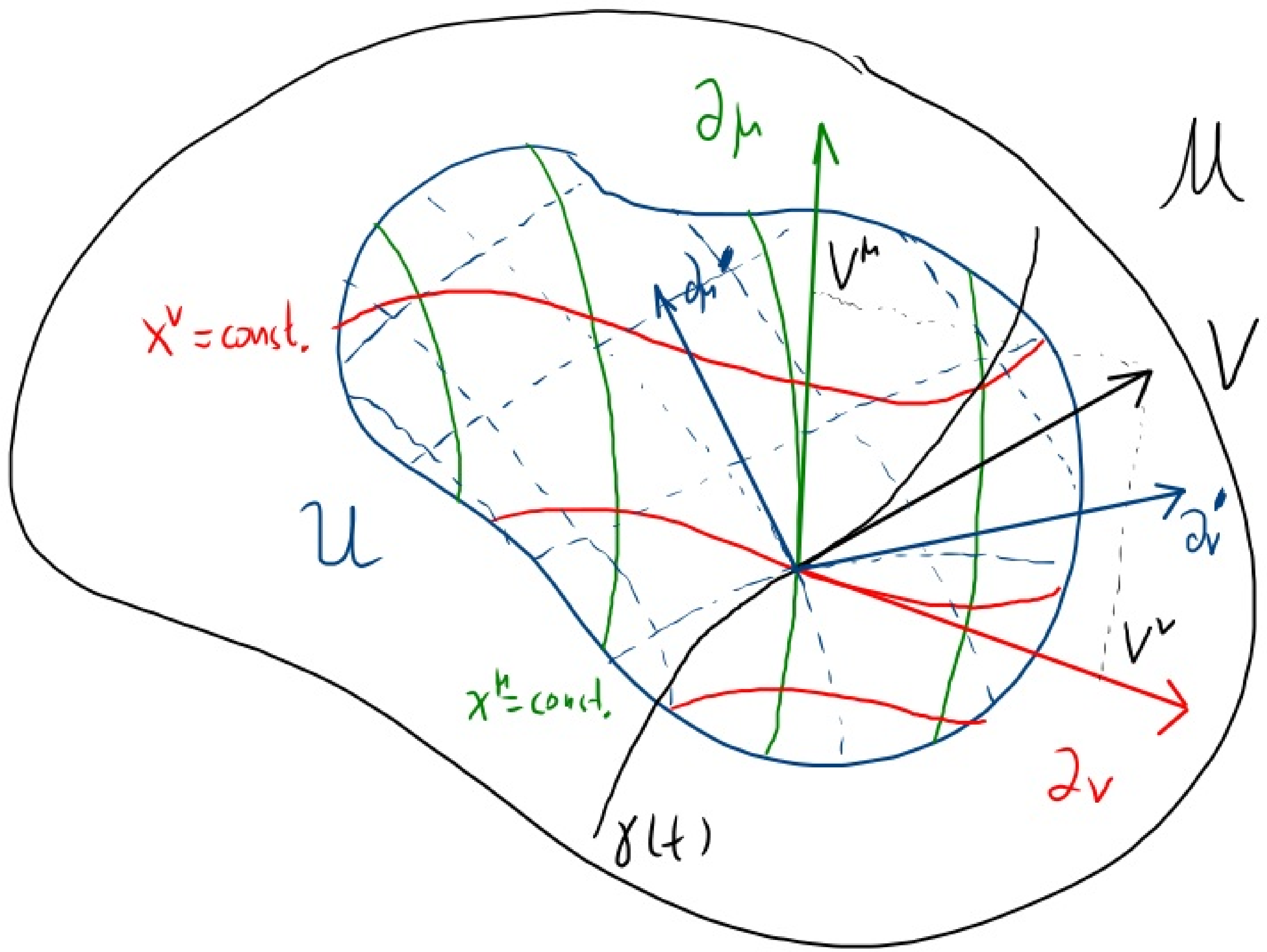

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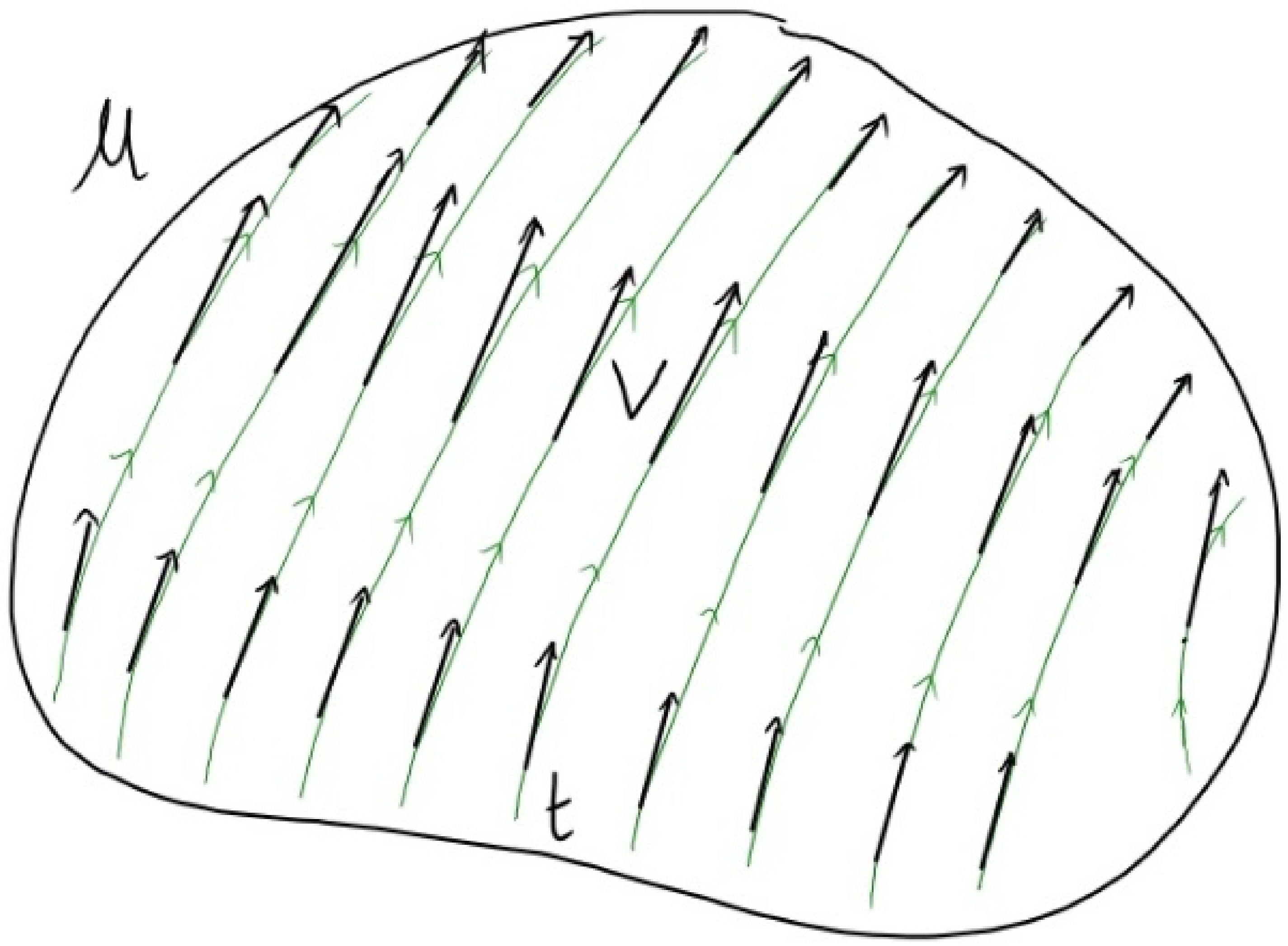
* General basis change $\{e_\alpha\} \to \{e_{\alpha'}\}$

$$e_\alpha = \Lambda_\alpha{}^{\alpha'} e_{\alpha'} \Rightarrow V^{\alpha'} = \Lambda_\alpha{}^{\alpha'} V^\alpha$$

row column

* **Vector fields:**

smoothly defined vectors $\forall\ P \in M$

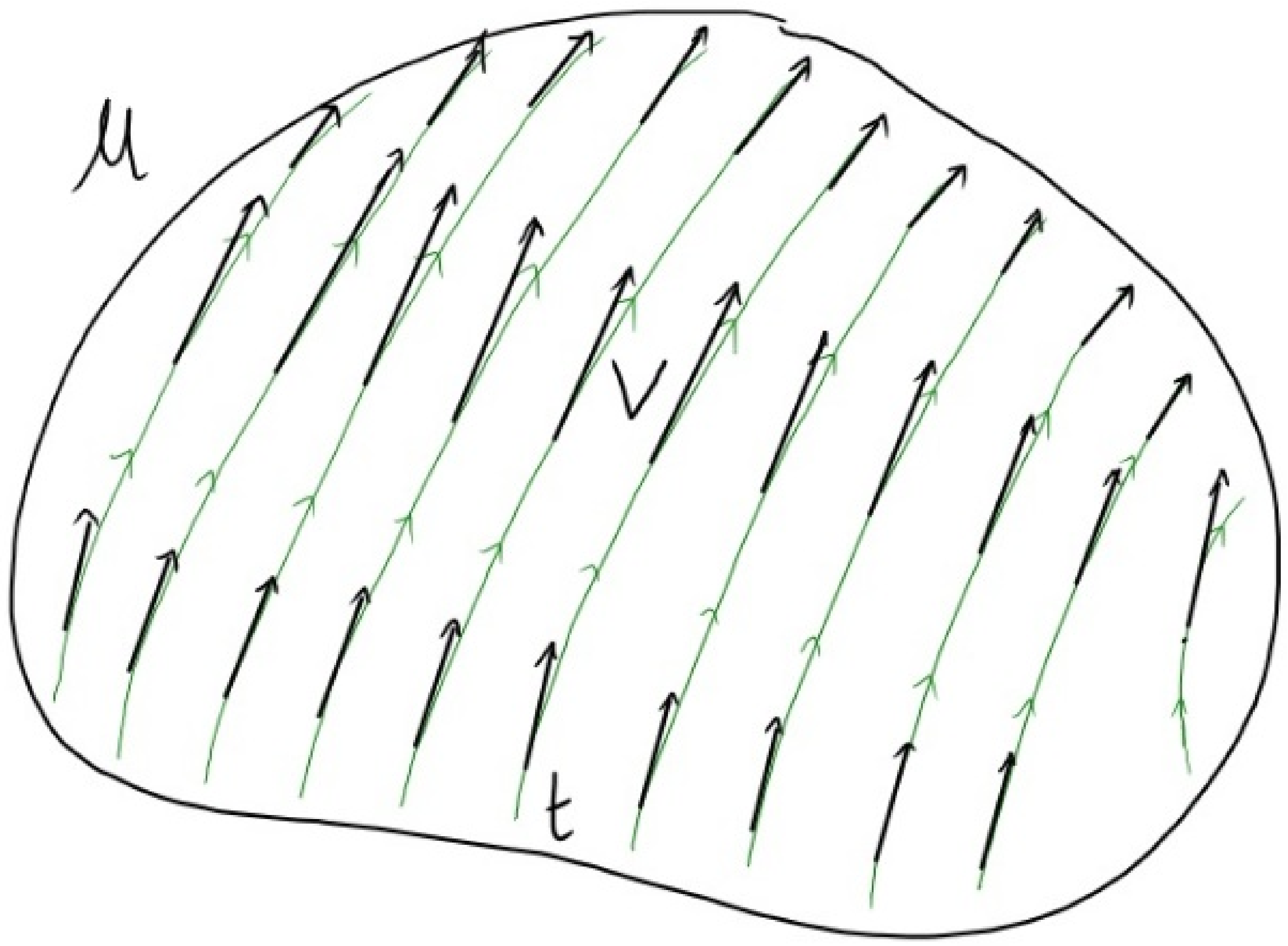


* Vector fields:

smoothly defined vectors $\forall\ p \in M \Rightarrow$

$$V(f) = \frac{df}{dt} \quad \text{a smooth function } \forall f \in \mathcal{F}(M)$$

along integral curves of V



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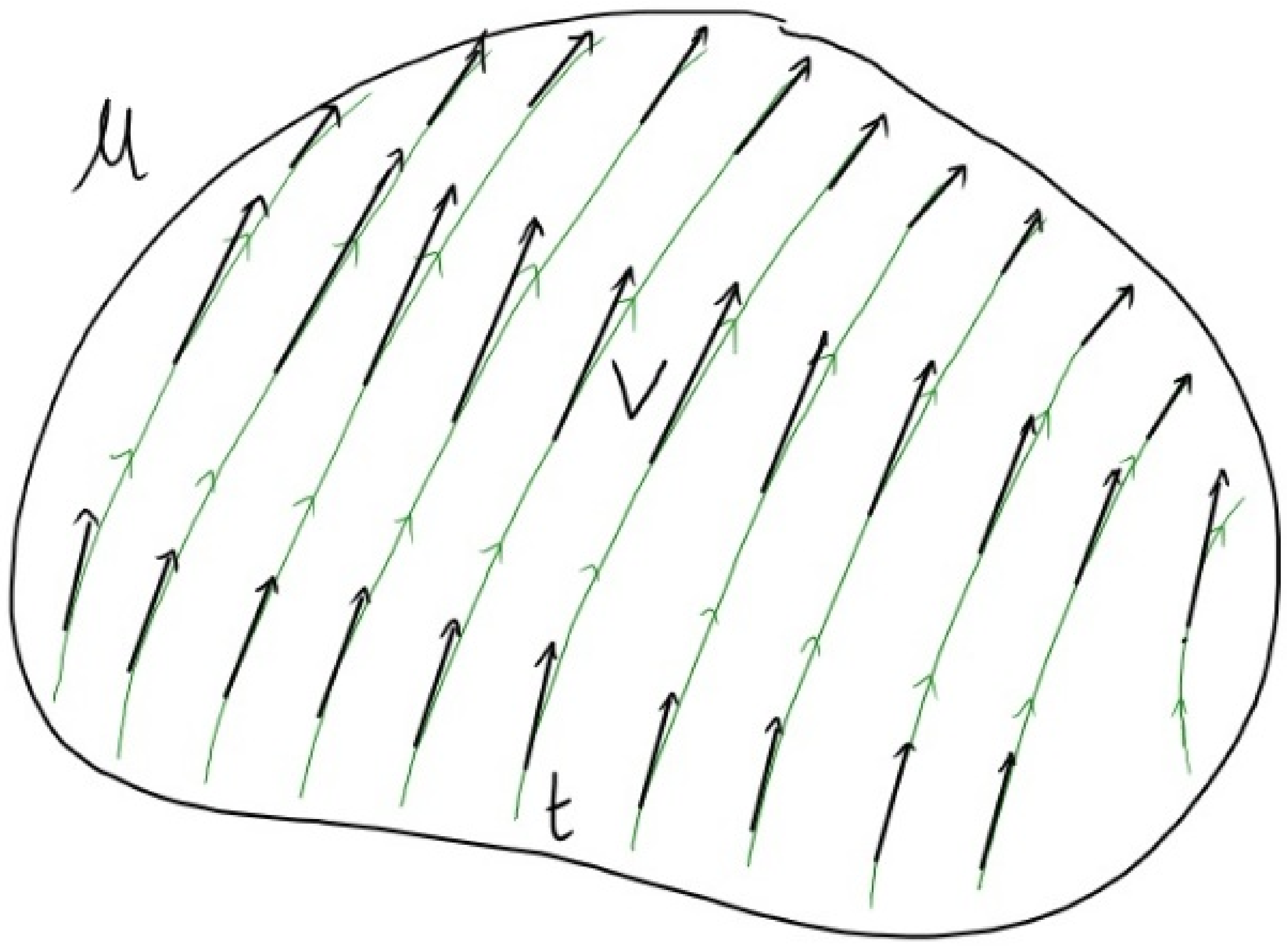
$$\Rightarrow V^\mu = \frac{dx^\mu}{dt} \quad \text{smooth functions}$$



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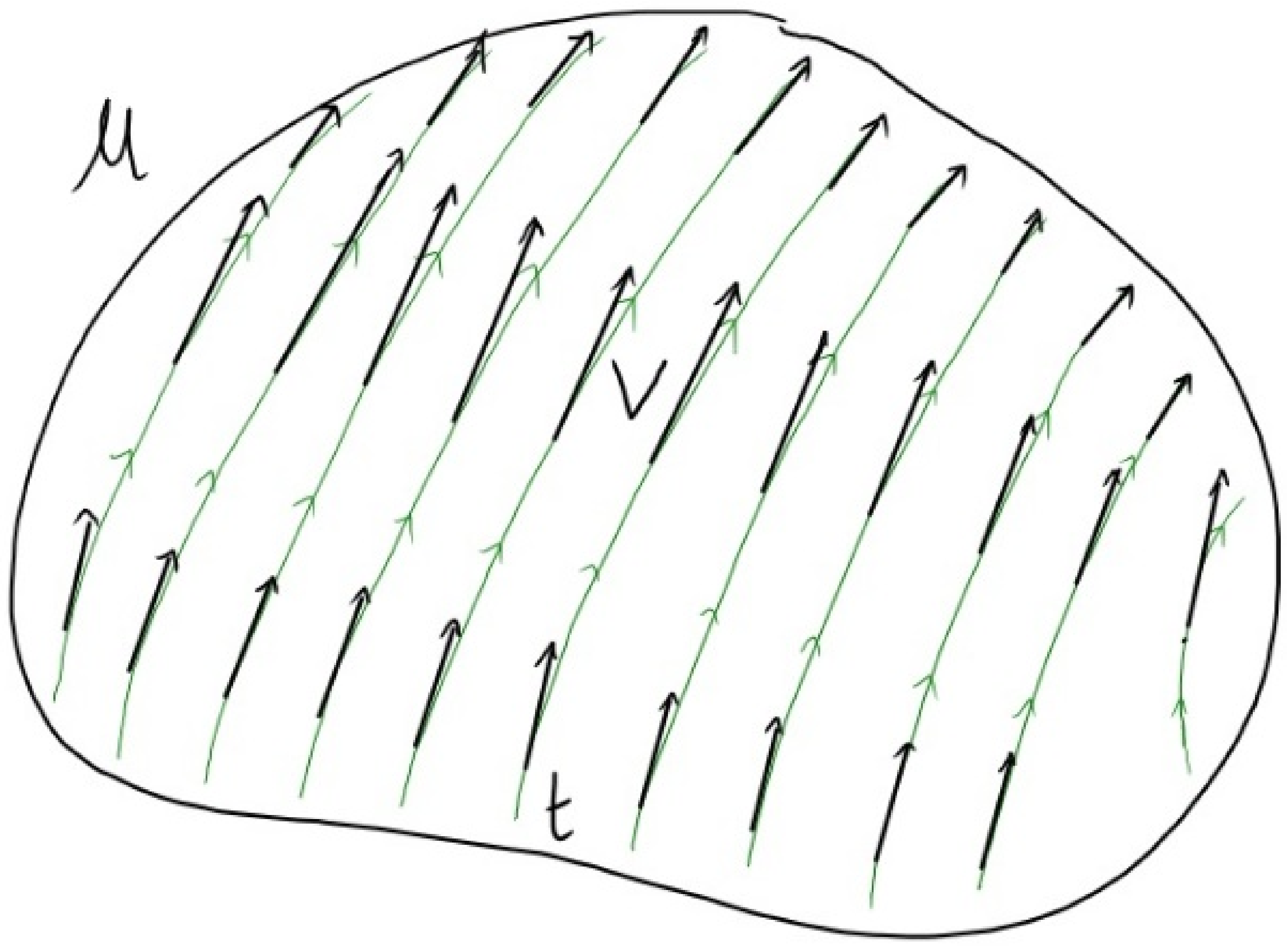

* Integral curves of a nonzero v.f. in $U \in M$ form a congruence
  - a unique one passing through every $P \in U$
  - they never cross

One-forms:

- defined @ $p \in M$
- linear maps on T_pM
- they form a dual vector space T_p^*M to T_pM

One-forms: acting on vectors $\rightarrow$ numbers

One-forms: acting on vectors $\to$ numbers

* Linear maps on $T_p M$:

$$\omega : T_p M \to \mathbb{R}$$
$$V \mapsto \omega(V)$$

One-forms: acting on vectors $\rightarrow$ numbers

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- Linearity:

$$\omega(\alpha V + \beta W) = \alpha\,\omega(V) + \beta\,\omega(W) \qquad \alpha, \beta \in \mathbb{R}$$

One-forms: acting on vectors $\to$ numbers

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- Linearity:

$$\omega(\alpha V + \beta W) = \alpha\,\omega(V) + \beta\,\omega(W) \qquad \alpha, \beta \in \mathbb{R}$$

- They form a vector space:

ω, σ 1-forms $\to$ $\alpha\omega + \beta\sigma$ a 1-form

One-forms: acting on vectors $\to$ numbers

* Linear maps on T_pM:

$$\omega : T_pM \to \mathbb{R}$$
$$V \mapsto \omega(V)$$

- Linearity: $\omega(\alpha V + \beta W) = \alpha\,\omega(V) + \beta\,\omega(W)$

- Addition + scalar multiplication:

$$(\alpha\omega + \beta\sigma)(V) = \alpha\,\omega(V) + \beta\,\sigma(V)$$

Exercise: Prove that $\alpha\omega + \beta\sigma$ is a 1-form

One-forms: acting on vectors $\rightarrow$ numbers

* Linear maps on $T_p\mathcal{M}$:

$$\omega : T_p\mathcal{M} \rightarrow \mathbb{R} \qquad V \mapsto \omega(V) \in \mathbb{R}$$

* They form an n-dim vector space $T_p^*\mathcal{M}$:
  - Let $\{e_\alpha\}$ be a basis in $T_p\mathcal{M}$
  - Define its dual $\{e^\alpha\}$, e^α 1-forms, from

$$e^\alpha(e_\beta) = \delta^\alpha{}_\beta$$

(Careful: not an orthonormality relation, e^α, e_β are different objects)

One-forms: acting on vectors $\to$ numbers

* Linear maps on $T_p\mathcal{M}$:

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$$e^\alpha(e_\beta) = \delta^\alpha{}_\beta$$

use linearity of maps

$$\Rightarrow e^\alpha(V) = e^\alpha(V^\beta e_\beta) = V^\beta e^\alpha(e_\beta) = V^\beta \delta^\alpha{}_\beta = V^\alpha$$

* We can always write

$$\underbrace{\omega}_{\text{1-form}} = \underbrace{\omega_\alpha}_{\text{numbers}} \underbrace{e^\alpha}_{\text{1-forms}} \Rightarrow \{e^\alpha\} \text{ a basis in } T^*_p\mathcal{M}$$

---

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* We can always write

$$\omega = \omega_\alpha e^\alpha \quad \Rightarrow \{e^\alpha\} \text{ a basis in } T_p^* M$$

Indeed:

$$\omega = \omega_\beta e^\beta \quad \Rightarrow \quad \omega(e_\alpha) = \omega_\beta \, e^\beta(e_\alpha) = \omega_\beta \, \delta^\beta{}_\alpha = \omega_\alpha$$

$\forall e_\alpha$

use duality relation

$$e^\alpha(e_\beta) = \delta^\alpha{}_\beta$$

$$\Rightarrow e^\alpha(V) = e^\alpha(V^\beta e_\beta) = V^\beta e^\alpha(e_\beta) = V^\beta \delta^\alpha{}_\beta = V^\alpha$$

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$$\Rightarrow \omega(V) = \omega_\beta e^\beta(V) \quad \forall V \in T_p^*\mathcal{M} \Leftrightarrow \omega = \omega_\beta e^\beta$$

* We can always write

$\omega = \omega_\alpha e^\alpha \Rightarrow \{e^\alpha\}$ a basis in $T_p^* M$

$\Rightarrow T_p^* M$ is n-dimensional

$\omega(V) = \omega_\alpha V^\alpha \in \mathbb{R}$ is the contraction of ω and V

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* The spaces $T_p M$ and $T_p^* M$ are dual:

A vector can be viewed as a linear map

$$T_p^* M \to \mathbb{R} \quad \text{s.t}$$

$$\omega \to V(\omega) \equiv \omega(V)$$

(in fact, it is possible to define 1-forms to be the fundamental geometric objects on M)

* The gradient df

Given $f \in \mathcal{F}(M)$, define the 1-form:

$$df : T_P M \to \mathbb{R}$$

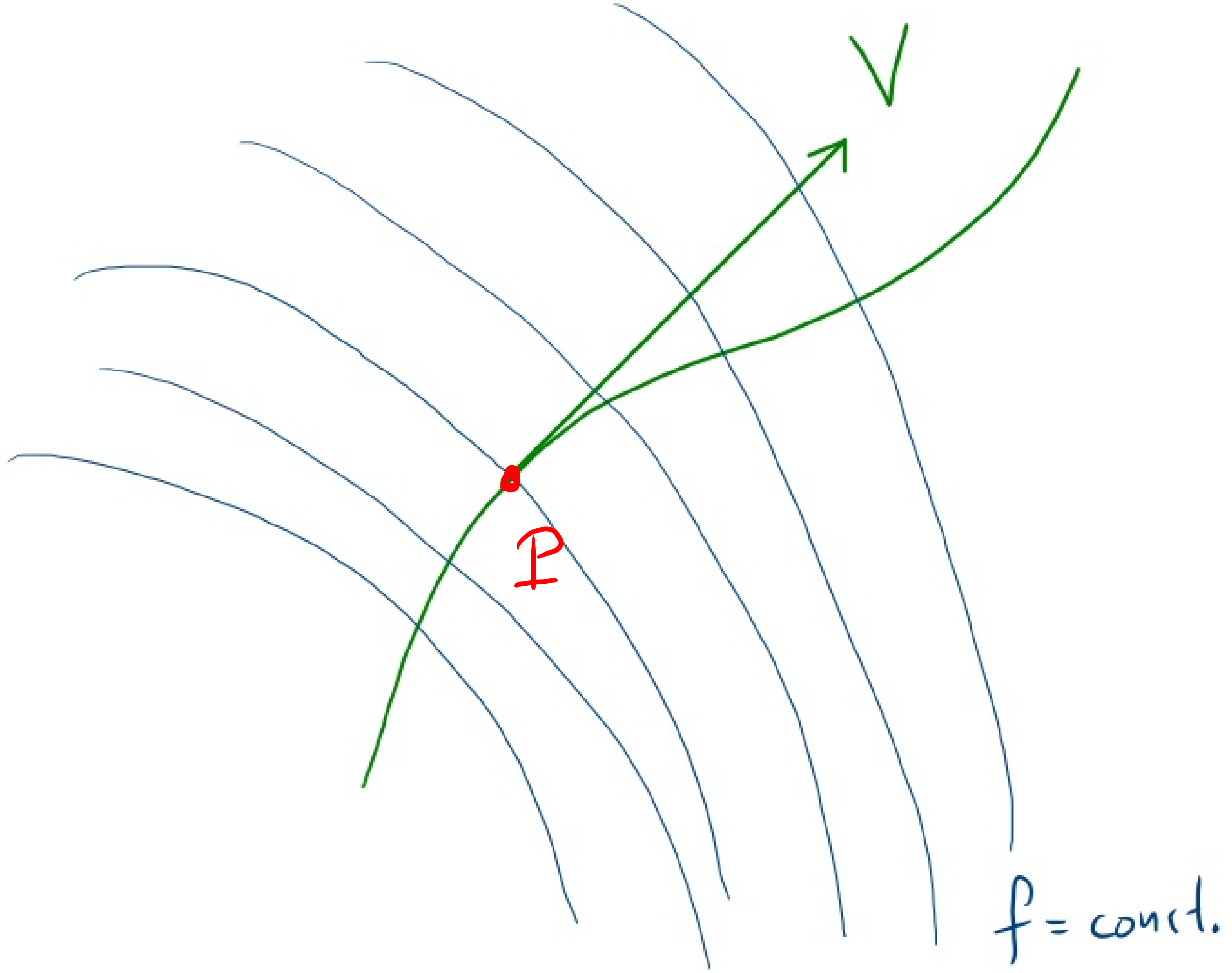
$$V \to df(V) \equiv V(f) = \frac{df}{dt}$$

along the curve of V

choice of V

$\Updownarrow$

choice of equiv. classes of curves through P



* The gradient df

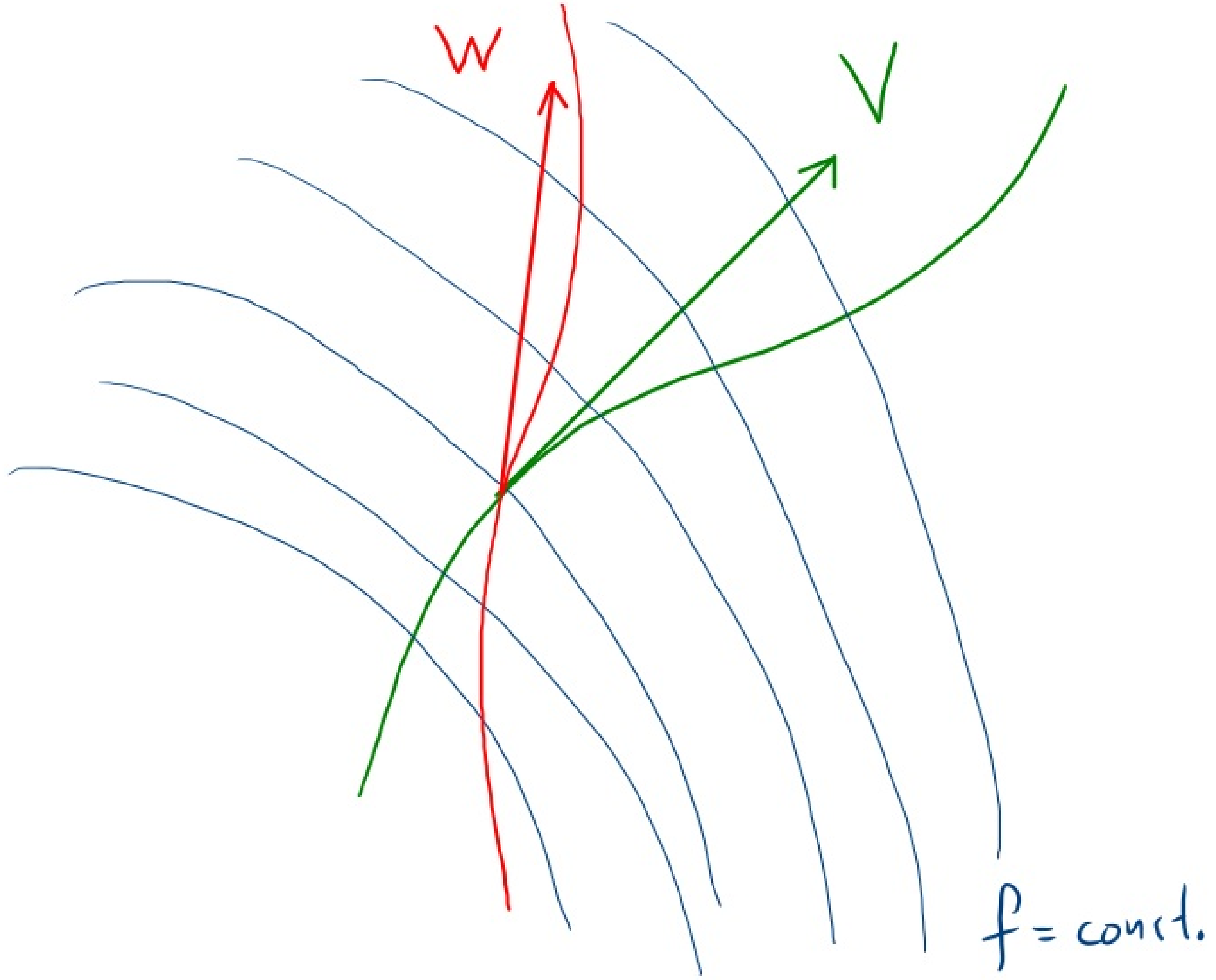
Given $f \in \mathcal{F}(\mathcal{M})$, define the 1-form:

$$df : T_p\mathcal{M} \to \mathbb{R}$$

a function

$\forall$ vector

$$\begin{cases} V \to df(V) \equiv V(f) \\ W \to df(W) = W(f) = \dfrac{df}{d\lambda} \end{cases}$$



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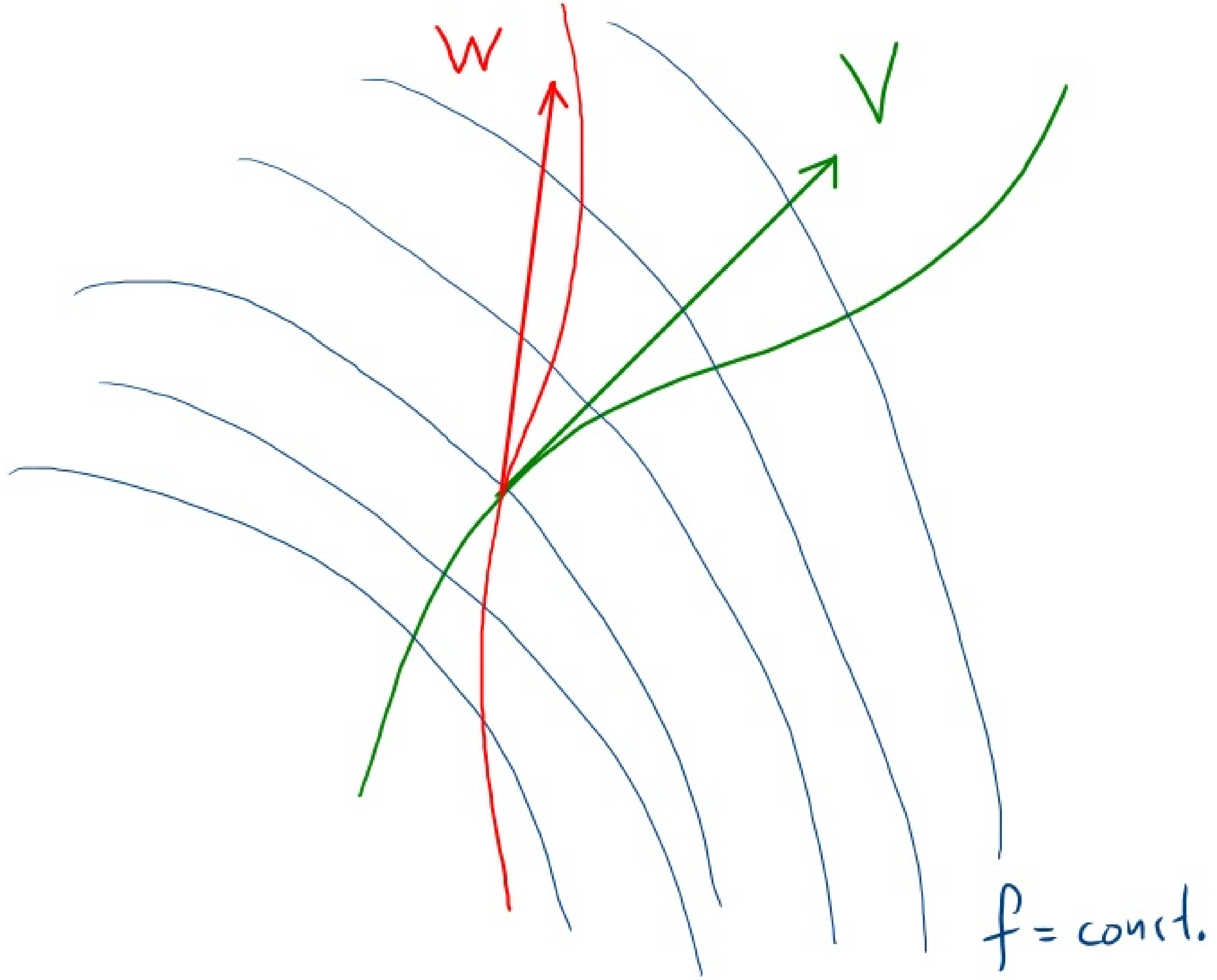

A linear function:

$$df(\alpha V + \beta W) = (\alpha V + \beta W)(f)$$
$$= \alpha V(f) + \beta W(f)$$
$$= \alpha\, df(V) + \beta\, df(W)$$

* If $\{\partial_\mu\}$ is a coordinate basis, then

$$df(\partial_\mu) = \partial_\mu f$$

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For $f = x^\mu$

$$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta_\nu{}^\mu \Rightarrow \{dx^\mu\} \text{ the dual basis to } \{\partial_\mu\}$$

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For $f = x^\mu$

$$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta_\nu{}^\mu \quad \Rightarrow \{dx^\mu\} \text{ the dual basis to } \{\partial_\mu\}$$

$$\Rightarrow \quad \omega = \omega_\mu dx^\mu \quad , \quad \omega_\mu = \omega(\partial_\mu)$$ (as shown before for a general basis)

ω_μ → the components of ω in coordinate (dual) basis $\{dx^\mu\}$

* If $\{\partial_\mu\}$ is a coordinate basis, then

$$\omega = \omega_\mu dx^\mu$$

* Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

$$\omega = \omega_{\mu'} dx^{\mu'}$$

* If $\{\partial_\mu\}$ is a coordinate basis, then

$$\omega = \omega_\mu \, dx^\mu$$

* Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

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But:

$$dx^{\mu'}(\partial_\nu) = \partial_\nu (x^{\mu'}) = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

* If $\{\partial_\mu\}$ is a coordinate basis, then

$$\omega = \omega_\mu \, dx^\mu$$

* Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

$$\omega = \omega_{\mu'} \, dx^{\mu'}$$

But: components of $dx^{\mu'}$ in $\{dx^\nu\}$ basis

$$dx^{\mu'}(\partial_\nu) = \partial_\nu (x^{\mu'}) = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

$$\Rightarrow dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} \, dx^\nu$$

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* $dx^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$

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* $$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

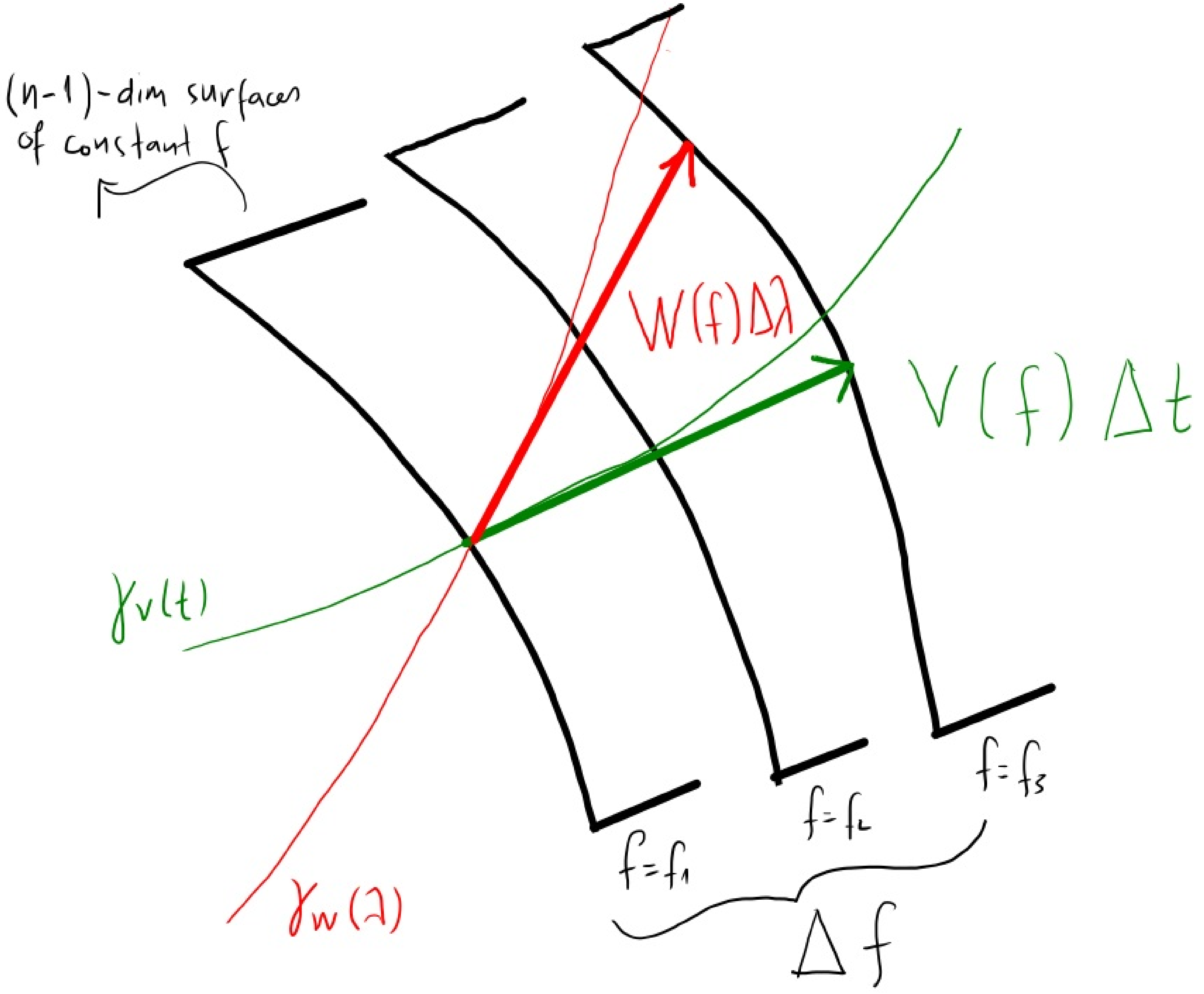
$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

compare!

The Geometry of df

$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta\lambda$$



$$V(f) = \frac{df}{dt} \qquad W(f) = \frac{df}{d\lambda}$$

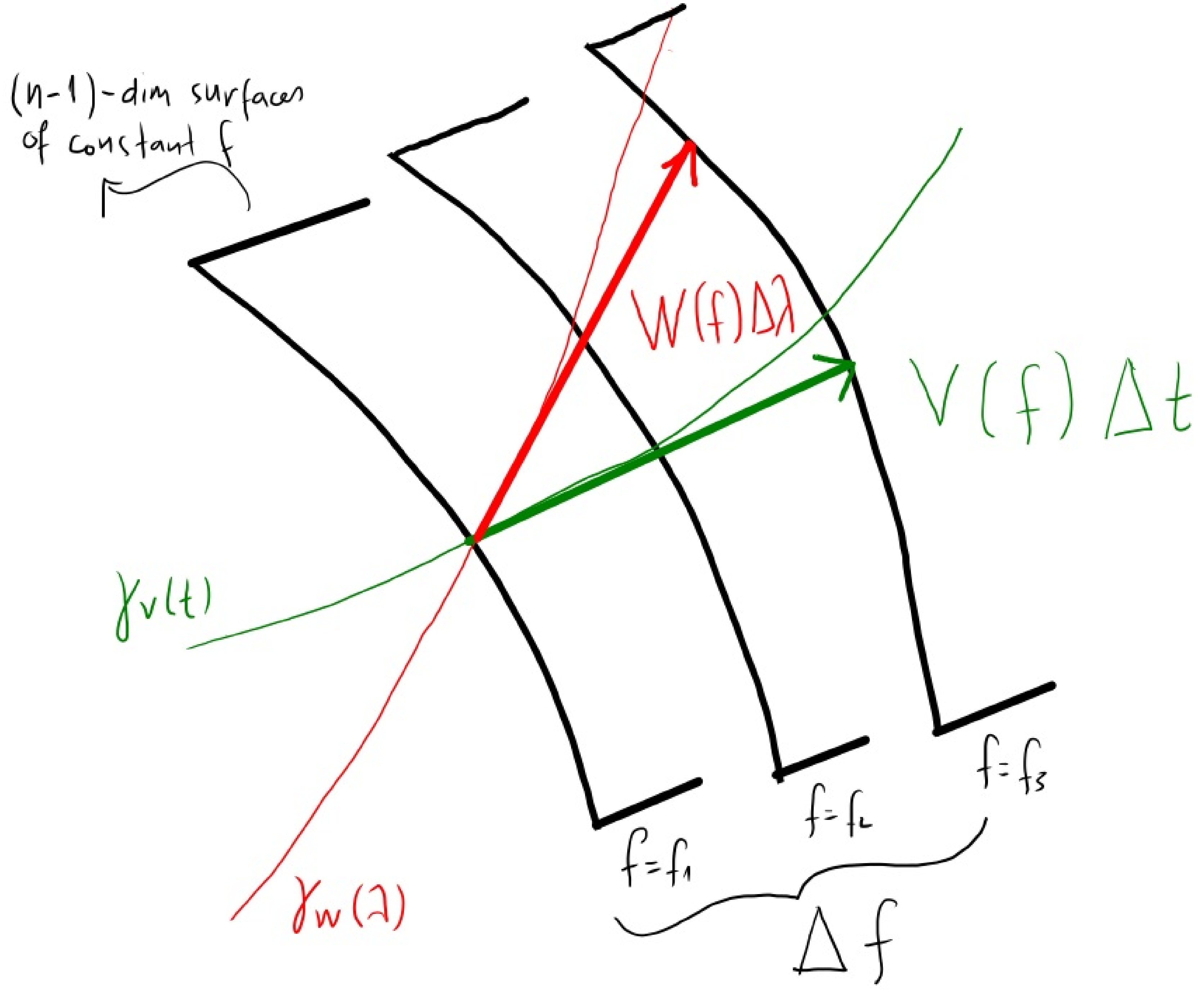
The Geometry of df

$\Delta f = V(f)\,\Delta t$

$\Delta f = W(f)\,\Delta\lambda$

For all vectors:

$\Delta f = V(f)\,\Delta t = df(V)\,\Delta t$



The Geometry of df

$$\Delta f = V(f)\,\Delta t$$

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For all vectors:

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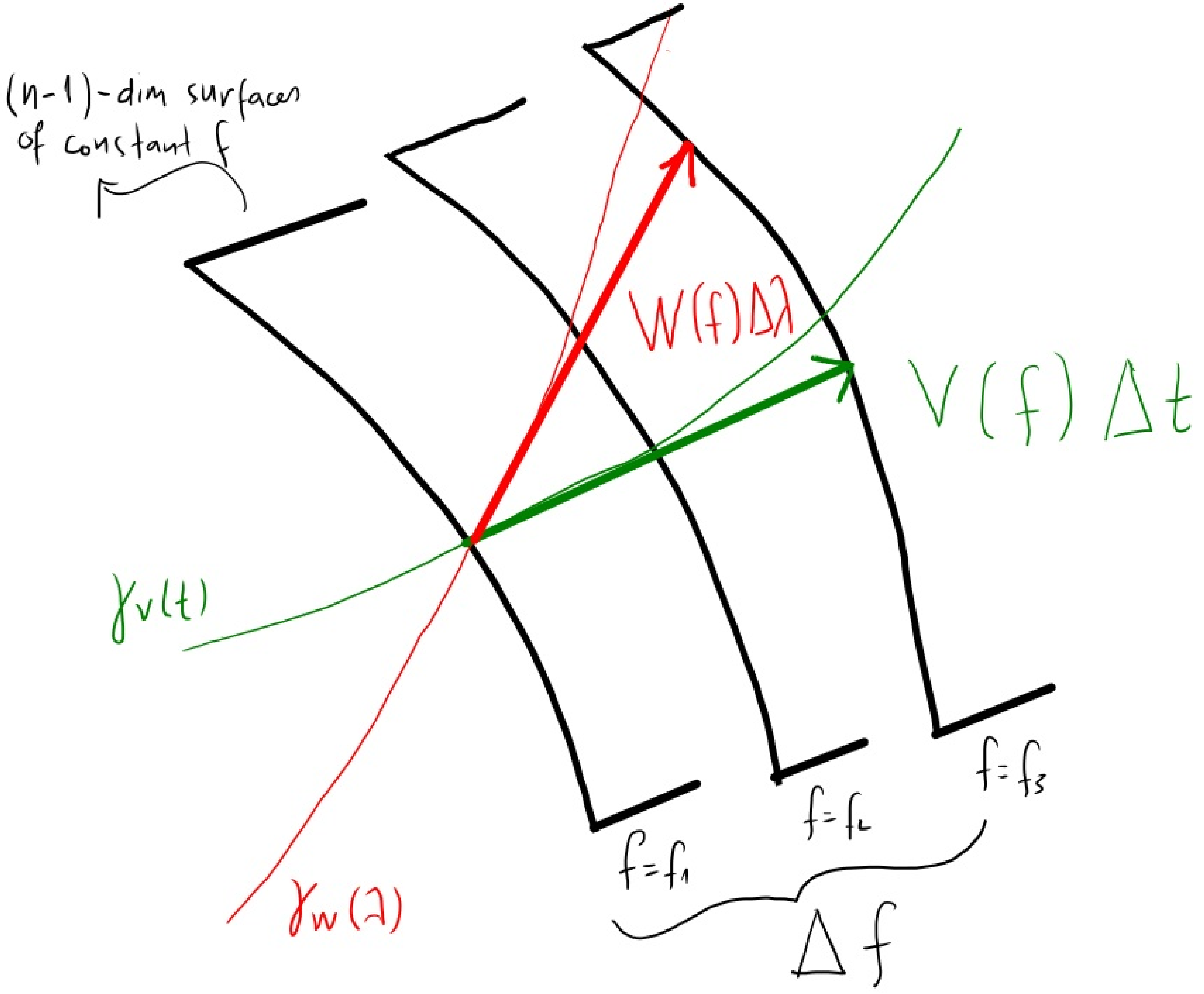
$$= \frac{\partial f}{\partial x^\mu} V^\mu \Delta t$$



The Geometry of df

$$\Delta f = V(f)\,\Delta t$$

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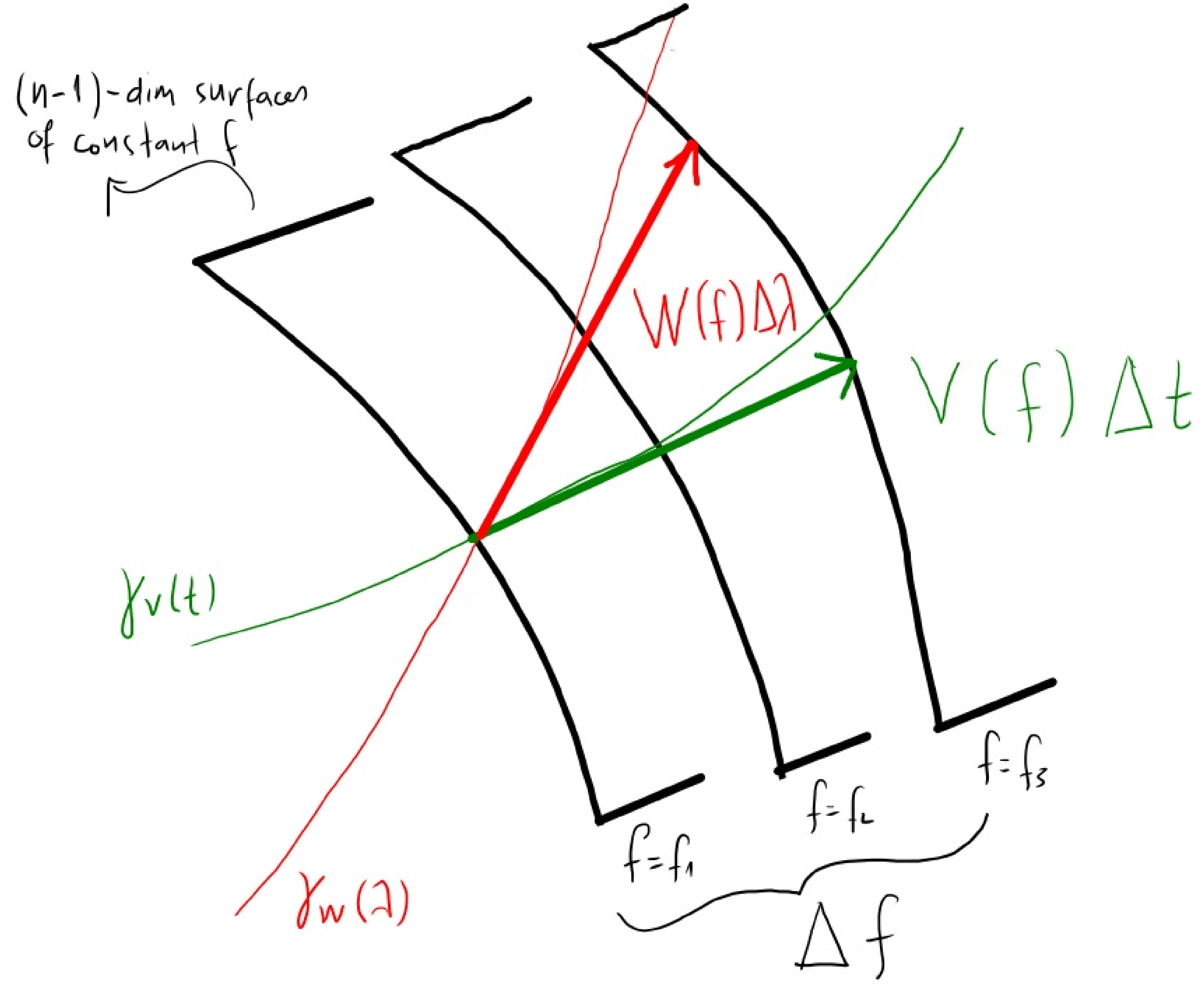
For all vectors:

$$\Delta f = V(f)\,\Delta t = df(V)\,\Delta t$$

$$= \frac{\partial f}{\partial x^\mu} V^\mu \Delta t = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}\Delta t$$



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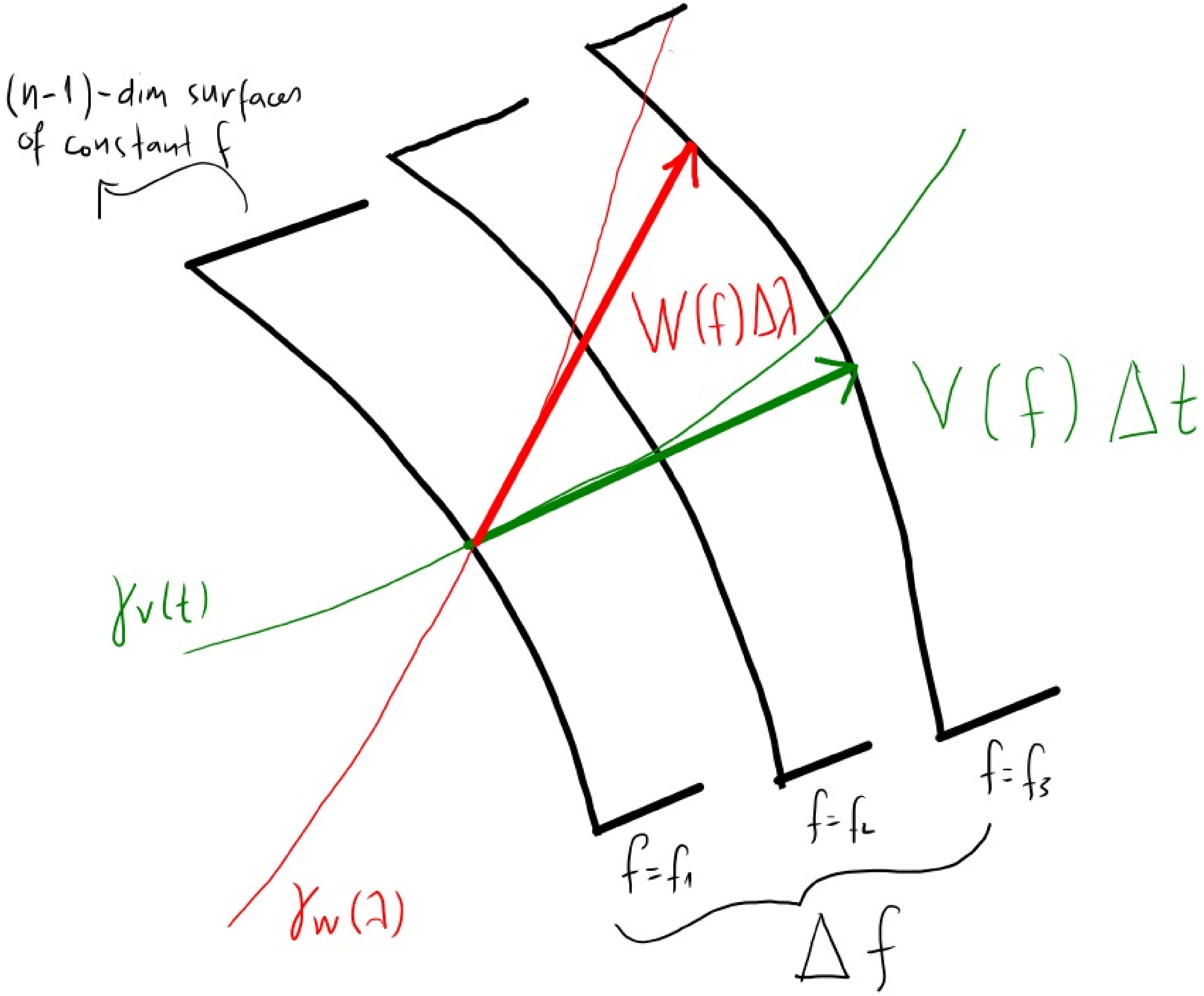


For all vectors:

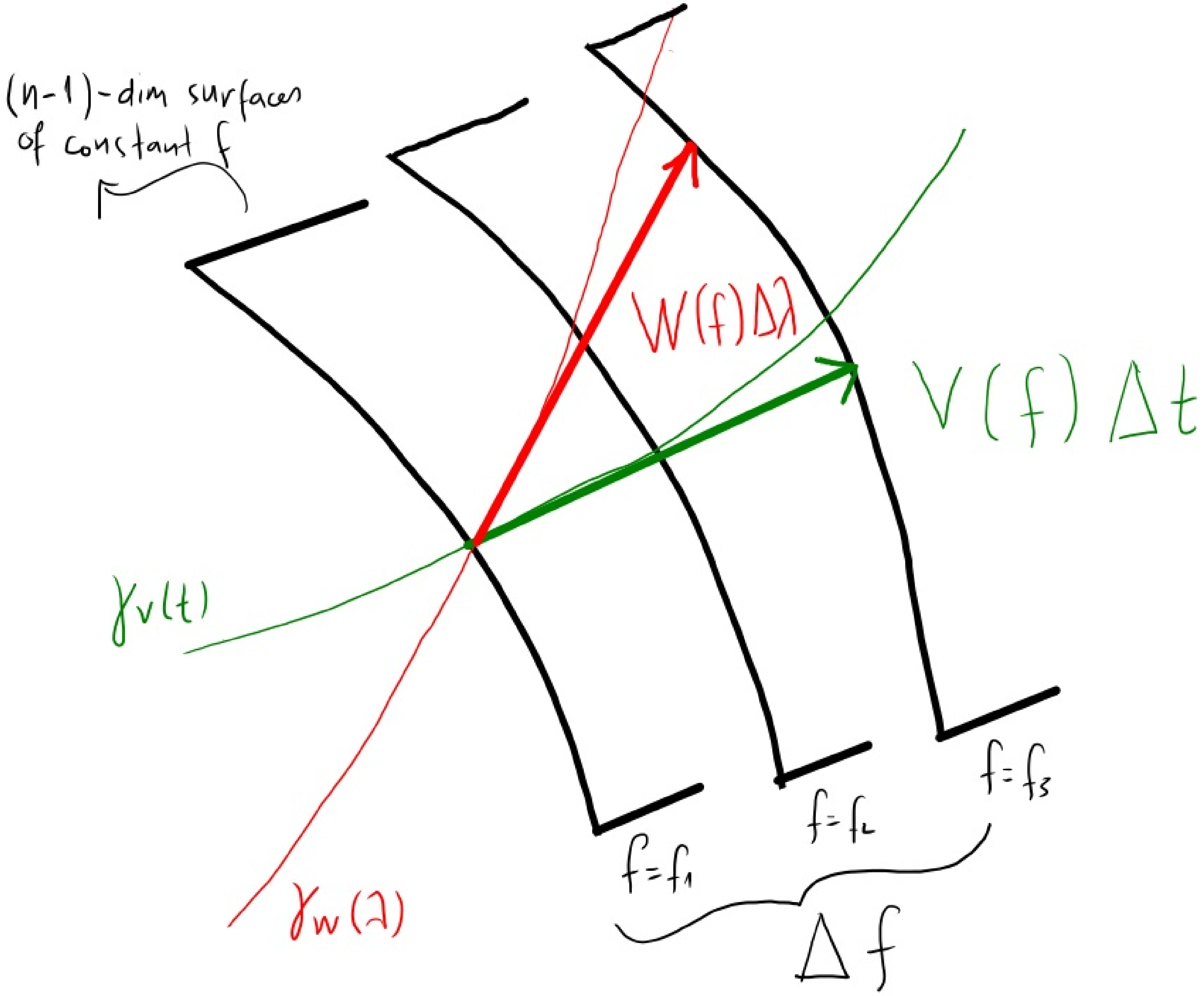
$$\Delta f = V(f)\,\Delta t = df(V)\,\Delta t$$

$$= \frac{\partial f}{\partial x^\mu} V^\mu \Delta t = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}\Delta t = \frac{\partial f}{\partial x^\mu}\Delta x^\mu$$

The Geometry of df

$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta\lambda$$



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No reference to the curve –
just choose $\{\Delta x^\mu\}$

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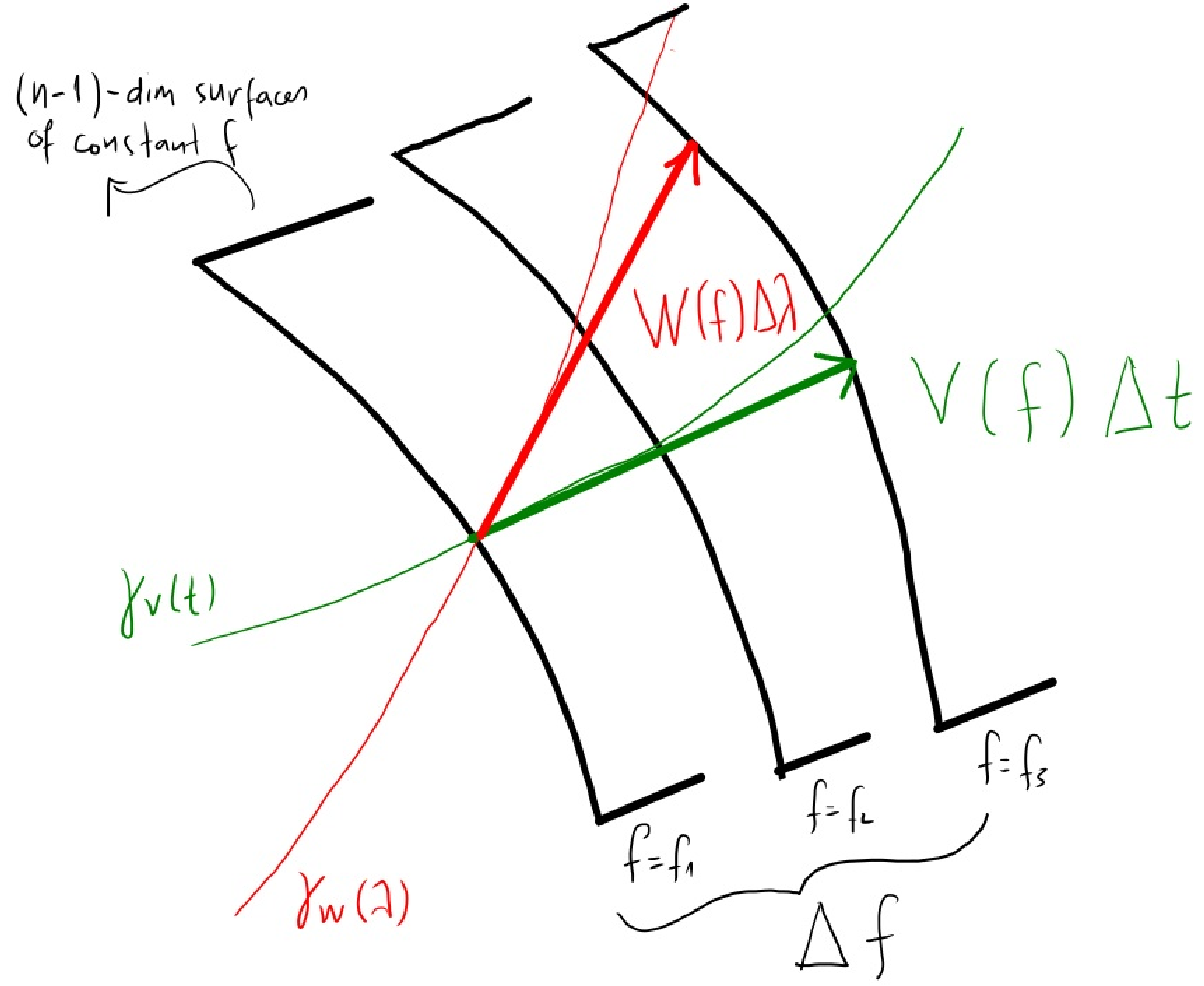
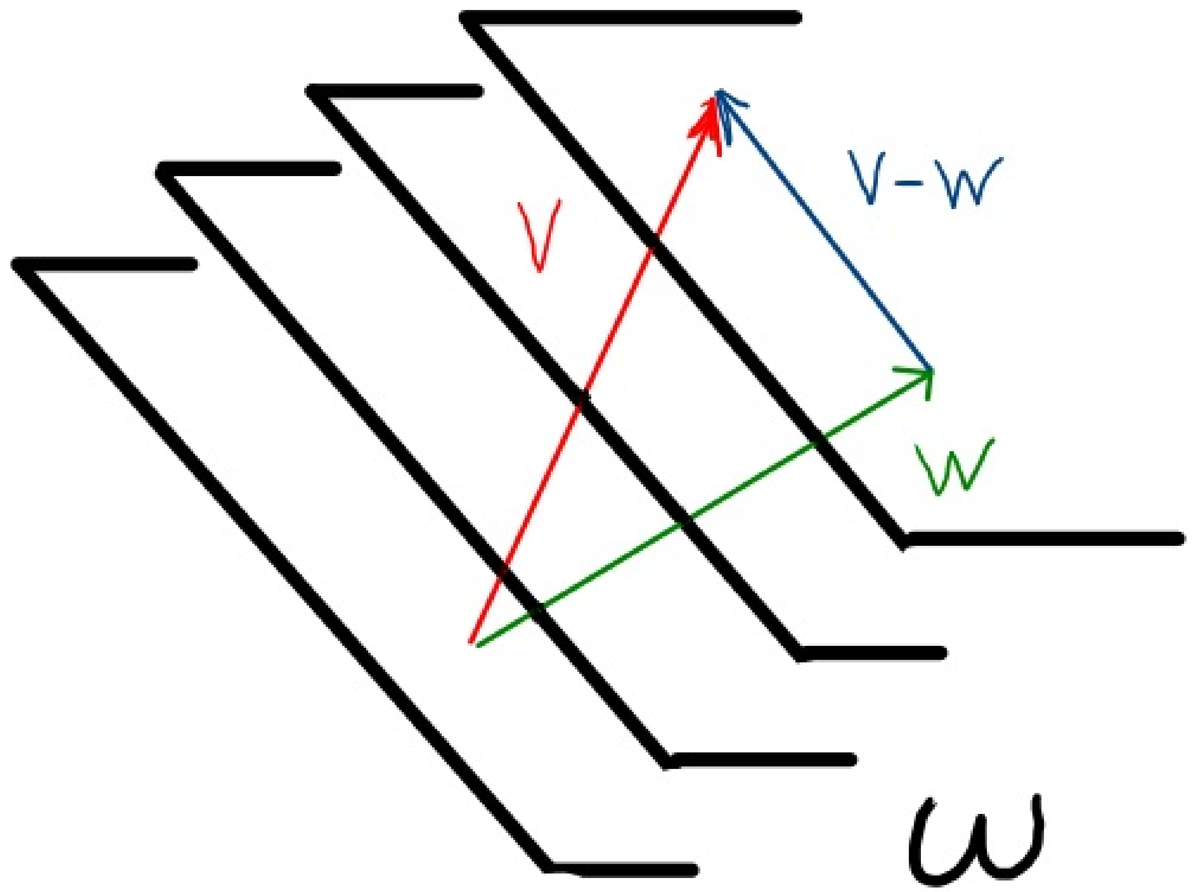


For all vectors:

$$\Delta f = V(f)\,\Delta t = \underbrace{df(V)}_{}\,\Delta t$$

$$\left(\begin{array}{l}\text{\# of pierced surfaces of constant } f \\ \text{by } V \text{ per unit parameter}\end{array}\right) \times \Delta t$$

* Vectors depicted as arrows in $T_p M$
* 1-forms depicted as $(n-1)$-dim hyperplanes in $T_p M$



* Vectors depicted as arrows in $T_p\mathcal{M}$
* 1-forms depicted as $(n-1)$-dim hyperplanes in $T_p\mathcal{M}$, s.t.

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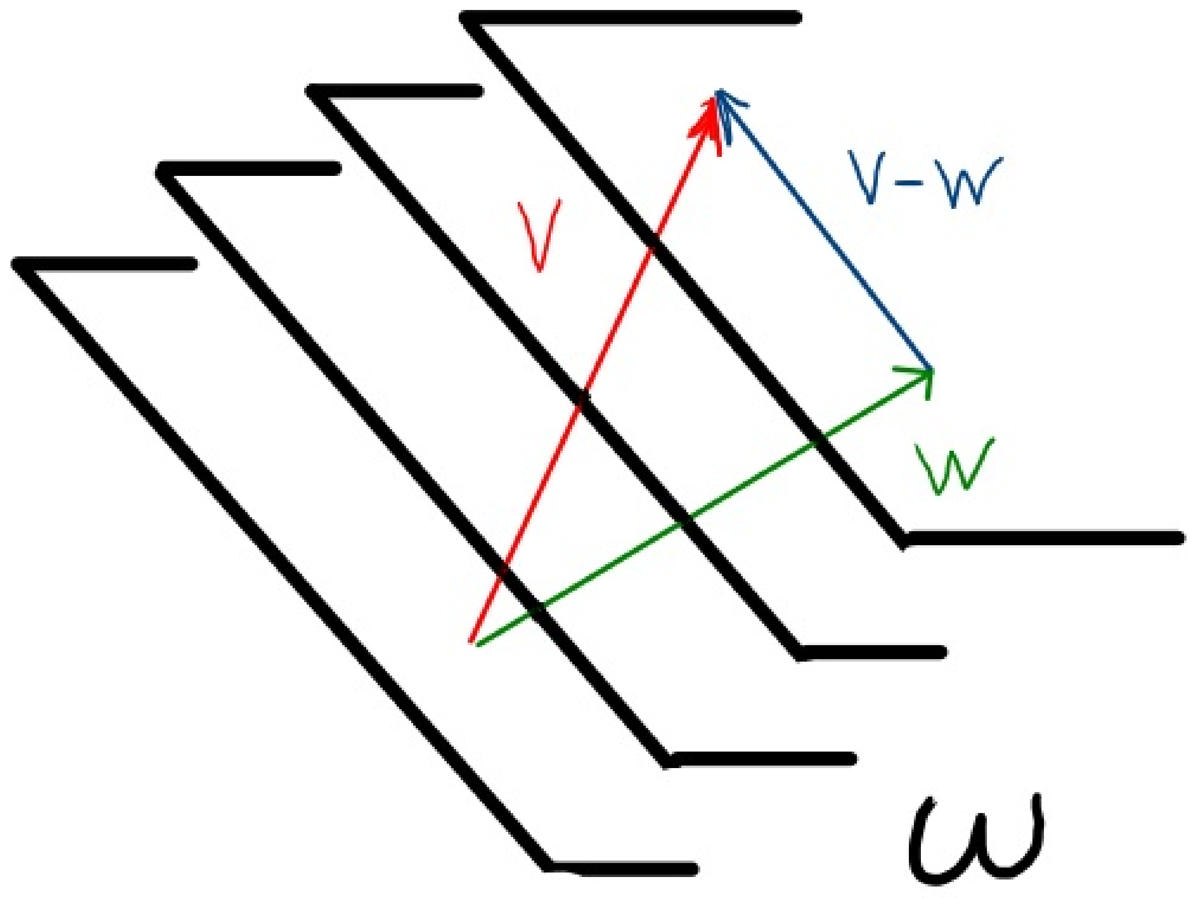

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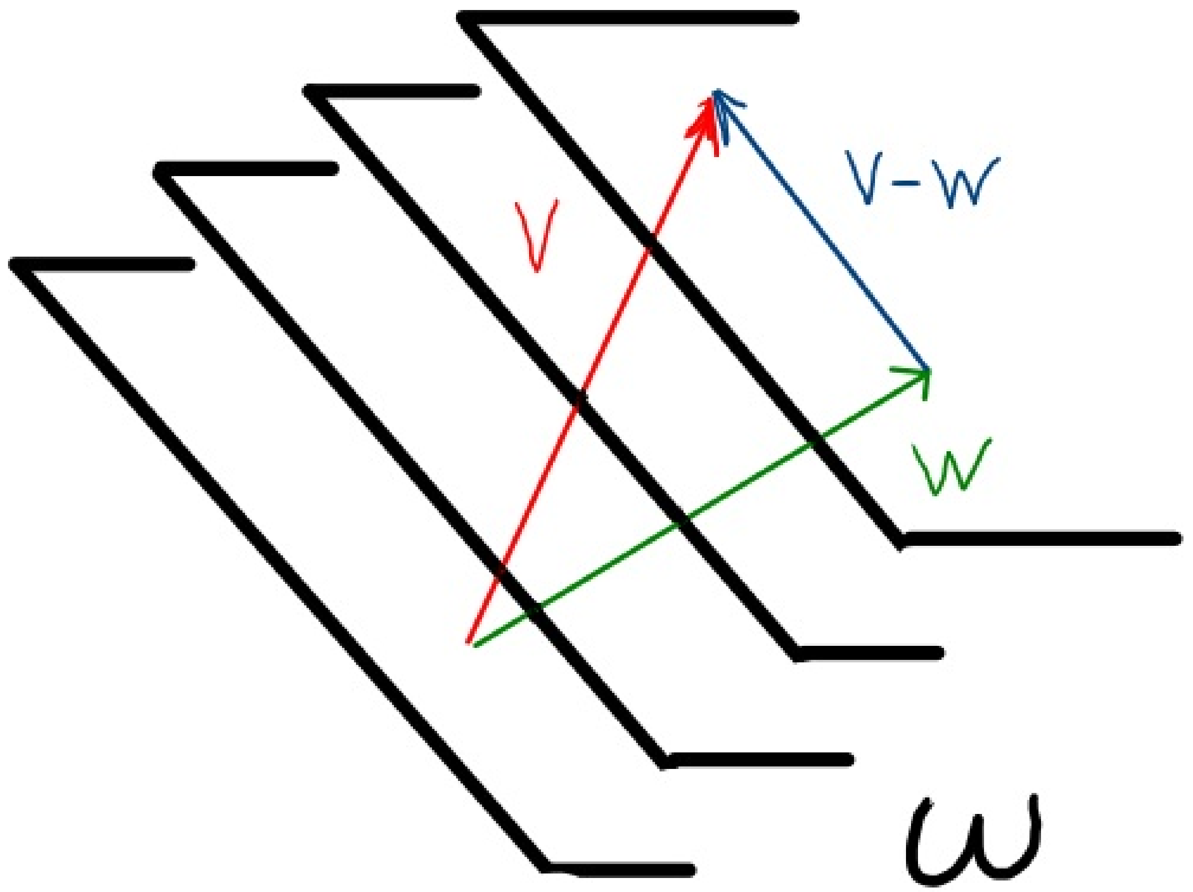

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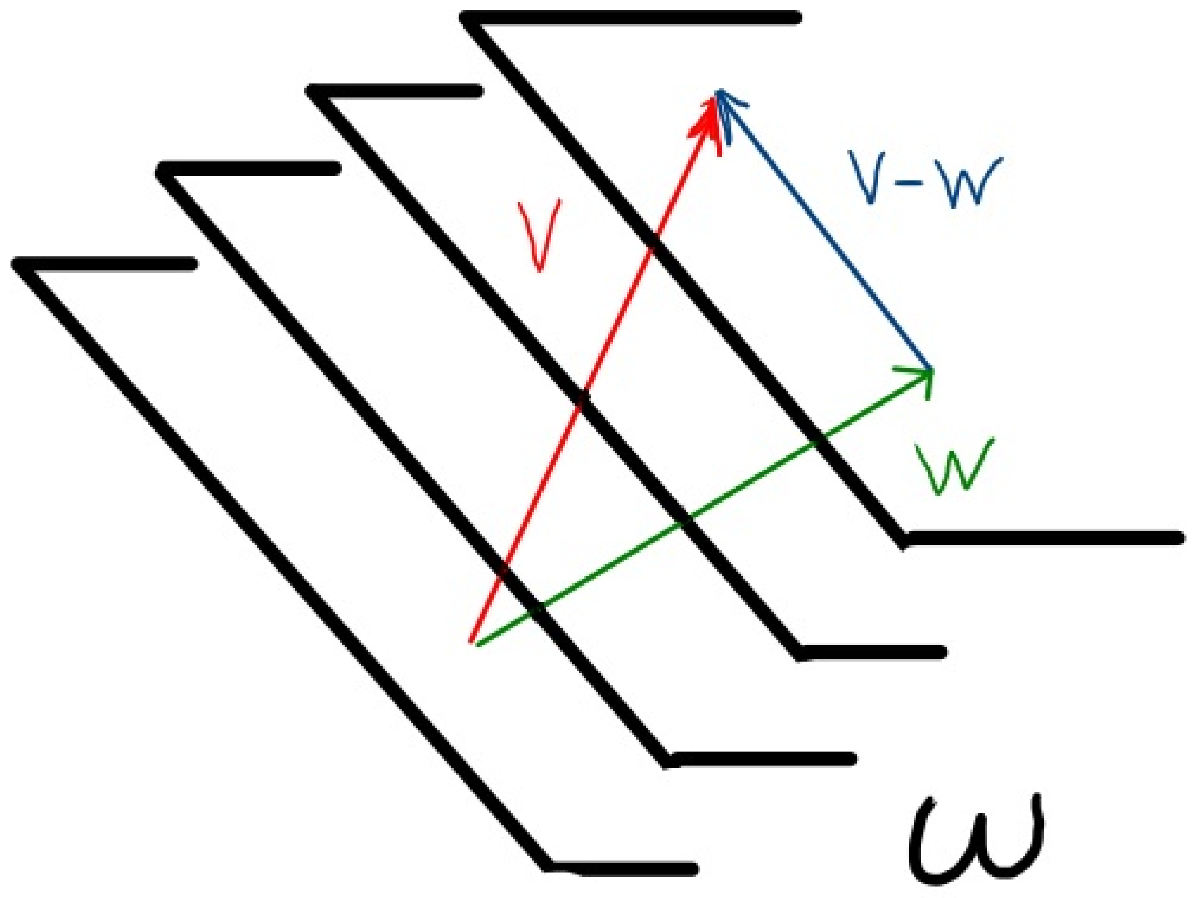

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* If $\omega(V) = \omega(W) \Leftrightarrow \omega(V-W) = 0 \Leftrightarrow V-W \parallel$ hyperplanes

means: "does not pierce"

* Vectors depicted as arrows in $T_p\mathcal{M}$

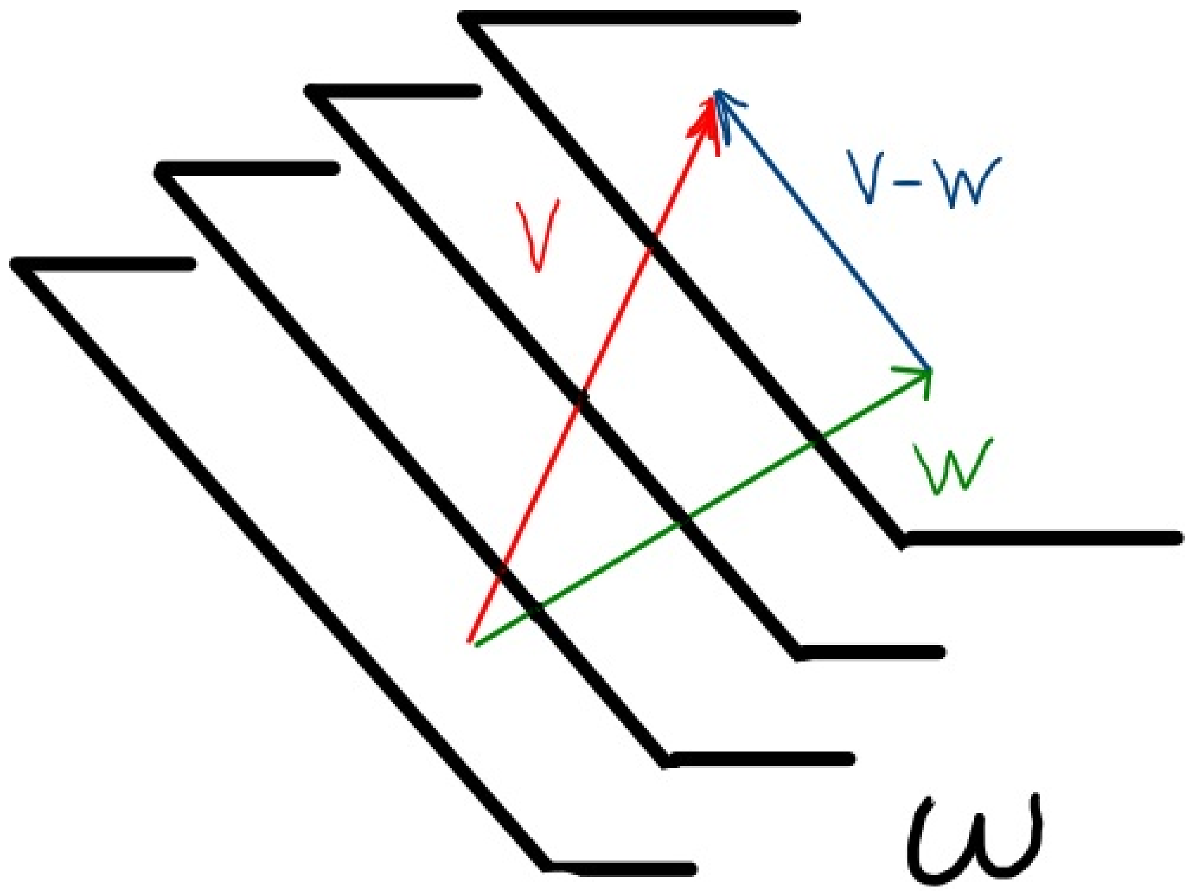
* 1-forms depicted as $(n-1)$-dim hyperplanes in $T_p\mathcal{M}$, s.t.

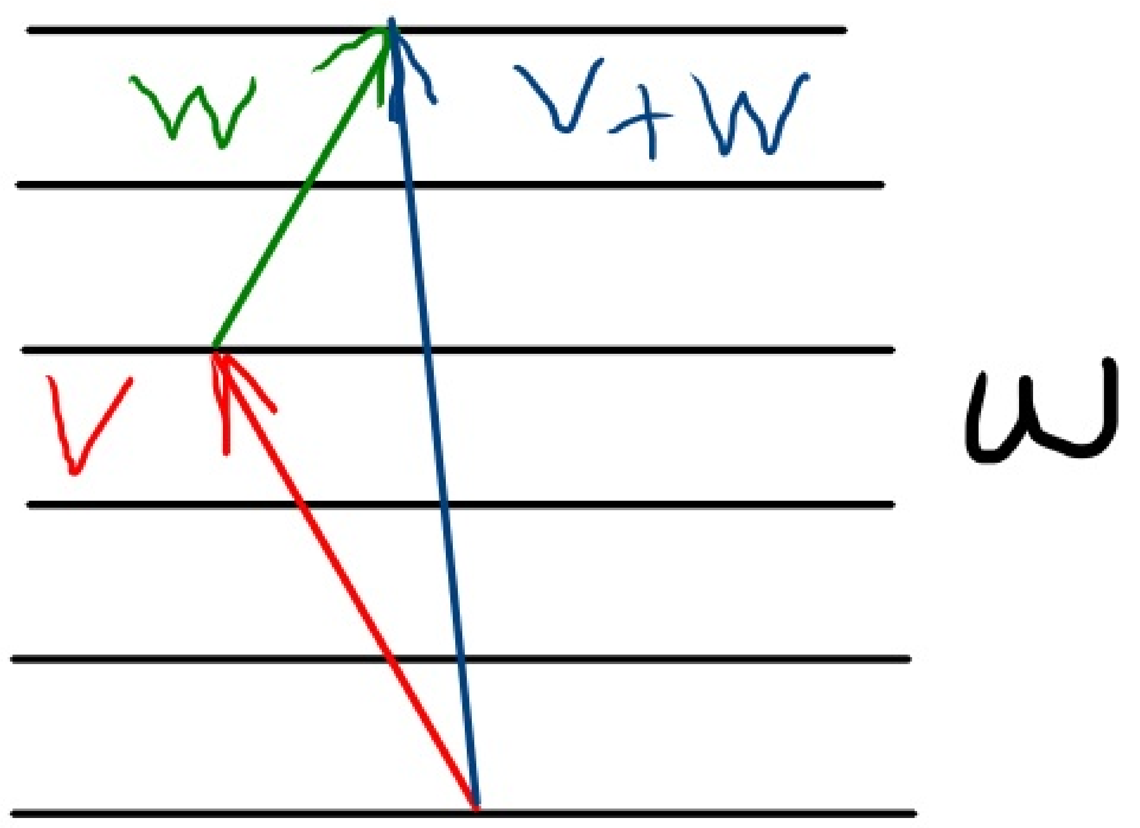
$$\omega(V) = \begin{pmatrix} \#\text{ of pierced hyperplanes} \\ \text{by } V \end{pmatrix}$$

* depicts linearity of action of ω

$$\omega(\alpha V) = \alpha\, \omega(V)$$

$$\omega(V+W) = \omega(V) + \omega(W)$$





One form fields

* An assignment of a 1-form $\forall\, p \in M$ in a smooth way

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$$df(V) = V(f) = V^\mu \partial_\mu f$$

- $\omega = \omega_\mu dx^\mu$ smooth $\Leftrightarrow$ ω_μ smooth functions

$$\underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-times}} \times \underbrace{T_pM \times \dots T_pM}_{l\text{-times}}$$

– a vector space with vectors

$$(\omega^{(1)}, \dots, \omega^{(k)}, V_{(1)}, \dots, V_{(l)})$$

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– a vector space with vectors

$$\left(\omega^{(1)}, \; \dots, \omega^{(k)} \;, \; V_{(1)}, \dots, V_{(l)} \right)$$

– its dimension is

$$n^k \cdot n^l = n^{k+l}$$

$$T : \underbrace{T_p^* M \times \dots \times T_p^* M}_{k\text{-times}} \times \underbrace{T_p M \times \dots T_p M}_{l\text{-times}} \longrightarrow \mathbb{R}$$

If T is a linear map $\Rightarrow$ T is a tensor at p

of $\begin{pmatrix} \text{type} \\ \text{rank} \\ \text{order} \\ \text{valence} \end{pmatrix}$ (k, l)

degree

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If T is a linear map $\Rightarrow$ T is a (k,l) tensor at p

- (k,l) tensors form the vector space $T_p^{(k,l)}M$

vectors are type $(1,0)$

1-forms $(0,1)$

Example: $(1,1)$ tensor

$$T : T_p^* M \times T_p M \to \mathbb{R}$$

$$(\omega, V) \mapsto T(\omega; V) \in \mathbb{R}$$

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$$T(\omega; \alpha V + \beta W) = \alpha T(\omega; V) + \beta T(\omega; W)$$

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$$T(\omega;V) = T^{\mu}{}_{\nu}\,\omega_\mu V^\nu \quad , \quad T^{\mu}{}_{\nu} \equiv T(dx^\mu;\partial_\nu)$$

For a (k,l) tensor:

$$T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = T(dx^{\mu_1},\ldots,dx^{\mu_k};\partial_{\nu_1},\ldots,\partial_{\nu_l})$$

and

$$T(\omega,\ldots;V,\ldots) = T^{\mu_1\ldots}{}_{\nu_1\ldots}\,\omega_{\mu_1}\ldots\,V^{\nu_1}\ldots$$

Tensor Product

* used to construct higher rank tensors

S is of type (k_1, l_1)

T " " " (k_2, l_2)

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$S \otimes T$ " " " $(k_1 + k_2, l_1 + l_2)$, s.t.

$$S \otimes T(\omega^{(1)}, \dots, \omega^{(k_1)}, \sigma^{(1)}, \dots, \sigma^{(k_2)} ; V_{(1)}, \dots, V_{(l_1)}, W_{(1)}, \dots, W_{(l_2)}) =$$

$$= S(\omega^{(1)}, \dots, \omega^{(k_1)} ; V_{(1)}, \dots, V_{(l_1)}) \cdot T(\sigma^{(1)}, \dots, \sigma^{(k_2)} ; W_{(1)}, \dots, W_{(l_2)})$$

Note:
- $S \otimes T \neq T \otimes S$
- $S \otimes T$ a linear function of its arguments → prove!
- $(S \otimes T)^{\mu_1 \dots \mu_{k_1} \nu_1 \dots \nu_{k_2}}{}_{\lambda_1 \dots \lambda_{l_1} \rho_1 \dots \rho_{l_2}} =$

$$= S^{\mu_1 \dots \mu_{k_1}}{}_{\lambda_1 \dots \lambda_{l_1}} \cdot T^{\nu_1 \dots \nu_{k_2}}{}_{\rho_1 \dots \rho_{l_2}}$$

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$$= \underbrace{S^{\mu_1 \dots \mu_{k_1}}{}_{\lambda_1 \dots \lambda_{l_1}} \cdot T^{\nu_1 \dots \nu_{k_2}}{}_{\rho_1 \dots \rho_{l_2}}}_{\text{simple product of components}}$$

$\rightarrow$ also prove!

$$S \otimes T(\omega^{(1)}, \dots, \omega^{(k_1)}, \sigma^{(1)}, \dots, \sigma^{(k_2)}; V_{(1)}, \dots, V_{(l_1)}, W_{(1)}, \dots, W_{(l_2)}) =$$

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Indeed:

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$$\begin{aligned}
\omega_\lambda \sigma_\rho \, dx^\lambda \otimes dx^\rho (\partial_\mu, \partial_\nu) &= \omega_\lambda \omega_\rho \, dx^\lambda(\partial_\mu) \; dx^\rho(\partial_\nu) \\
&= \omega_\lambda \omega_\rho \, \delta^\lambda_{\underline{\mu}} \; \delta^\rho_{\underline{\nu}} \\
&= \omega_\mu \omega_\nu \\
&= \omega \otimes \sigma (\partial_\mu, \partial_\nu)
\end{aligned}$$

Now take any V, W:

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$$= \alpha\; \omega \otimes \sigma(V, W) + \beta\; \omega \otimes \sigma(U, W)$$

$\{ dx^\mu \otimes dx^\nu \}$ a coordinate basis of $T_p^{(0,2)} M$

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$$\Rightarrow \quad S = S_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

- $\lambda_{\mu\nu}\, dx^\mu \otimes dx^\nu = 0 \Rightarrow \lambda_{\mu\nu}\, dx^\mu \otimes dx^\nu (\partial_\rho, \partial_\sigma) = 0 \Rightarrow \lambda_{\rho\sigma} = 0$ ($dx^\mu \otimes dx^\nu$ linearly independent)

Change of coordinates: $\{dx^{\mu} \otimes dx^{\nu}\} \to \{dx^{\mu'} \otimes dx^{\nu'}\}$

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$$\Rightarrow S_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} S_{\mu'\nu'} \iff S_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} S_{\mu\nu}$$

Example: $T \in T_p^{(2,1)} M$

$T(\omega, \sigma ; V) \in \mathbb{R}$

$T^{\mu\nu}{}_{\rho} = T(dx^{\mu}, dx^{\nu} ; \partial_{\rho})$

$T = T^{\mu\nu}{}_{\rho} \, \partial_{\mu} \otimes \partial_{\nu} \otimes dx^{\rho}$

Example: $T \in T_p^{(2,1)} M$

$T(\omega, \sigma ; V) \in \mathbb{R}$

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$\{dx^{\mu}\}, \{\partial_{\mu}\} \rightarrow \{dx^{\mu'}\}, \{\partial_{\mu'}\}$

$T = T^{\mu'\nu'}{}_{\rho'} \, \partial_{\mu'} \otimes \partial_{\nu'} \otimes dx^{\rho'}$

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$\{dx^\mu\}, \{\partial_\mu\} \to \{dx^{\mu'}\}, \{\partial_{\mu'}\}$

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$$

$$\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

$$T = T^{\mu'\nu'}{}_{\rho'}\, \partial_{\mu'} \otimes \partial_{\nu'} \otimes dx^{\rho'}$$

$$= T^{\mu'\nu'}{}_{\rho'} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\rho'}}{\partial x^\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho$$

$$\Rightarrow T^{\mu\nu}{}_{\rho} = \frac{\partial x^\mu}{\partial x^{\mu'}} \cdot \frac{\partial x^\nu}{\partial x^{\nu'}} \cdot \frac{\partial x^{\rho'}}{\partial x^\rho}\, T^{\mu'}{}_{\nu'\rho'}$$

Example: $T \in T_p^{(2,1)} M \qquad R \in T_p^{(1,1)} M$

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad\qquad R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma$$

Example: $T \in T_p^{(2,1)} M \qquad R \in T_p^{(1,1)} M$

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma$$

- $T \otimes R$ is a $(2+1, 1+1) = (3,2)$ tensor
- $T \otimes R(\omega, \sigma, \rho; V, W) = T(\omega, \sigma; V)\, R(\rho; W)$
- $T \otimes R = T^{\mu\nu}{}_{\rho} R^{\lambda}{}_{\sigma}\, \underset{\omega}{\partial_\mu} \otimes \underset{\sigma}{\partial_\nu} \otimes \underset{V}{dx^\rho} \otimes \underset{\rho}{\partial_\lambda} \otimes \underset{W}{dx^\sigma}$ Not canonical positions!

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$$= T^{\mu\nu}{}_{\rho} R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\sigma \otimes \partial_\lambda \otimes dx^\rho \otimes dx^\sigma$$

$(\omega, \sigma, \rho; V, W)$ canonical order

Contractions

Example: $T = T^{\mu\nu}{}_{\rho}\ \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

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Define the $(1,0)$ tensor:

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Contractions

Example: $T = T^{\mu\nu}{}_{\rho}\ \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

Define the $(1,0)$ tensor:

$T(\ldots, dx^\lambda; \partial_\lambda)$ → Independent of choice of basis Same if $T(\ldots, dx^{\lambda'}; \partial_{\lambda'})$ – Prove!

empty slot for 1-form

Contractions

Example: $T = T^{\mu\nu}{}_{\rho}\ \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

Define the (1,0) tensor:

$$T(\ldots, dx^\lambda ; \partial_\lambda) = T^{\mu\nu}{}_{\rho}\ \partial_\nu(dx^\lambda)\, dx^\rho(\partial_\lambda)\ \ \partial_\mu$$

Contractions

Example: $T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

Define the (1,0) tensor:

$$T(\ldots, dx^\lambda ; \partial_\lambda) = T^{\mu\nu}{}_{\rho}\, \partial_\nu(dx^\lambda)\, dx^\rho(\partial_\lambda)\, \partial_\mu$$

$$= T^{\mu\nu}{}_{\rho}\, \delta_\nu{}^{\lambda}\, \delta^{\rho}{}_{\lambda}\, \partial_\mu$$

$$= T^{\mu\lambda}{}_{\lambda}\, \partial_\mu$$

ν, ρ indices have been contracted!

Contractions

Contracting two indices xfms a $(k,\ell) \to (k-1,\ell-1)$ tensor with components

$$T^{\mu_1 \ldots \lambda \ldots \mu_k}{}_{\nu_1 \ldots \lambda \ldots \nu_\ell}$$

Contractions

Contracting two indices between two tensors in $\otimes$

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad (2,1) \text{ tensor}$$

$$R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma \qquad (1,1) \text{ tensor}$$

$$T \otimes R = T^{\mu\nu}{}_{\rho} R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\nu \otimes \partial_\lambda \otimes dx^\rho \otimes dx^\sigma \qquad (3,2) \text{ tensor}$$

contract ρ, λ

$$S = T^{\mu\nu}{}_{\lambda} R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\nu \otimes dx^\sigma \qquad (2,1) \text{ tensor}$$

Symmetries of Tensors

$g_{\mu\nu} = g_{\nu\mu}$ totally symmetric

$A_{\mu\nu\rho} = A_{\nu\mu\rho}$ symmetric in its first two indices

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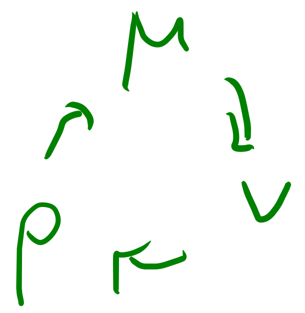
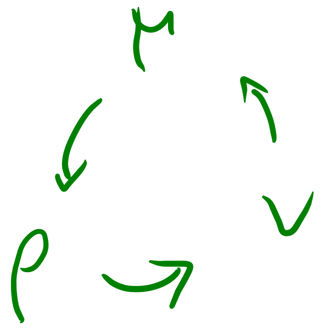
$$\begin{aligned} A_{\underline{\mu}\nu\rho} &= A_{\rho\underline{\mu}\nu} \\ &= A_{\nu\rho\underline{\mu}} \\ &= A_{\underline{\mu}\rho\nu} \\ &= A_{\nu\underline{\mu}\rho} \\ &= A_{\rho\nu\underline{\mu}} \end{aligned}$$





totally symmetric

$3! = 6$ permutations of indices

Symmetries of Tensors

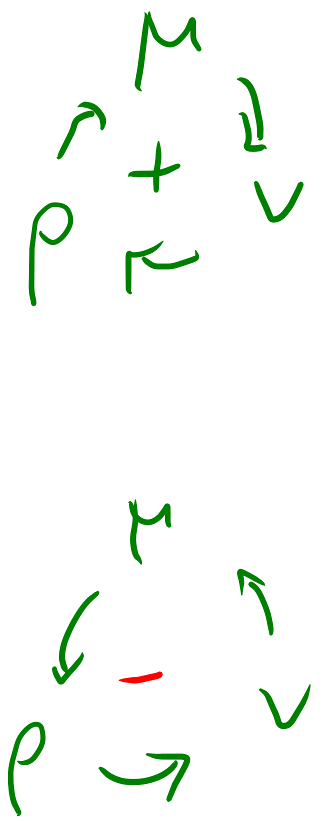
$$g_{\mu\nu} = - g_{\nu\mu}$$

totally anti symmetric

$$A_{\mu\nu\rho} = - A_{\nu\mu\rho}$$

anti symmetric in its first two indices

$$\begin{aligned} A_{\mu\nu\rho} = &+ A_{\rho\mu\nu} \\ &+ A_{\nu\rho\mu} \\ &- A_{\mu\rho\nu} \\ &- A_{\nu\mu\rho} \\ &- A_{\rho\nu\mu} \end{aligned}$$



totally anti symmetric

$3! = 6$ permutations of indices

Symmetrization

$$g_{(\mu\nu)} = \frac{1}{2}\left(g_{\mu\nu} + g_{\nu\mu} \right)$$

$$A_{(\mu\nu)\rho} = \frac{1}{2!}\left(A_{\mu\nu\rho} + A_{\nu\mu\rho} \right)$$

$$A_{(\mu\nu\rho)} = \frac{1}{3!}\left(A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} + A_{\mu\rho\nu} + A_{\nu\mu\rho} + A_{\rho\nu\mu} \right)$$

$$S_{(\mu_1 \mu_2 \cdots \mu_k)} = \frac{1}{k!} \sum_{\sigma} S_{\sigma(\mu_1)\sigma(\mu_2)\cdots\sigma(\mu_k)}$$

$$\sigma = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ \sigma(\mu_1) & \sigma(\mu_2) & \cdots & \sigma(\mu_k) \end{pmatrix}$$

1-1 map of k-integers

Antisymmetrization

$$g_{[\mu\nu]} = \frac{1}{2}\left(g_{\mu\nu} - g_{\nu\mu} \right)$$

$$A_{[\mu\nu]\rho} = \frac{1}{2!}\left(A_{\mu\nu\rho} - A_{\nu\mu\rho} \right)$$

$$A_{[\mu\nu\rho]} = \frac{1}{3!}\left(A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu} \right)$$

$$S_{[\mu_1\mu_2\cdots\mu_k]} = \frac{1}{k!}\sum_\sigma \mathrm{sign}(\sigma)\, S_{\sigma(\mu_1)\sigma(\mu_2)\cdots\sigma(\mu_k)}$$

$$\mathrm{sign}(\sigma) = (-1)^{(\#\text{ permutations})}$$

$$\sigma = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ \sigma(\mu_1) & \sigma(\mu_2) & \cdots & \sigma(\mu_k) \end{pmatrix}$$

1-1 map of k-integers

Antisymmetrization

$R^{\mu}{}_{[\nu\rho\lambda]}$

$C_{(\mu\nu|\rho|\lambda)}$

↳ only μ, ν, λ ρ excluded

$T^{(\mu\nu\rho)\lambda[\sigma\nu]}{}_{[\alpha|\beta\gamma|\delta](\varepsilon]})$

↳ β, γ excluded