- The Metric
- Causal Structure

The metric is an additional structure on a manifold

→ we measure distances using a metric

"geometry"

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- curvature = gravitation

- Line element: infinitesimal length
  - Euclidean in $\mathbb{R}^3$:

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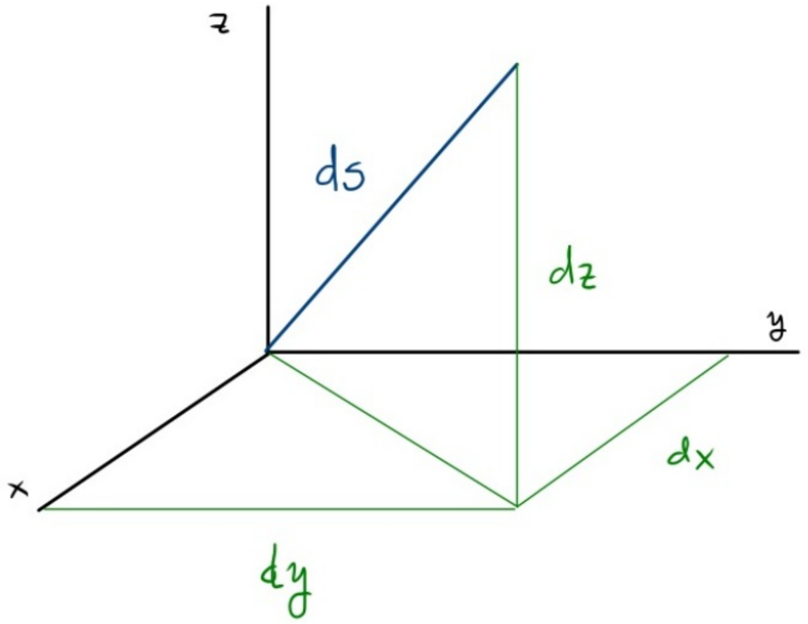

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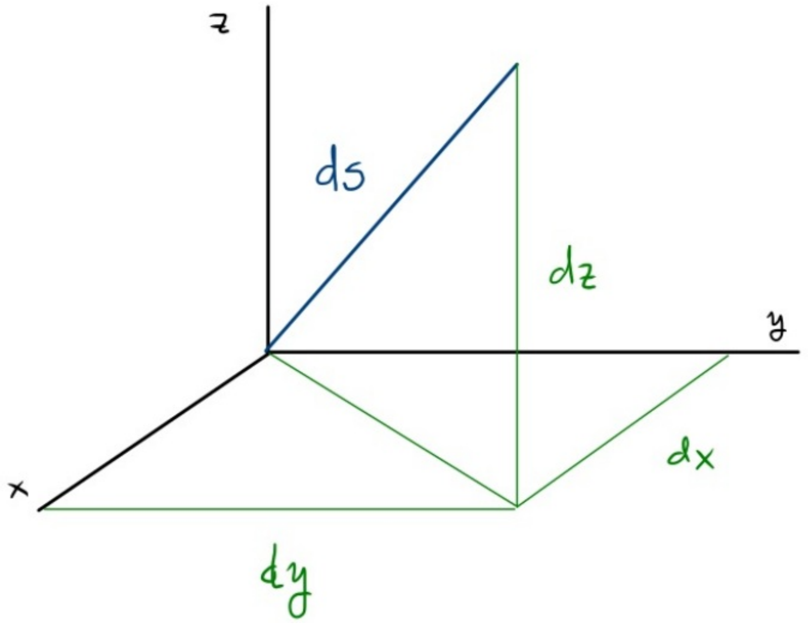

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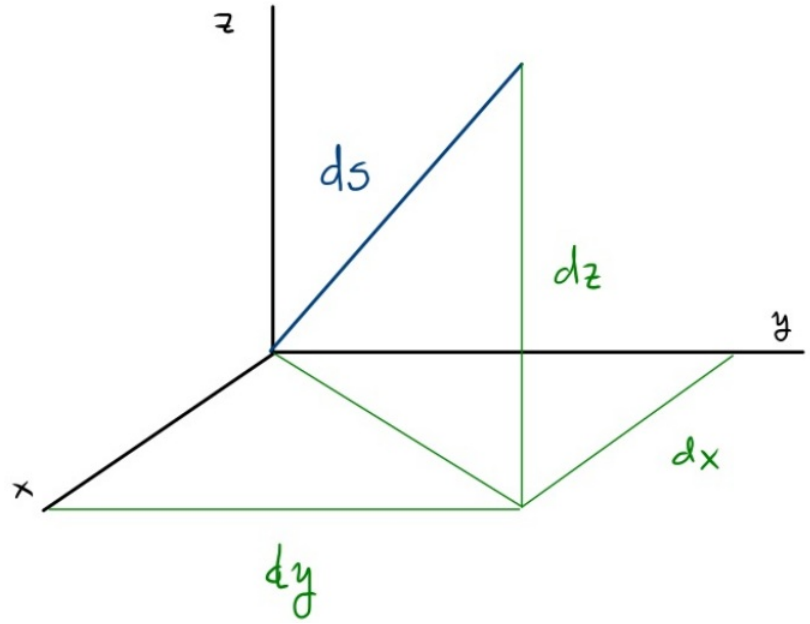

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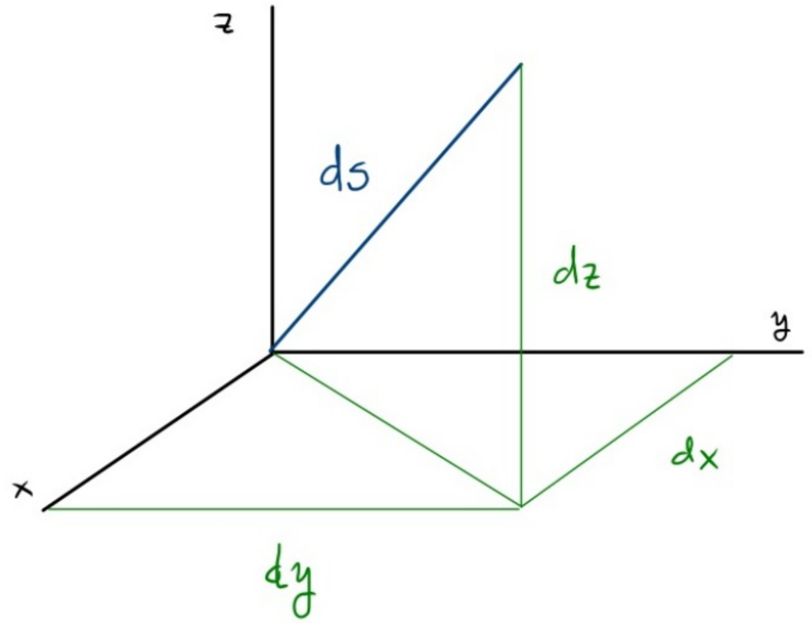

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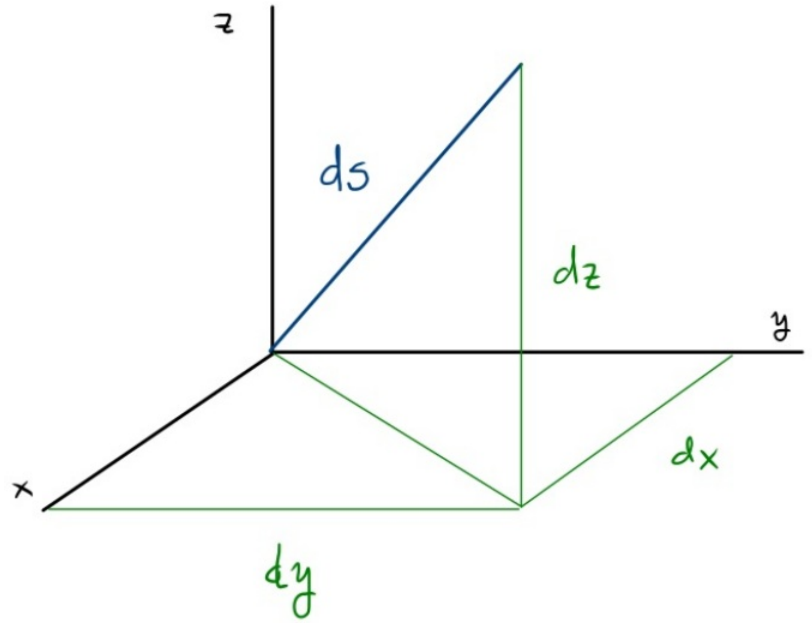

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all these expressions obtained by the usual rule:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

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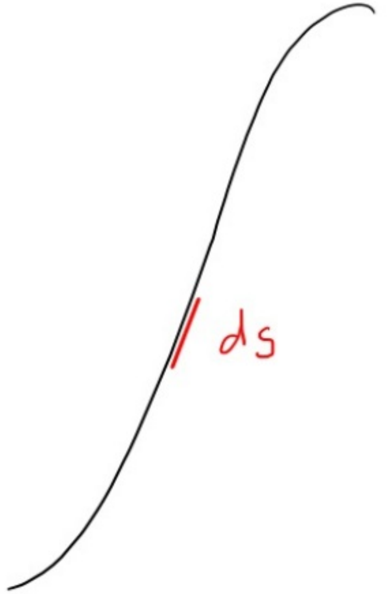

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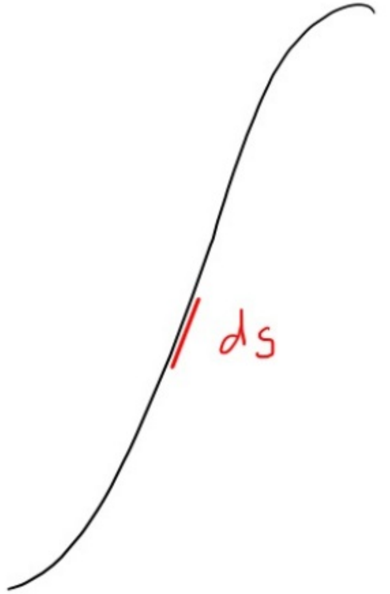

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    its norm

    a line integral on the curve $(x(t), y(t), z(t))$

    tangent vector $\left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt} \right)$

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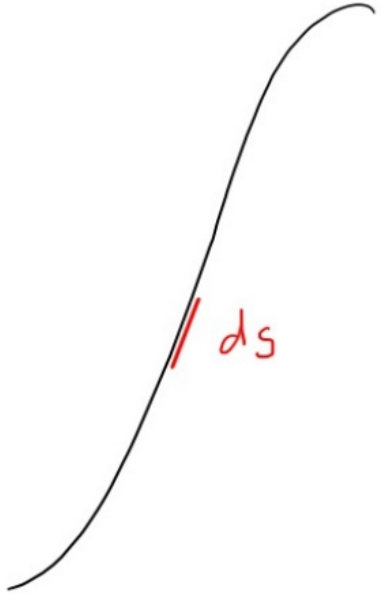

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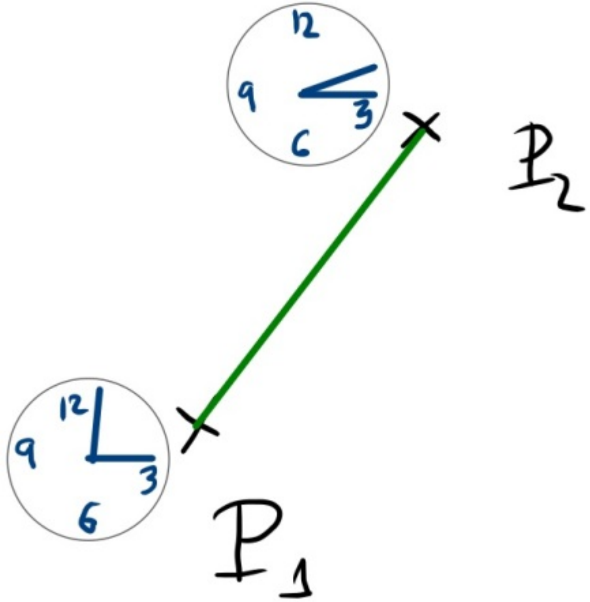
$$= \int \left(g_{\mu\nu}\, dx^\mu dx^\nu\right)^{1/2} = \int dt \left\{ g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right\}^{1/2}$$

↳ coordinate invariant

- Minkowski metric on $\mathbb{R}^4$:
  spacetime of events!



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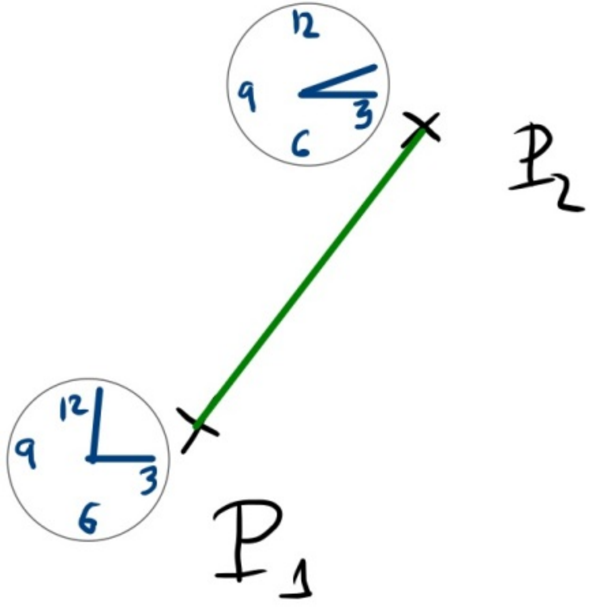
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$= (\text{spacetime distance of events})^2$$



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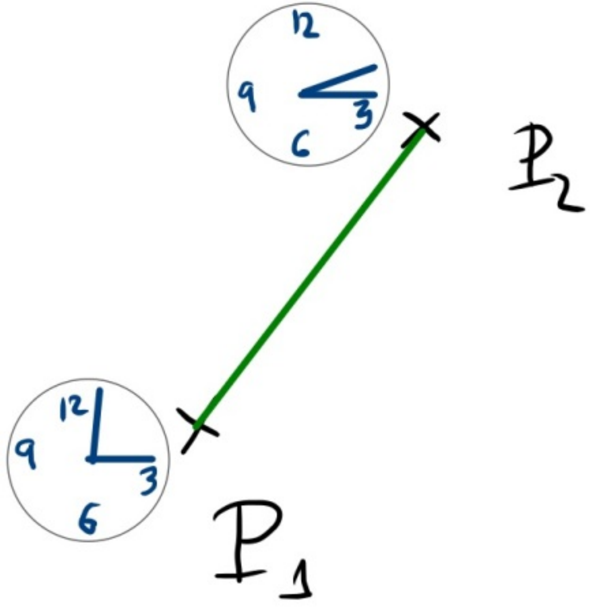

Worldline of observer: chooses coordinates w.r.t. she stays @ same place:

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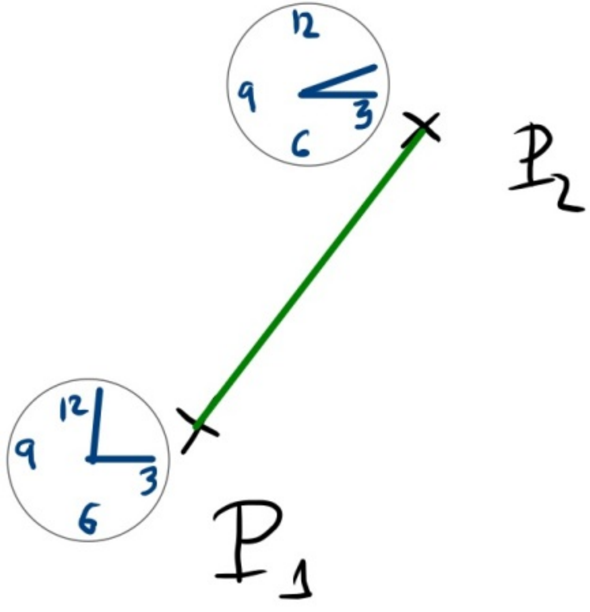

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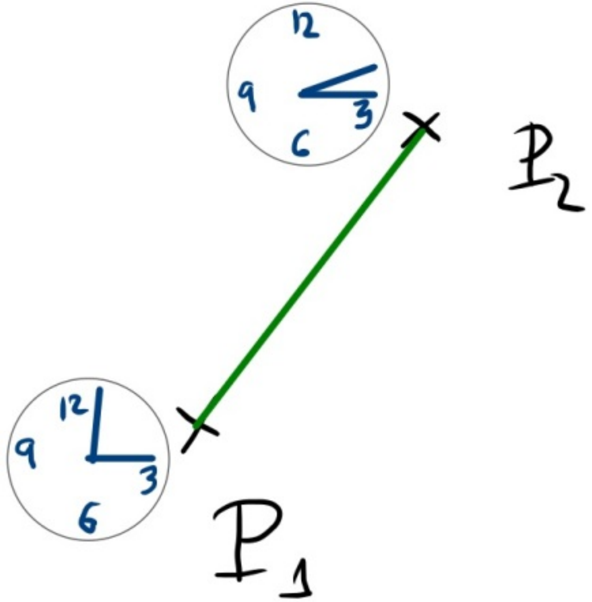

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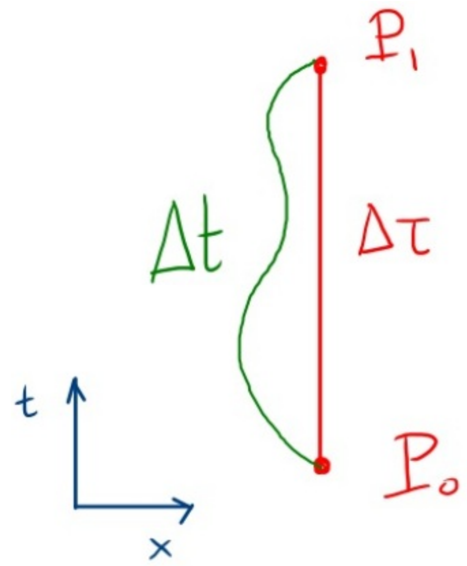
$d\tau$: $\begin{cases} \text{Her proper time} \\ \text{Her "biological" time} \\ \text{Her watch's time} \end{cases}$

- Proper time:
  - a geometric quantity, independent of $\begin{cases}\text{coordinates} \\ \text{other clocks}\end{cases}$
  - the longest proper time timelike curve connecting two nearby events is the straightest

these are the worldlines of free observers:

the lazier, the faster they age



$$\Delta t < \Delta\tau$$

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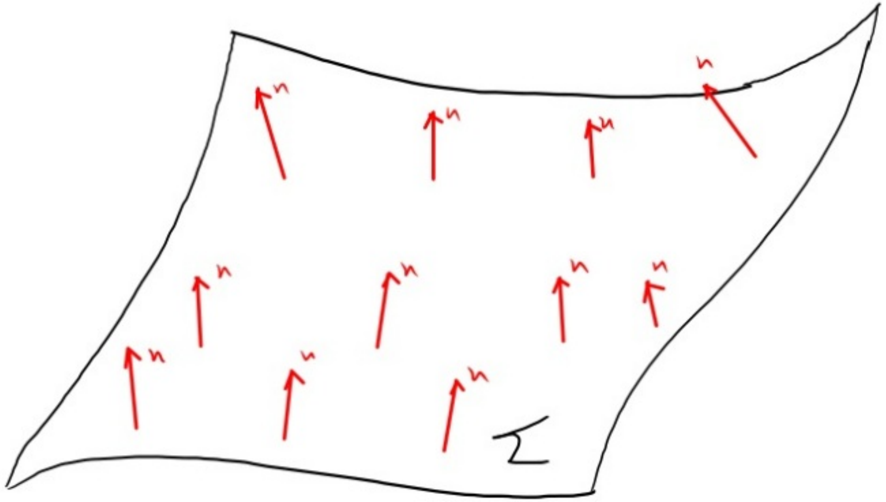
- Space:

The simultaneous event!

observer dependent!



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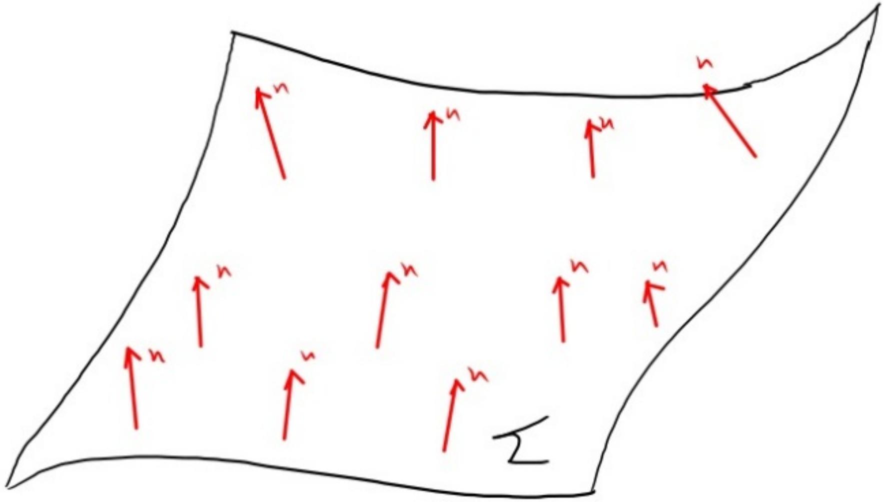

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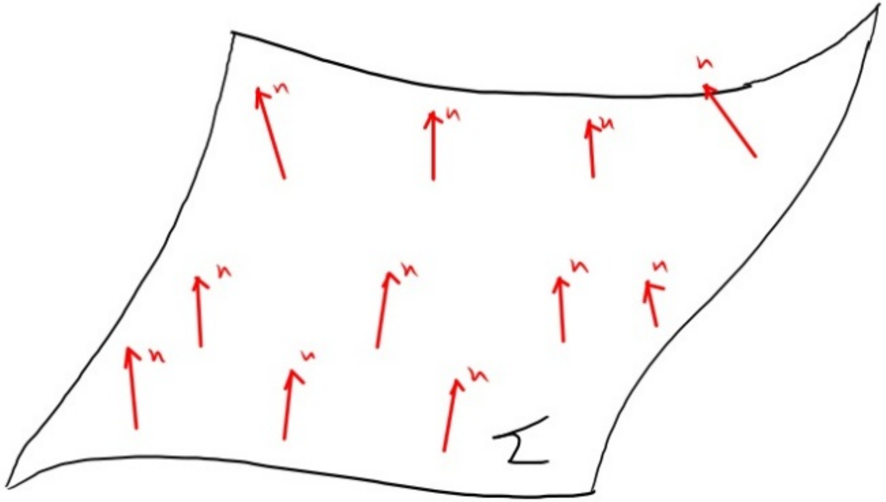

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spacelike surface: normal n^μ is timelike @ each point
all vectors tangent to surface spacelike

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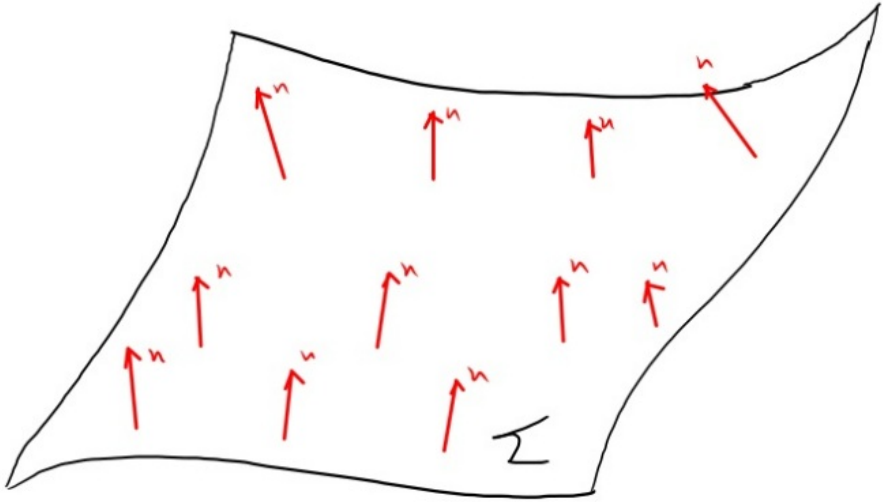

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no natural way to choose $\left\{ \begin{array}{c} \text{space} \\ \text{or} \\ \text{time} \end{array} \right\}$ globally

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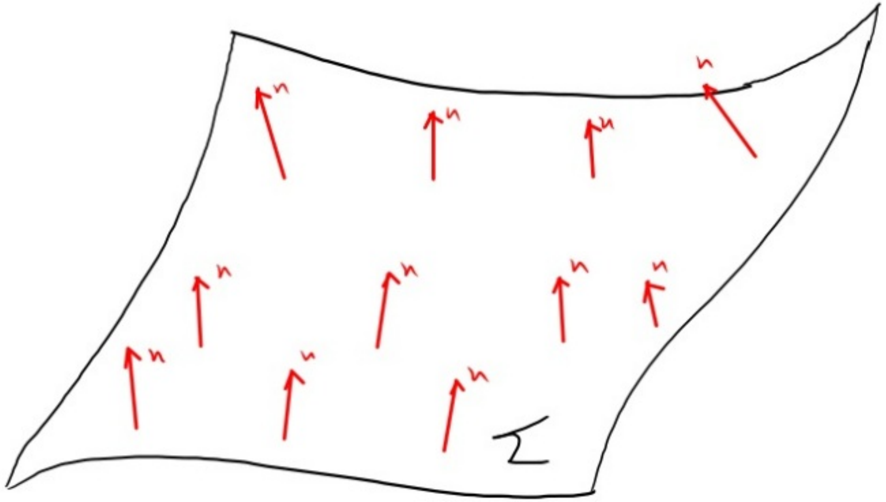

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- a metric is Riemannian when (iii) becomes

(iii)' $g_p(U,U) \geq 0 \quad \forall\, U \in T_pM$, and $g(U,U) = 0 \Leftrightarrow U = 0$

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- Nature makes a dynamical choice: In GR, solutions to Einstein equations

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(iii) $\Leftrightarrow$ the matrix $(g_{\mu\nu})$ is invertible $\Leftrightarrow \det g \neq 0$

↳ exercise (or watch video)

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we denote g^{-1} by $g^{\mu\nu}$, so that $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu}$

↳ also a symmetric tensor (prove)

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One form: maps linearly vectors to numbers

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but $\tilde{V}_\mu = \tilde{V}(\partial_\mu) = g(V, \partial_\mu) = g(V^\nu \partial_\nu, \partial_\mu)$

$$= V^\nu g(\partial_\nu, \partial_\mu)$$

$$= V^\nu g_{\nu\mu} = \underbrace{g_{\mu\nu}}_{\text{symmetric}} V^\nu$$

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  for simplicity $\tilde{V}_\mu$ is written as V_μ (index lowering)

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$$g(\tilde{\omega}, V) = \omega(V) \quad \forall V \in T_pM$$

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$$\Leftrightarrow g^{\mu\lambda} \omega_\mu = g^{\mu\lambda} g_{\nu\mu} \tilde{\omega}^\nu$$

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$$\Leftrightarrow g_{\nu\mu} \tilde{\omega}^\nu = \omega_\mu$$

$$\Leftrightarrow g^{\mu\lambda} \omega_\mu = g^{\mu\lambda} g_{\nu\mu} \tilde{\omega}^\nu = \delta^\lambda{}_\nu \tilde{\omega}^\nu = \tilde{\omega}^\lambda$$

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- If $V \in T_pM$, then $g(V, \cdot) \in T_p^*M$, $\tilde{V} = g(V, \cdot)$ $\tilde{V}_\mu = V_\mu = g_{\mu\nu} V^\nu$
- If $\omega \in T_p^*M$, we define $\tilde{\omega} \in T_pM$, s.t.

$$g(\tilde{\omega}, V) = \omega(V) \qquad \forall V \in T_pM$$

$$\Leftrightarrow \quad g_{\nu\mu} \tilde{\omega}^\nu V^\mu = \omega_\mu V^\mu \qquad \forall V^\mu$$

$$\Leftrightarrow \quad g_{\nu\mu} \tilde{\omega}^\nu = \omega_\mu$$

$$\Leftrightarrow \quad g^{\mu\lambda} \omega_\mu = g^{\mu\lambda} g_{\nu\mu} \tilde{\omega}^\nu = \delta^\lambda{}_\nu \tilde{\omega}^\nu = \tilde{\omega}^\lambda \Leftrightarrow \tilde{\omega}^\nu = g^{\nu\mu} \omega_\mu$$

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$$\tilde{g}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}$$ 2-index raising

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$$\tilde{g}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta} \text{ , then}$$

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(so, inverse matrix)

$$\tilde{g}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta}, \text{ then}$$

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$\tilde{\tilde{V}} = V$, $\tilde{\tilde{\omega}} = \omega$, etc i.e. duality

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- $(T, \tilde{T})$ does not depend on choice of basis or coordinates

($e^\alpha(e_\beta) = \delta^\alpha{}_\beta$ duality is basis-dependent)

Component xfm

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}$$

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- $\exists$ orthogonal matrix $O^{-1} = O^T$ s.t.

$$g_d = O^{-1} g O = O^T g O$$

with $g_d = \mathrm{diag}(g_0, g_1, \dots, g_{n-1}) = \begin{pmatrix} g_0 & 0 & \cdots & 0 \\ 0 & g_1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & g_{n-1} \end{pmatrix}$

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2. Choose a basis $e_{\mu'} = \Lambda^{\mu}{}_{\mu'} e_{\mu}$ for $\Lambda = OD$, with

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3. Compute $(g_{\mu'\nu'}) = \Lambda^T g \Lambda = \operatorname{diag}\left(\frac{g_0}{|g_0|}, \frac{g_1}{|g_1|}, \dots, \frac{g_{n-1}}{|g_{n-1}|} \right)$

$$= \operatorname{diag}(\pm 1, \pm 1, \dots, \pm 1)$$

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$$(g_{\mu'\nu'}) = \mathrm{diag}(-1, -1, \dots, -1, +1, +1, \dots, +1)$$

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$\rightsquigarrow$ the number of -1s is independent of basis: The signature of the metric

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Not unique: any other basis

$$\tilde{e}_{\mu} = \Lambda^{\mu'}{}_{\mu}\, e_{\mu'} \qquad \text{s.t.} \quad \Lambda^T \eta \Lambda = \eta$$

Lorentz xfms!

xfm orthonormal $\to$ orthonormal

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If s is the signature of the metric, and

$s=0$; the metric is Euclidean, $\Lambda \in O(n)$ orthogonal group

$$\Lambda\Lambda^T = \mathbb{1}$$

$$\eta = \mathbb{1}_{n\times n} \qquad \eta = \Lambda^T \eta \Lambda \iff \mathbb{1} = \Lambda^T \mathbb{1} \Lambda = \Lambda^T \Lambda$$

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Lorentz xfm

Lorentz group

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$s=1$: the metric has Minkowski signature, $\Lambda \in O(1, n-1)$
the manifold is pseudo-Riemannian

- Orthonormal basis fields are not coordinate bases

unless M is flat

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coordinate bases may consist of orthogonal vectors, but not of constant length

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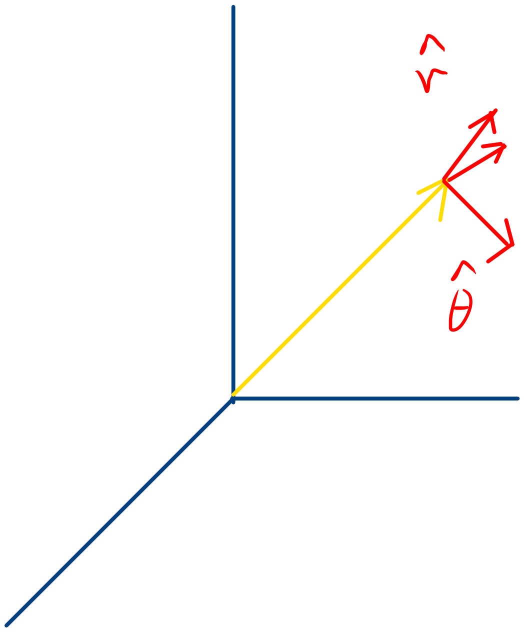
coordinate bases may consist of orthogonal vectors, but not of constant length

e.g. $(\hat{r}, \hat{\theta}, \hat{\varphi})$ orthonormal

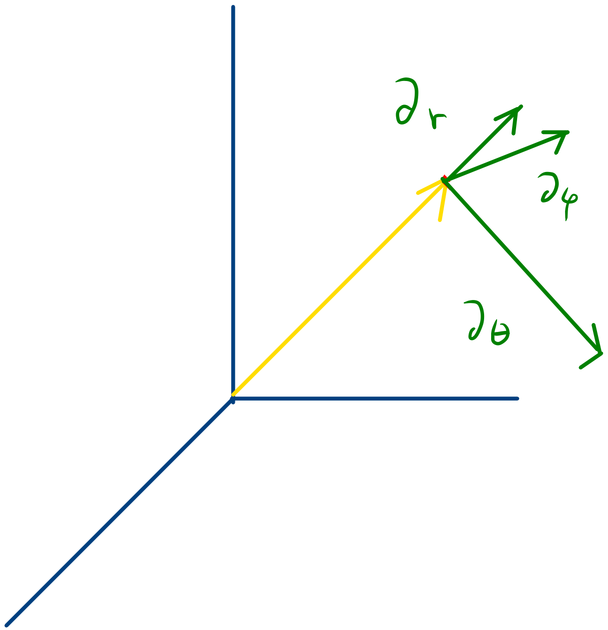


- Orthonormal basis fields are not coordinate bases

coordinate bases may consist of orthogonal vectors, but not of constant length

e.g. $(\hat{r}, \hat{\theta}, \hat{\varphi})$ orthonormal

$(\partial_r, \partial_\theta, \partial_\varphi)$ are not!



$g(\partial_r, \partial_r) = 1 \quad g(\partial_\theta, \partial_\theta) = r^2 \quad g(\partial_\varphi, \partial_\varphi) = r^2 \sin^2\theta$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

- Orthonormal basis fields are not coordinate bases

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- We can always choose a coordinate system s.t. at one point P:

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but $\partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \neq 0$

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such coordinate basis is orthonormal at P and defines a local Lorentz frame

locally inertial coordinates

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such coordinate basis is orthonormal at P and defines a local Lorentz frame (see proof in Carroll's book)

- effects of curvature negligible in a small enough lab

- In a locally inertial frame gravity is negligible
  - curvature effects may be ignored

  → physics is simple!

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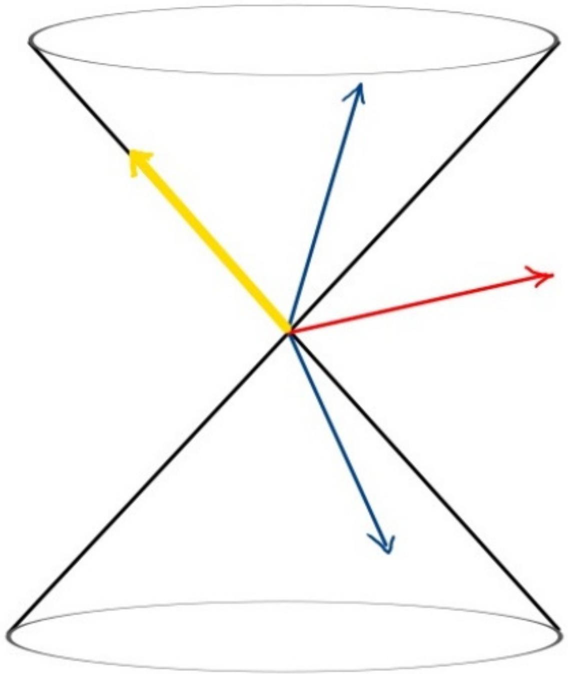

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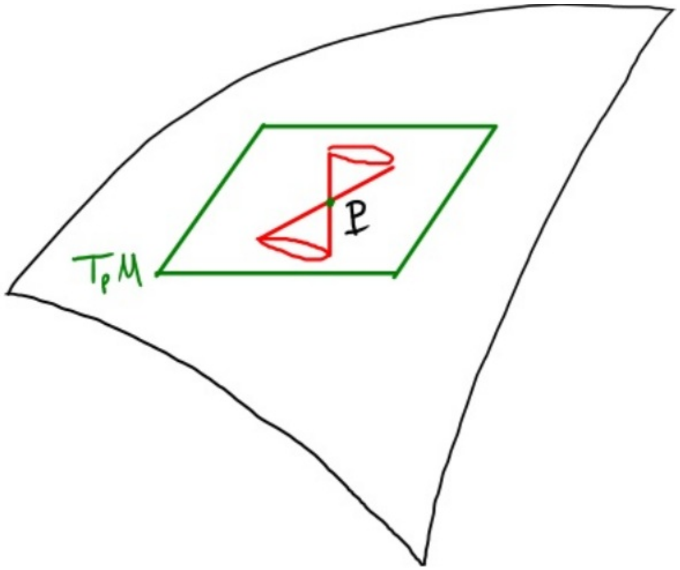

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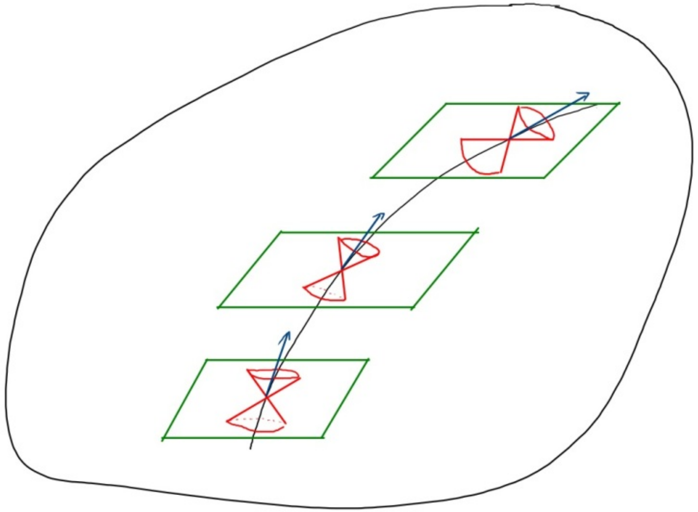

massive particles move on worldlines w/ tangents everywhere timelike

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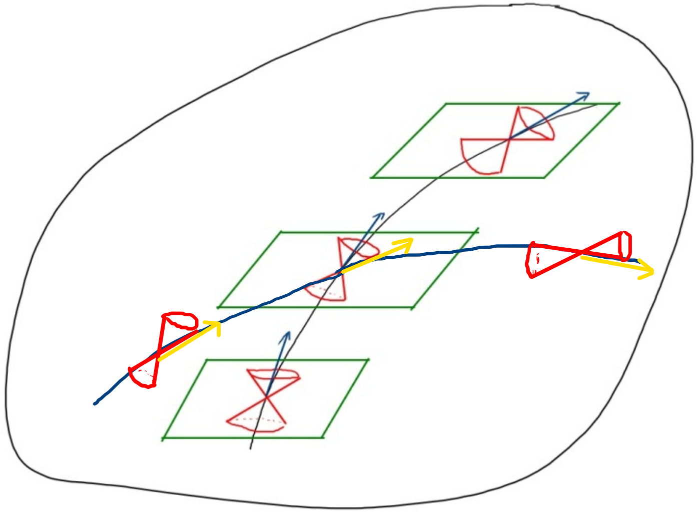

massless particles move on worldlines w/ tangents everywhere null

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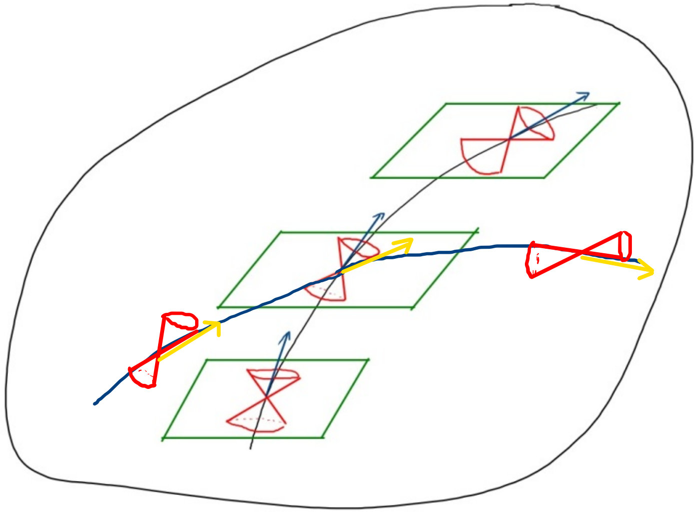
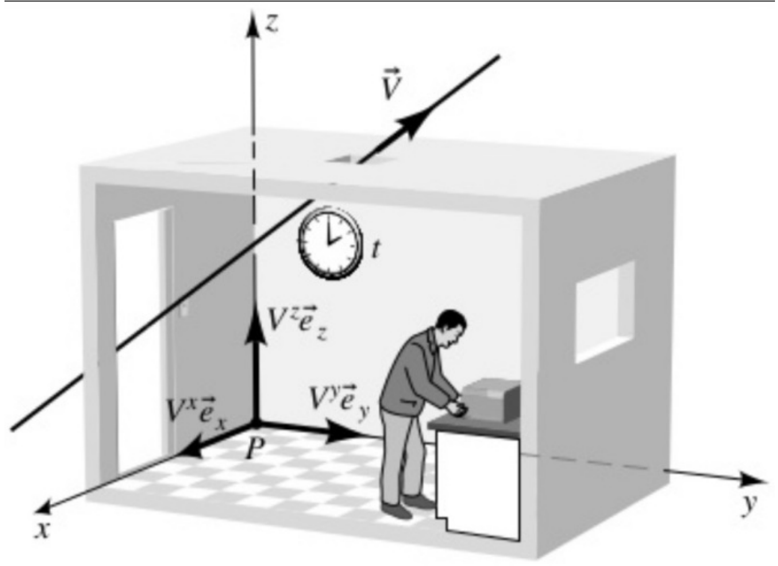


There are no worldlines that change category

Local Frames

Orthonormal bases define "observers"



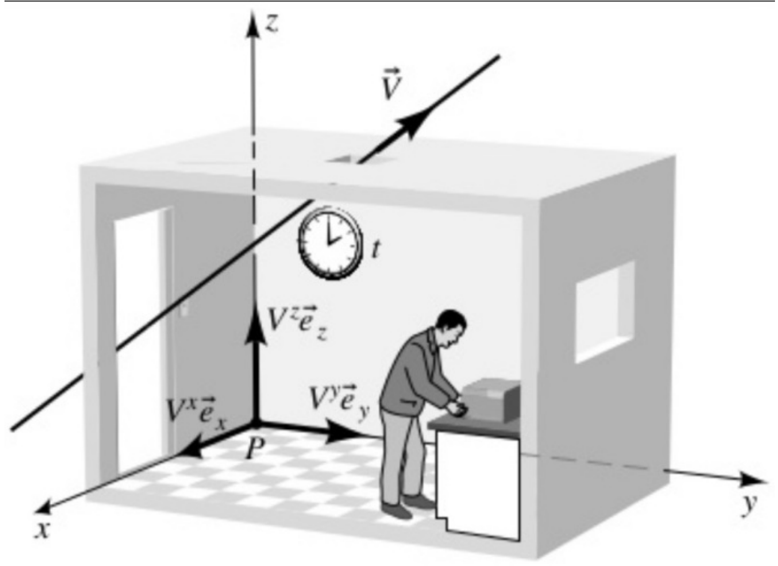
Hartle, Fig 7.6

Local Frames

Orthonormal bases define "observers"

4-velocity of observer $u = e_0$

local cartesian axes $\{e_1, e_2, e_3\}$



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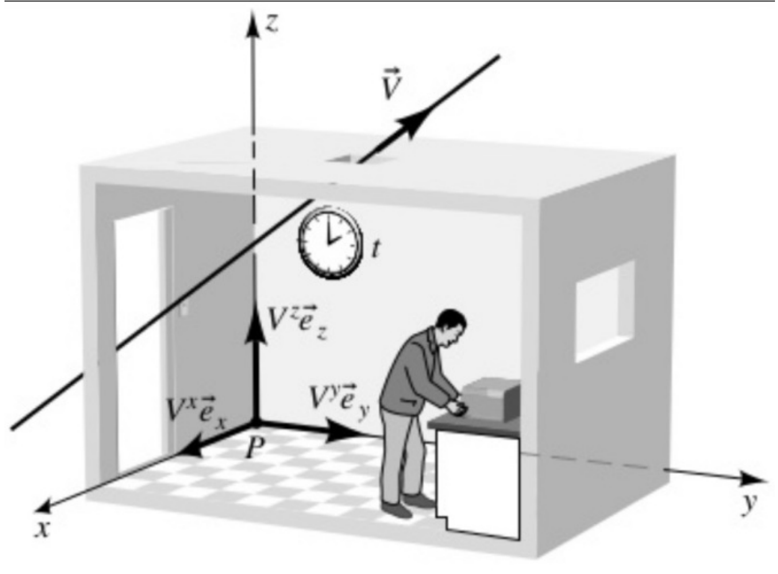

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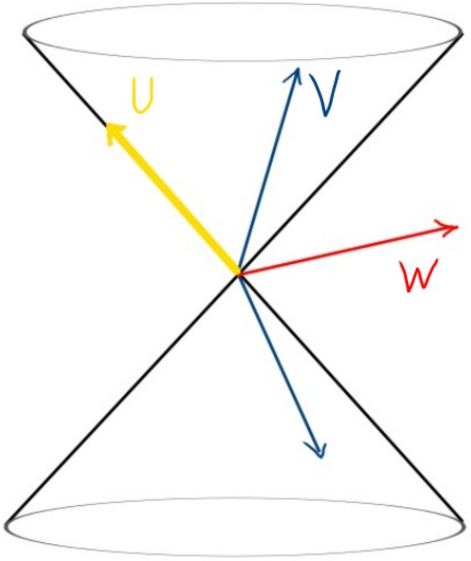

$\{e_0, e_1, e_2, e_3\}$ define the local lightcone:

$g(V,V) < 0$ timelike

$g(U,U) = 0$ light like

$g(W,W) > 0$ spacelike

Local Frames

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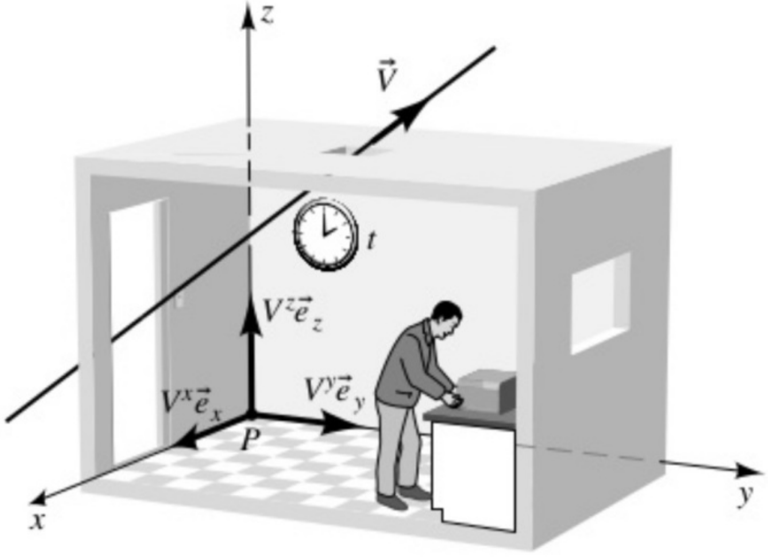
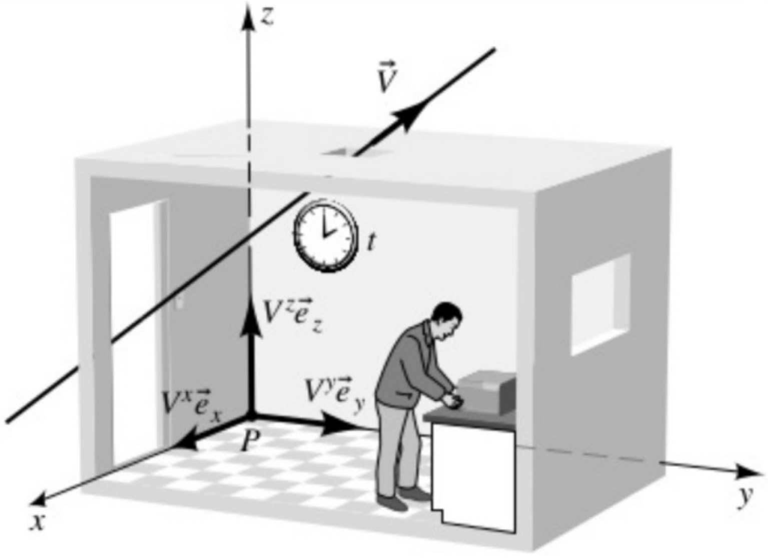


Hartle, Fig 7.6

4-velocity of observer $u = e_0$

$u = (1,0,0,0)$ in $\{e_0, e_1, e_2, e_3\}$ basis

Local Frames



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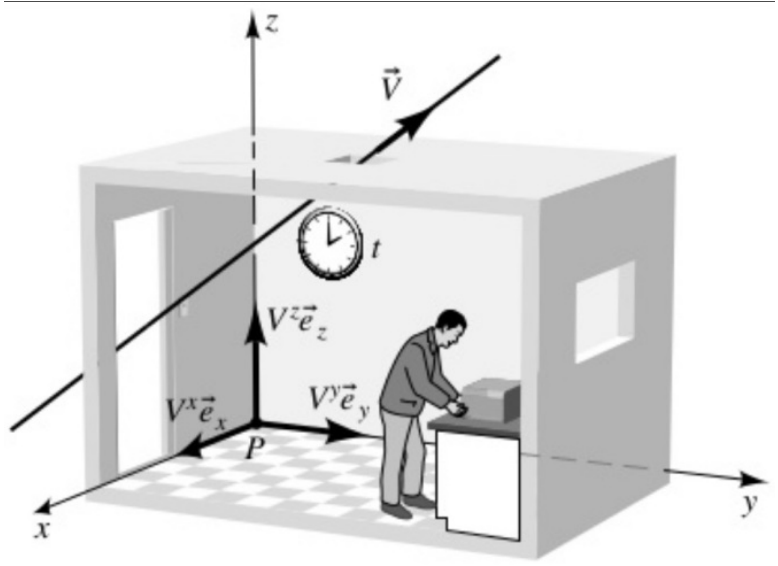
Local Frames



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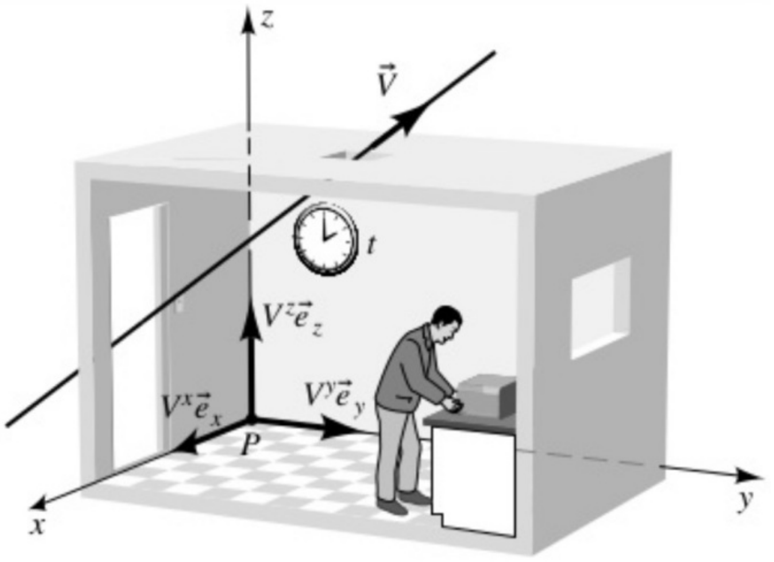

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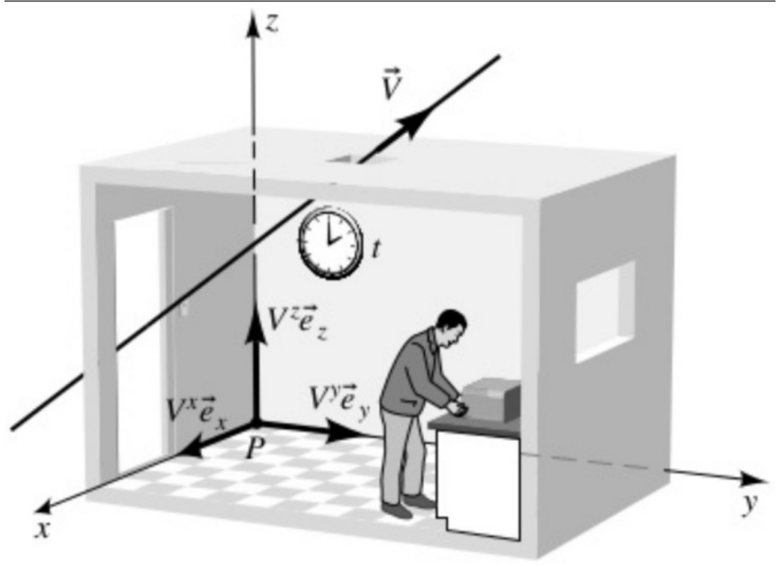

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coordinate invariant formula

relative speed of particle in frame

!! a LOCAL concept only !!

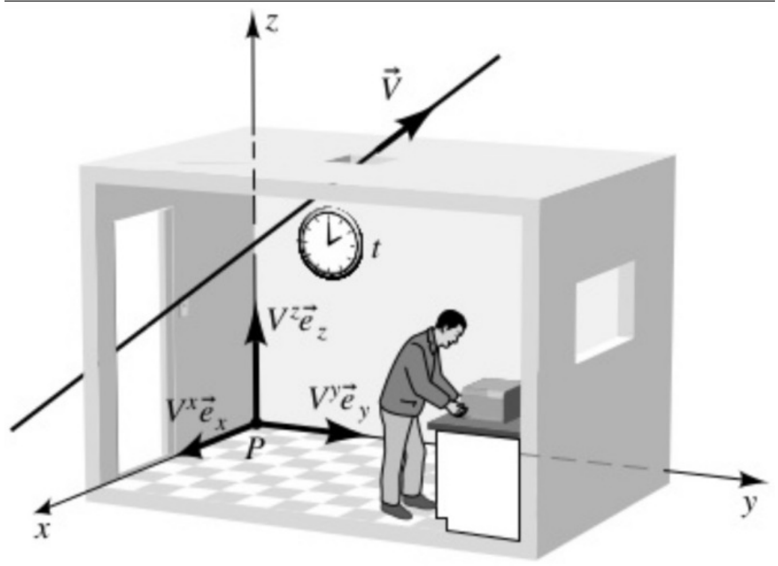
Local Frames

Orthonormal bases define "observers"

4-velocity of observer $\quad u = e_0$

Four-momentum of particle:

$p^\mu = (E, p^1, p^2, p^3)$



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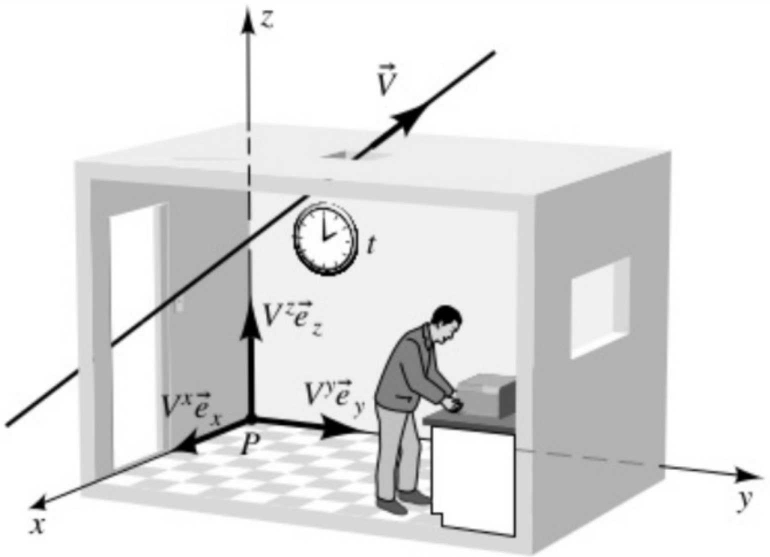
Four-momentum of particle:

$$\left.\begin{aligned} p^\mu &= (E, p^1, p^2, p^3) \\ u^\mu &= (1, 0, 0, 0) \end{aligned}\right\} \Rightarrow p_\mu u^\mu = -E$$



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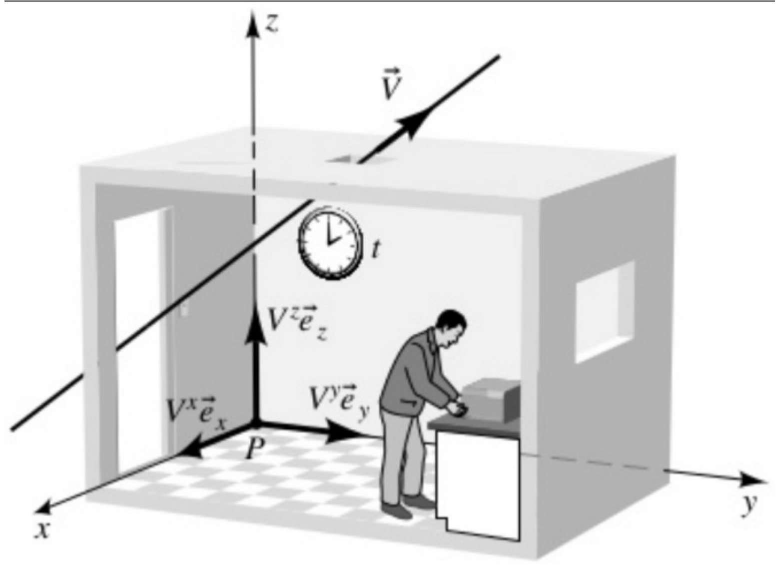

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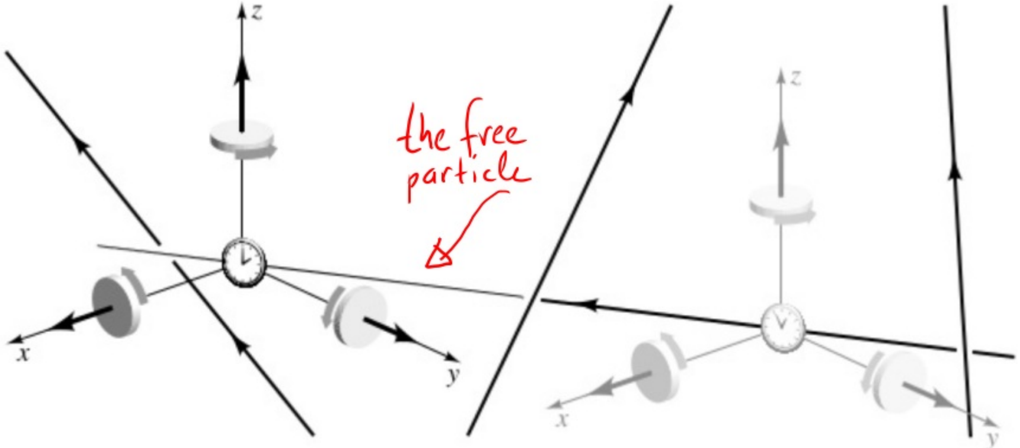
- a coordinate independent expression
- E, v defined for local observers, make no sense for distant particles

Local Inertial Observers

Observers observing free particles moving on straight lines @ constant rate are inertial observers

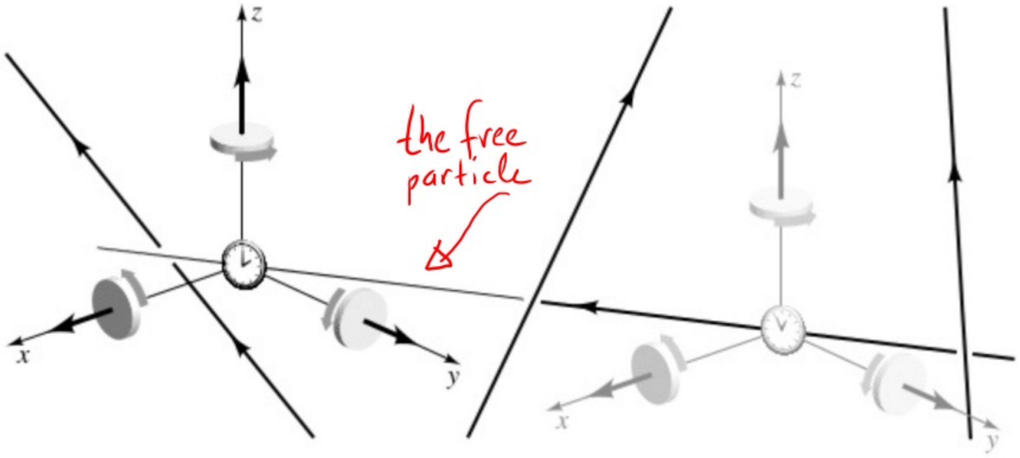


Hartle Fig 3.3

Local Inertial Observers

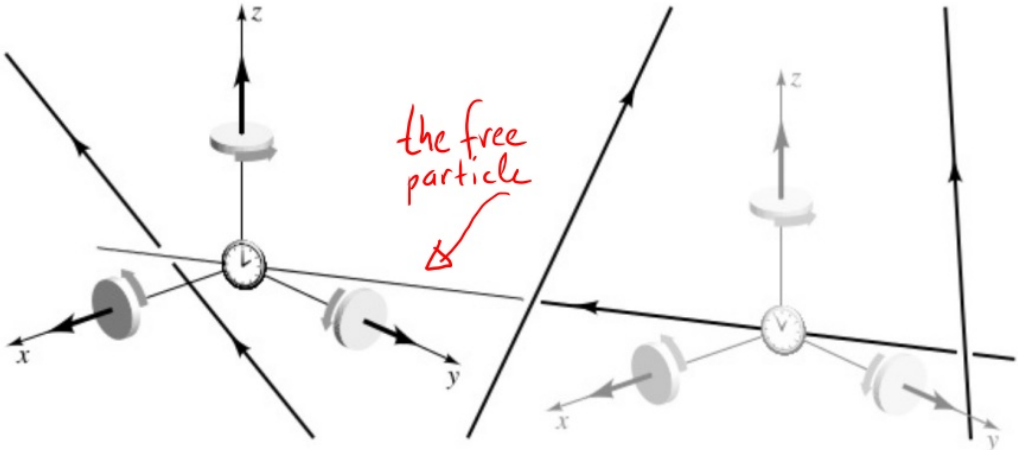
Observers observing free particles moving on straight lines @ constant rate are inertial observers

"freely falling"



Hartle Fig 3.3

Local Inertial Observers



Hartle Fig 3.3

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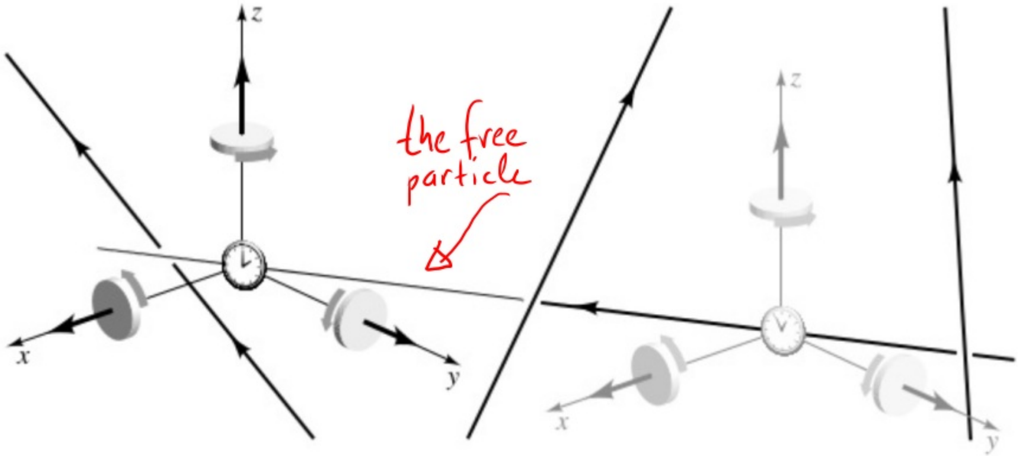
"freely falling"

How to become one:

1. follow a free massive particle and set axes' origin on it

Local Inertial Observers



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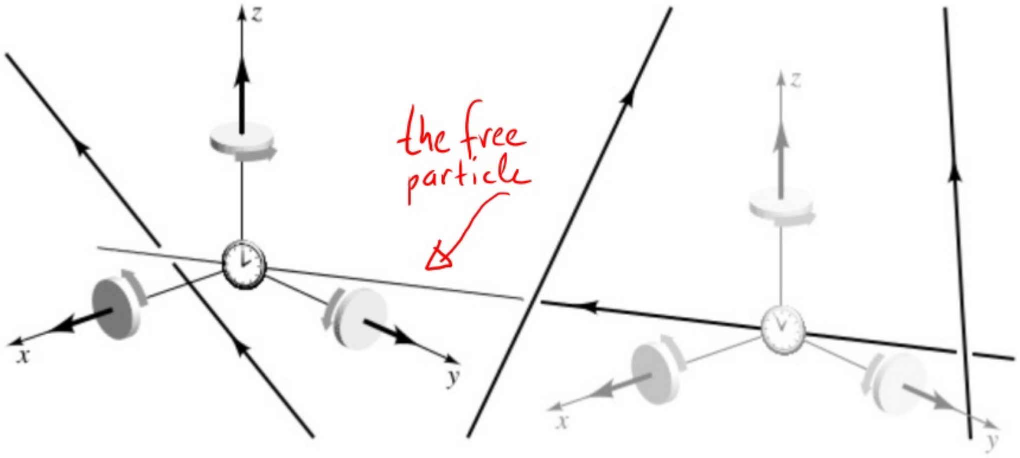

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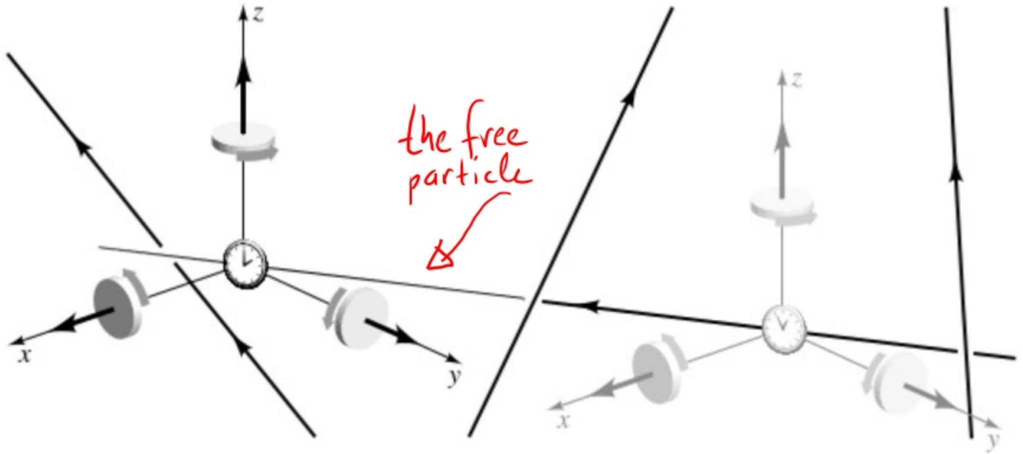

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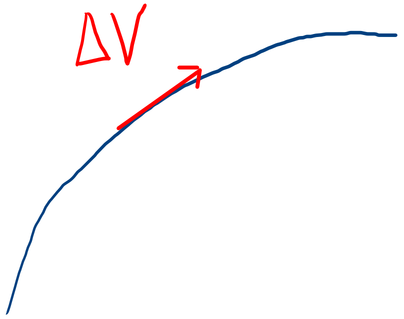
$\Rightarrow$ $g_{\mu\nu}|_0 = \eta_{\mu\nu}$, and $\partial_\sigma g_{\mu\nu}|_0 = 0$! Do SR physics !

Line Element

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$\Delta V = \Delta x^\mu \partial_\mu$$



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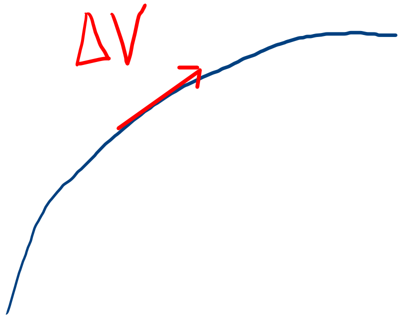
$$\left.\begin{aligned} g &= g_{\mu\nu}\, dx^\mu \otimes dx^\nu \\ \Delta V &= \Delta x^\mu \partial_\mu \end{aligned}\right\} \Rightarrow g(\Delta V, \Delta V) = g(\Delta x^\mu \partial_\mu, \Delta x^\nu \partial_\nu)$$



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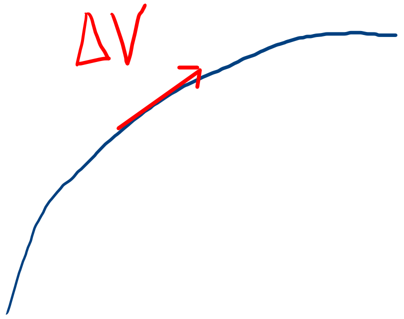

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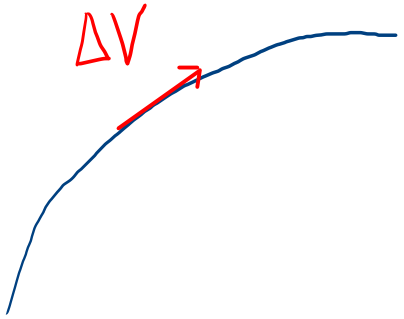
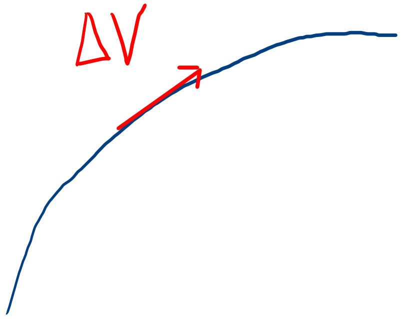


Line Element



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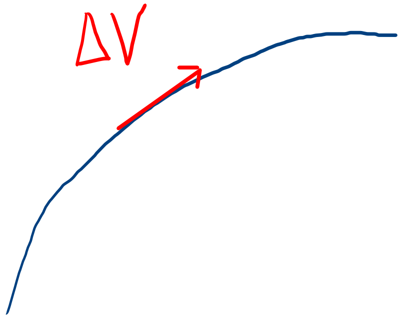

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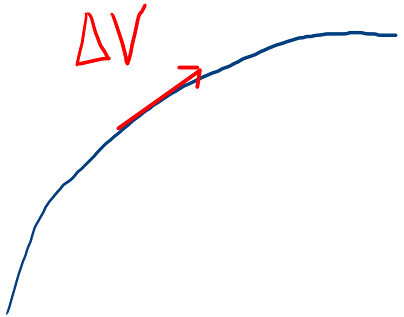
$$ds^2 \equiv g$$

$$dx^\mu dx^\nu \equiv dx^\mu \otimes dx^\nu \neq dx^\nu \otimes dx^\mu$$

Line Element



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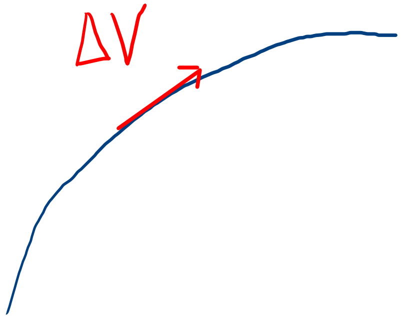

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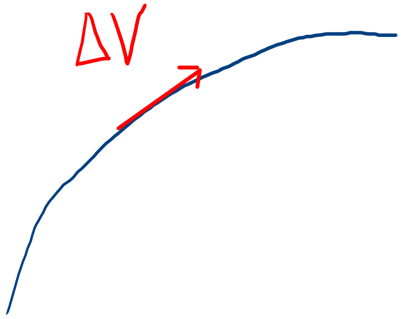
$$S_{AB} = \int_A^B |ds|$$

Line Element



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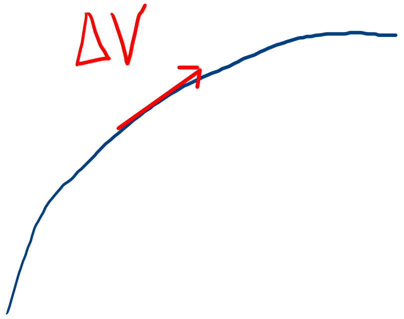
$$S_{AB} = \int_A^B |ds| = \int_A^B |g_{\mu\nu}\, dx^\mu dx^\nu|^{1/2}$$

Line Element



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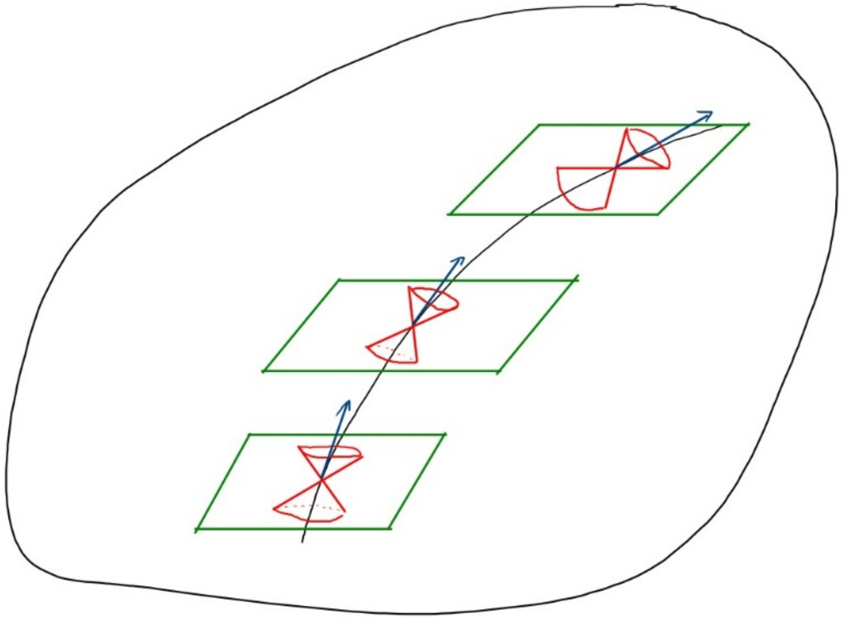
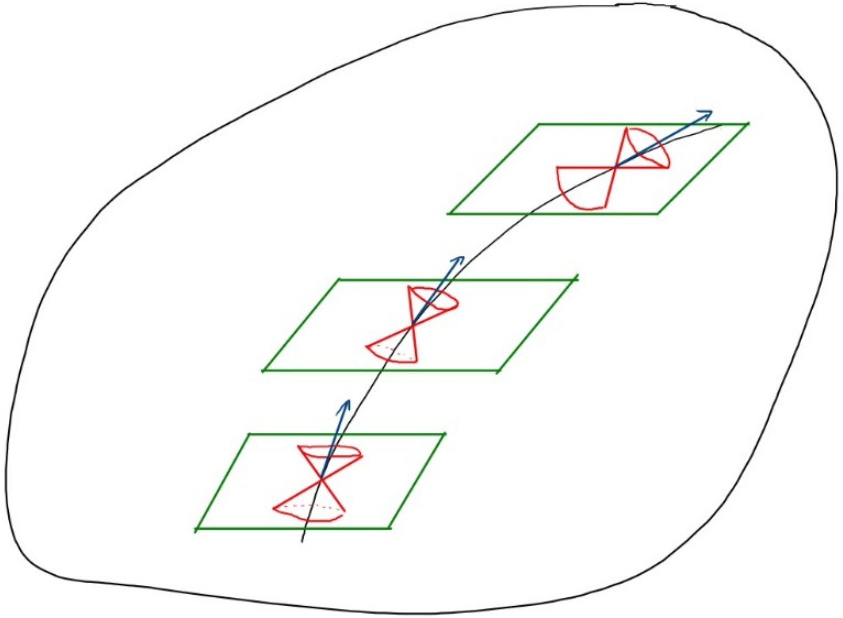
$$S_{AB} = \int_A^B |ds| = \int_A^B |g_{\mu\nu}\, dx^\mu dx^\nu|^{1/2} \equiv \int_{t_A}^{t_B} dt \left| g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right|^{1/2}$$

We focus on 3 types of curves:



We focus on 3 types of curves:

$ds^2 < 0$ everywhere   timelike curve



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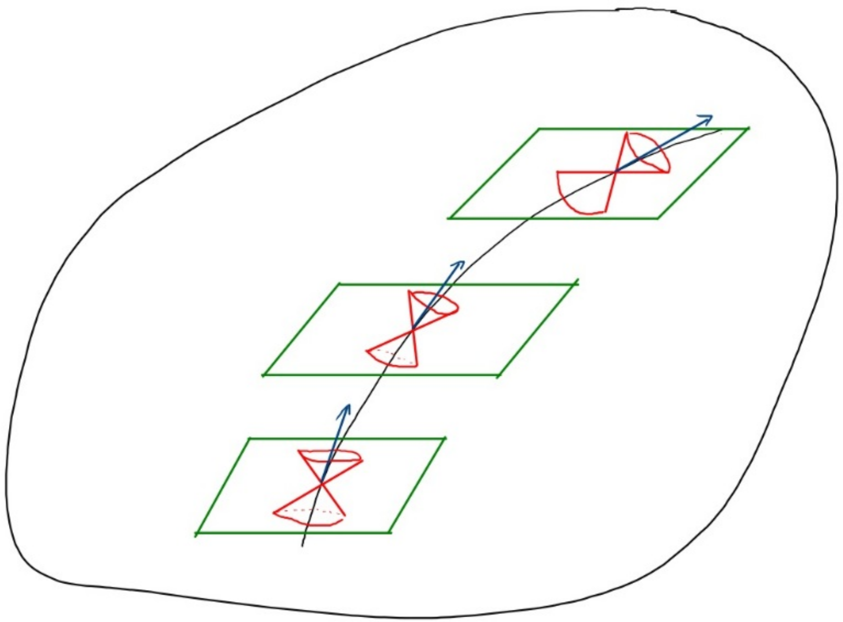

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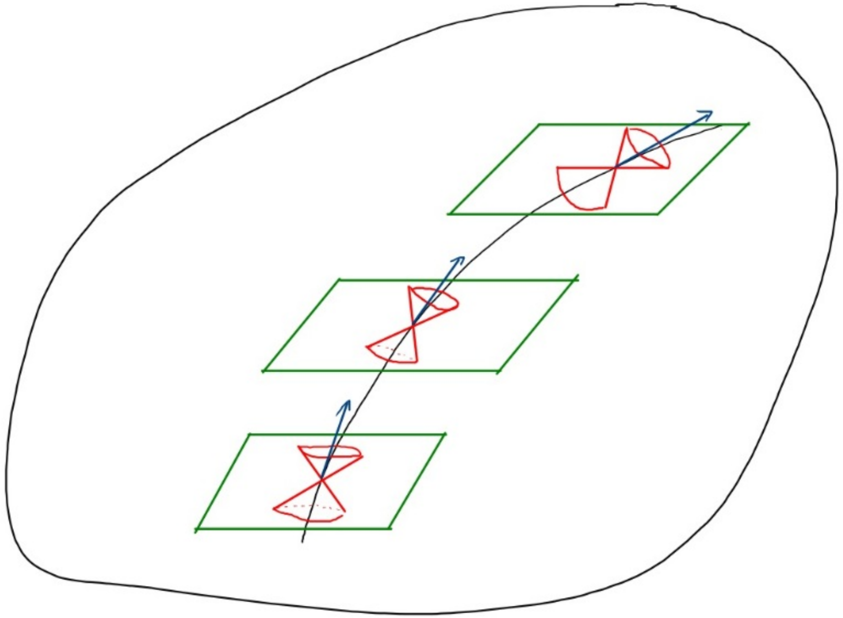

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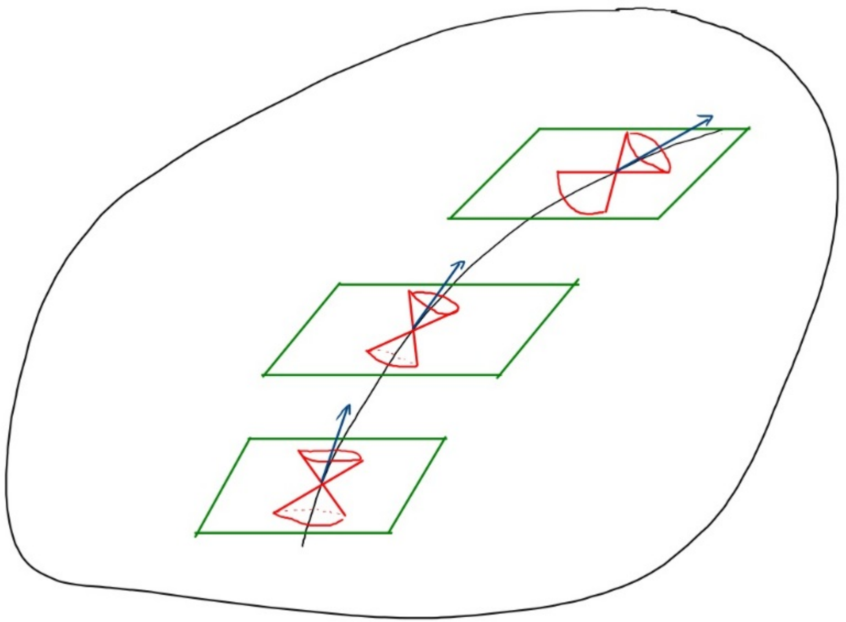

$ds^2 < 0$ everywhere — timelike curve

$ds^2 = 0$ " — null "

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The tangent vector V is of the same type at each point:

$g(V,V)$ does not change sign/0

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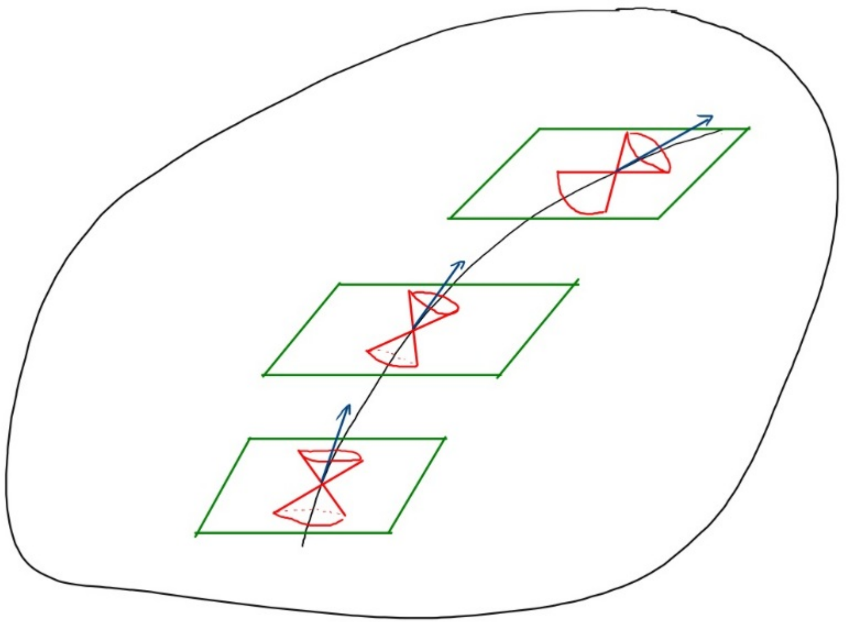


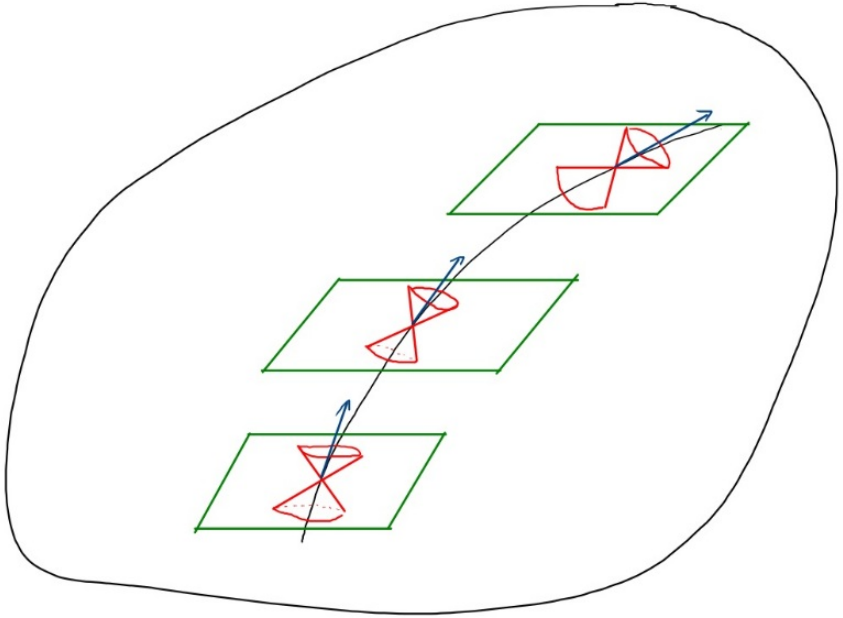
The tangent vector V is of the same type at each point:
$g(V,V)$ does not change sign/0

The tangent vectors are the 4-velocities of particles

$$V^\mu = \frac{dx^\mu}{dt} \qquad (ds^2 \leq 0)$$

We focus on 3 types of curves:

$ds^2 < 0$ everywhere	timelike curve	
$ds^2 = 0$ "	null "	
$ds^2 > 0$ "	spacelike "	



Timelike curves are worldlines of massive particles

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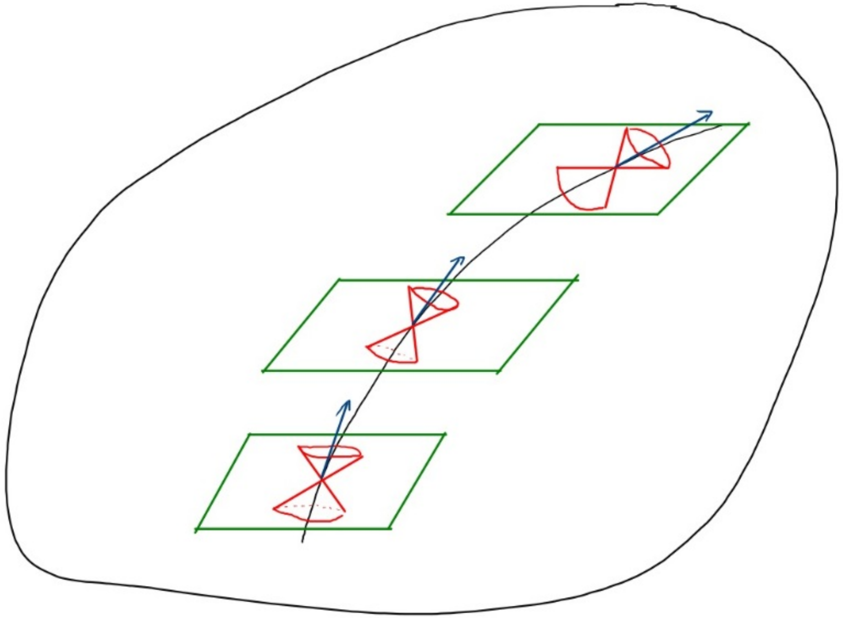

Timelike curves are worldlines of massive particles

Null " " " " massless "

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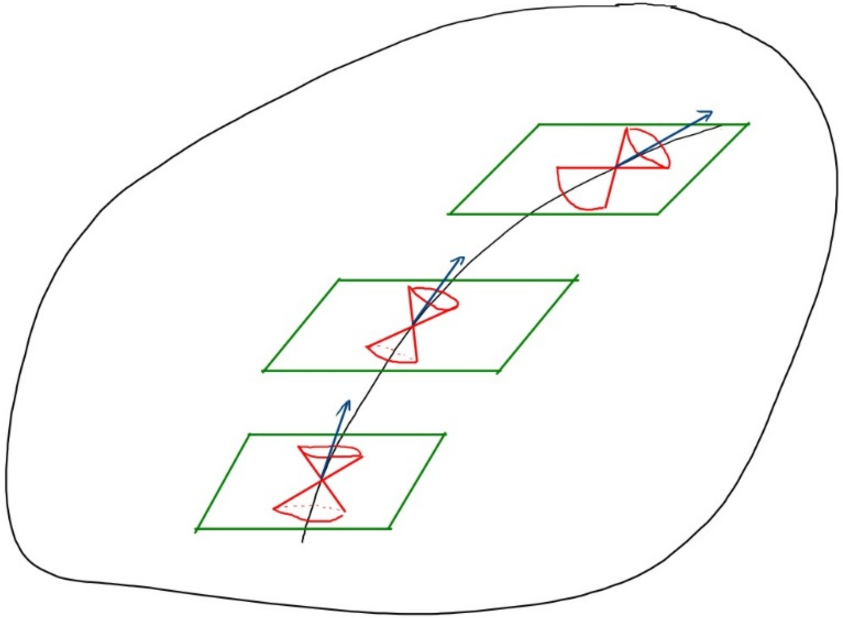

Timelike curves are worldlines of massive particles
Null   "   "   "   "   massless   "

They are causal curves: any event on curve can influence/be influenced by any other event on curve

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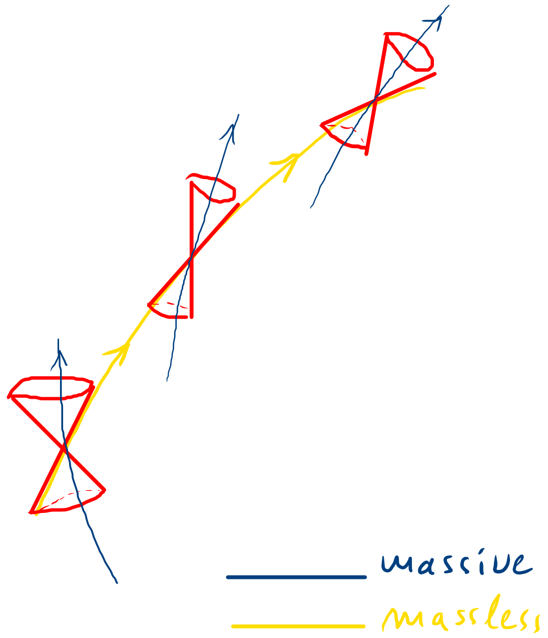

Light travels in a direction on local lightcone

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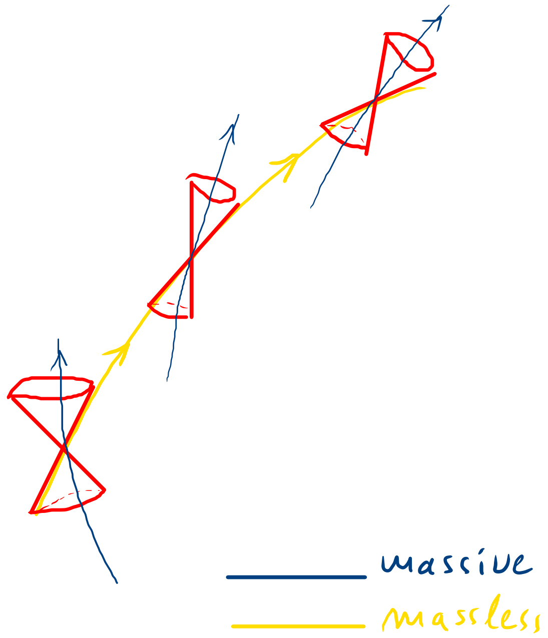
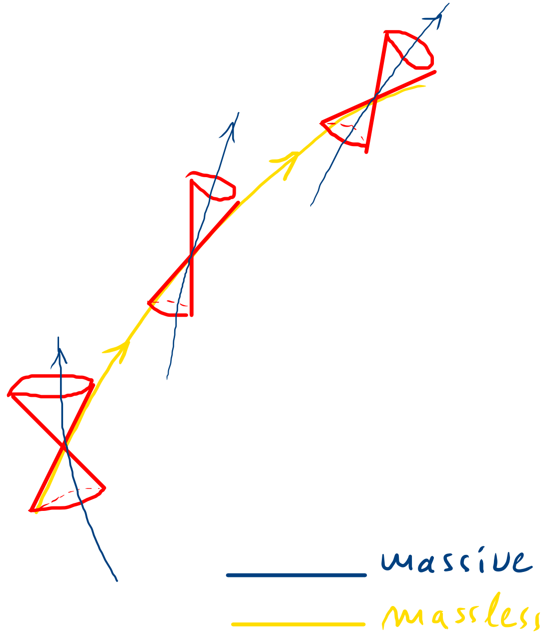


Light travels in a direction on local lightcone

Speed compares only locally: Not exceeding the speed of light means massive particles move in the direction within the local light cone

We focus on 3 types of curves:



$ds^2<0$	everywhere	timelike	curve
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Light travels in a direction on local lightcone

Speed compares only locally: Not exceeding the speed of light means massive particles move in the direction within the local light cone

Distances of faraway particles can increase at a rate >1 !