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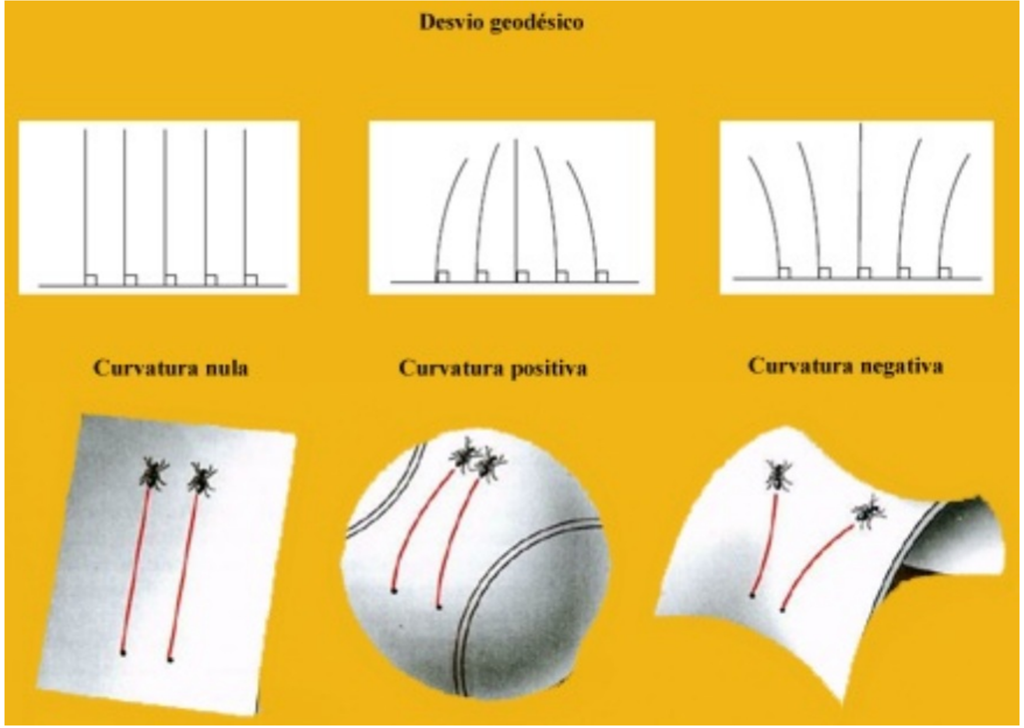
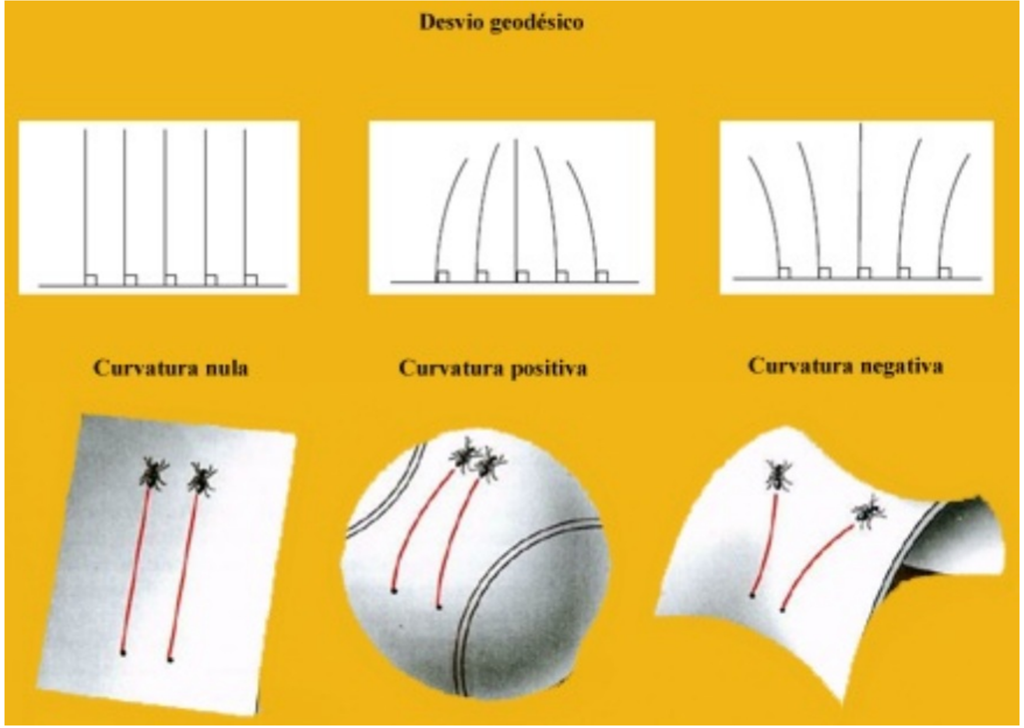
- curvature related to properties of parallel transport
  (choice of) affine connection $\Rightarrow$ curvature
- (choice of) metric $\rightarrow$ Levi-Civita connection $\rightarrow$ curvature
  but curvature can be defined w/o metric, e.g. gauge theories

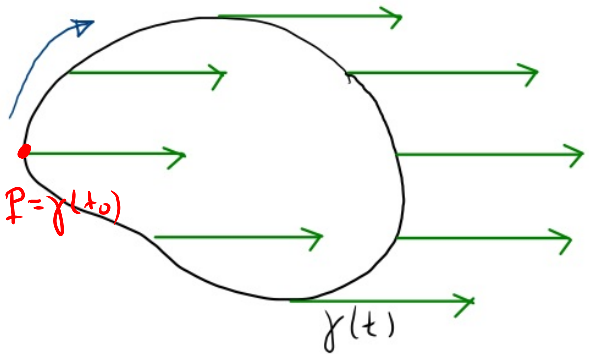


- In flat space, parallel geodesics remain parallel



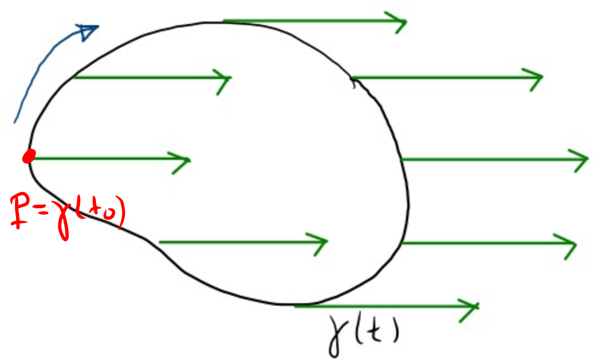
- In flat space, parallel geodesics remain parallel
- Curvature has the effect of making initially parallel geodesics to deviate

(relative acceleration) $\propto$ (curvature)



Flat Space Parallel Transport

* Flat Space : Parallel Transport of vector along closed curve leaves vector invariant at P



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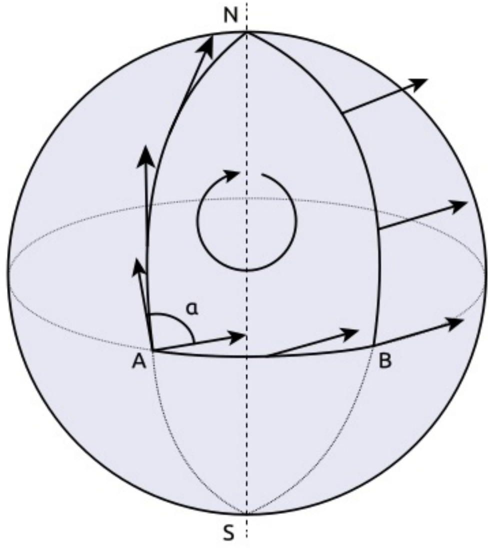

Wikipedia

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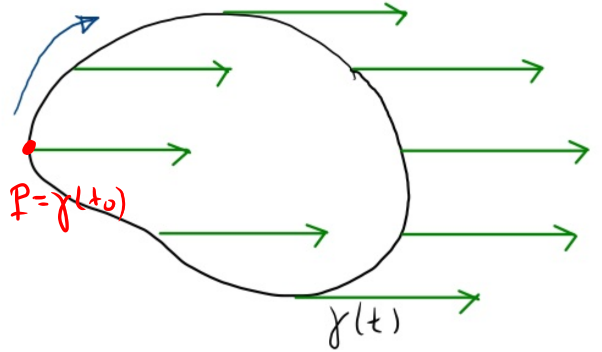
$$\delta V \propto (\text{curvature})$$



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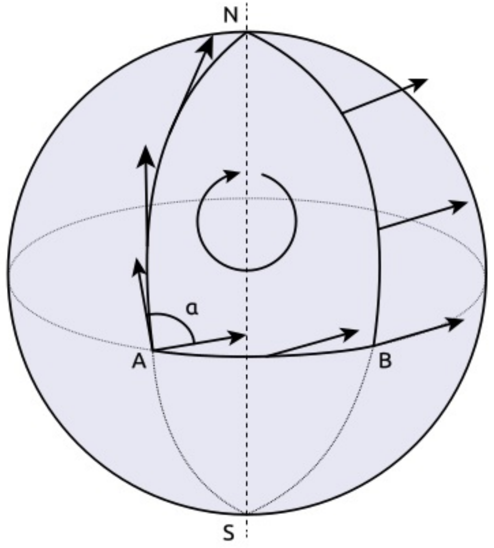

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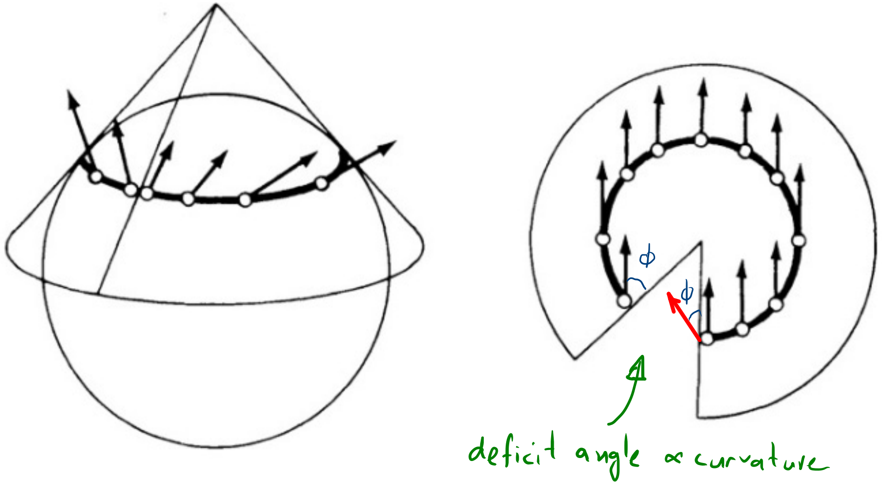

Vladimir I. Arnold, Mathematical Methods of Classical Mechanics (New York: Springer, 1989), 302, Fig. 231.

Cone with metric dx^2+dy^2 on plane

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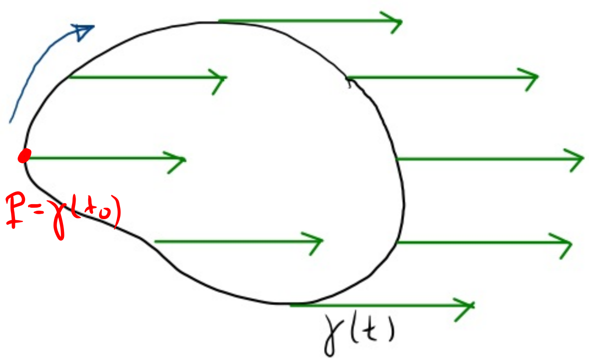

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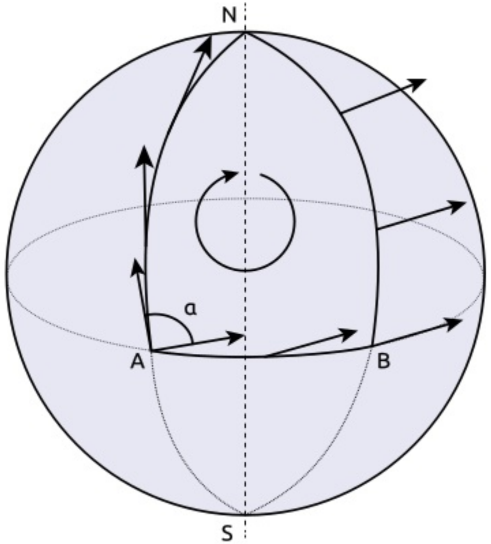

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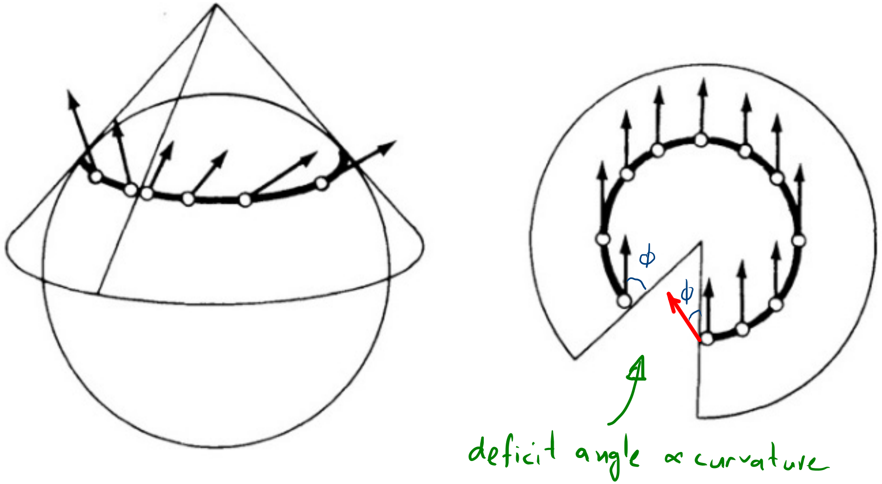

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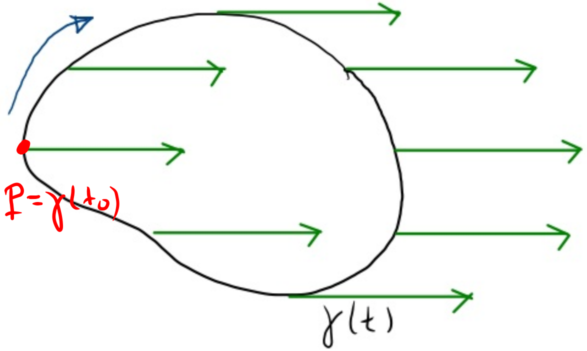
- measures deviation from flatness



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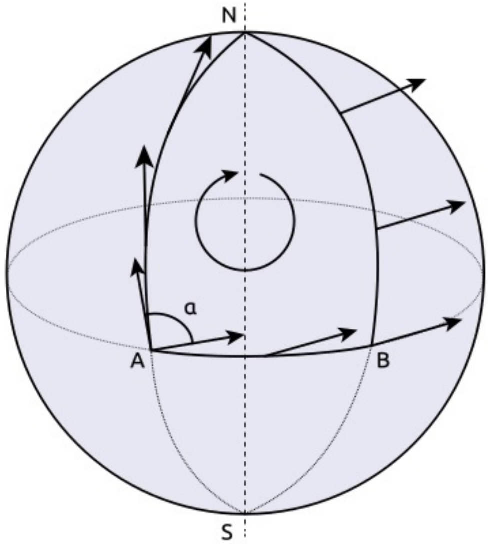

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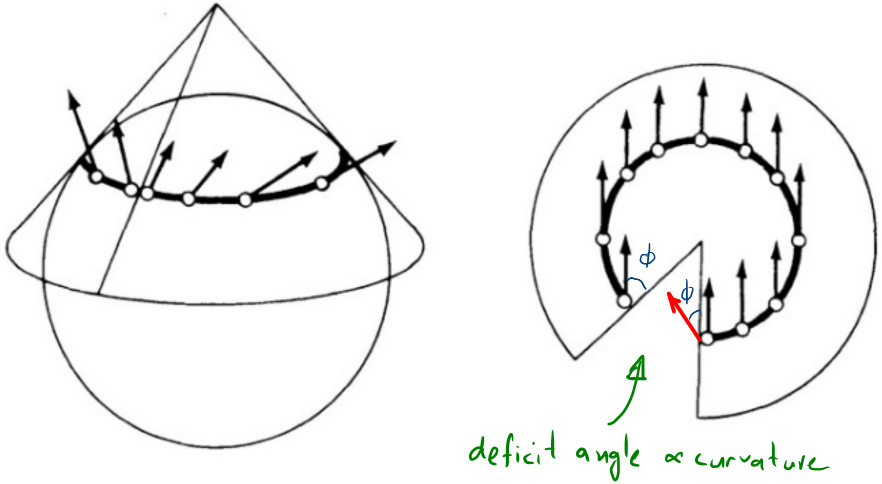

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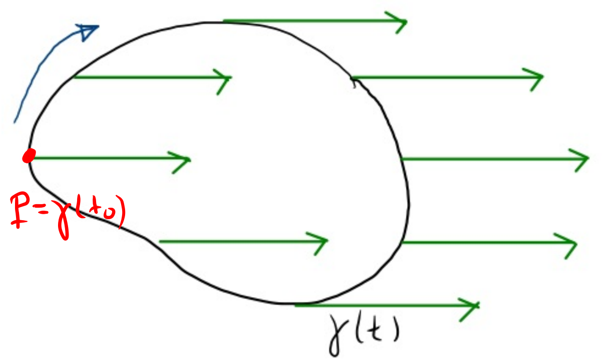
- measures deviation from flatness
- intrinsic notion, geometric property



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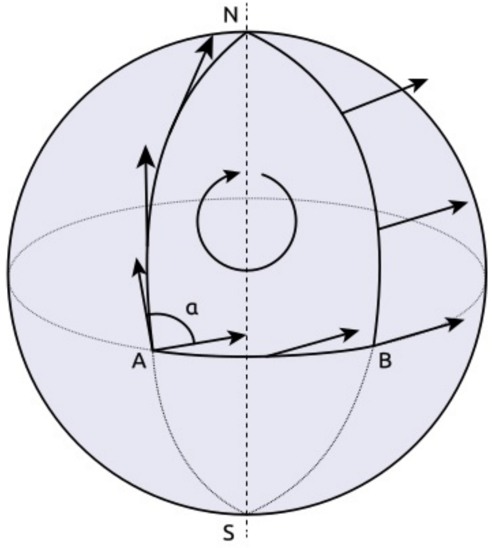

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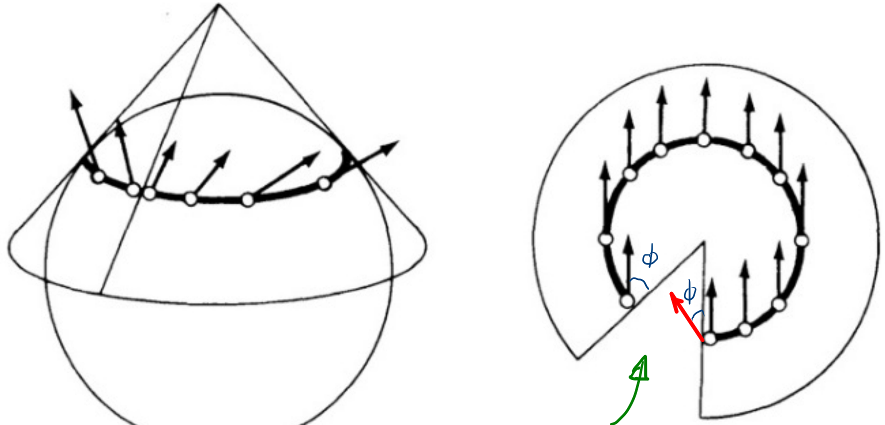

deficit angle ∝ curvature

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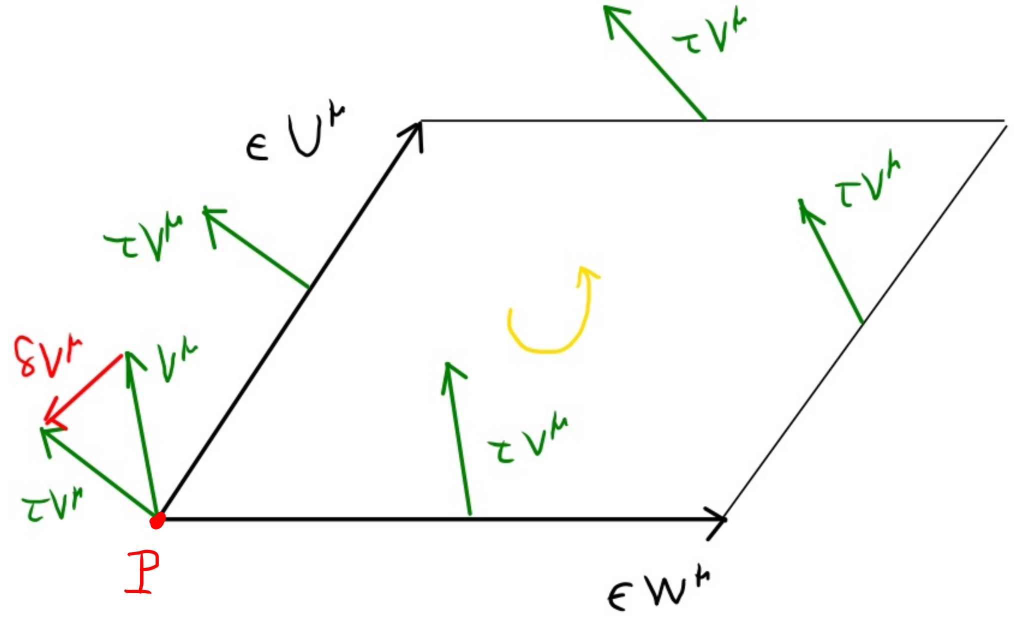
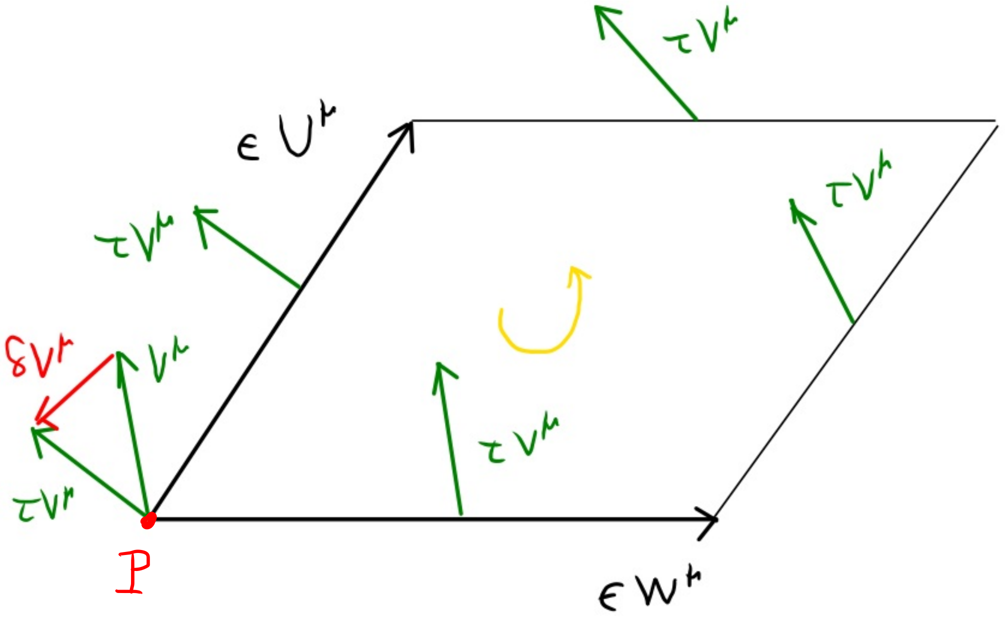
↳ global notion, shrink to get a local one

- shrink to infinitesimal curves
  - closed curve defined by ϵW^μ, ϵU^μ



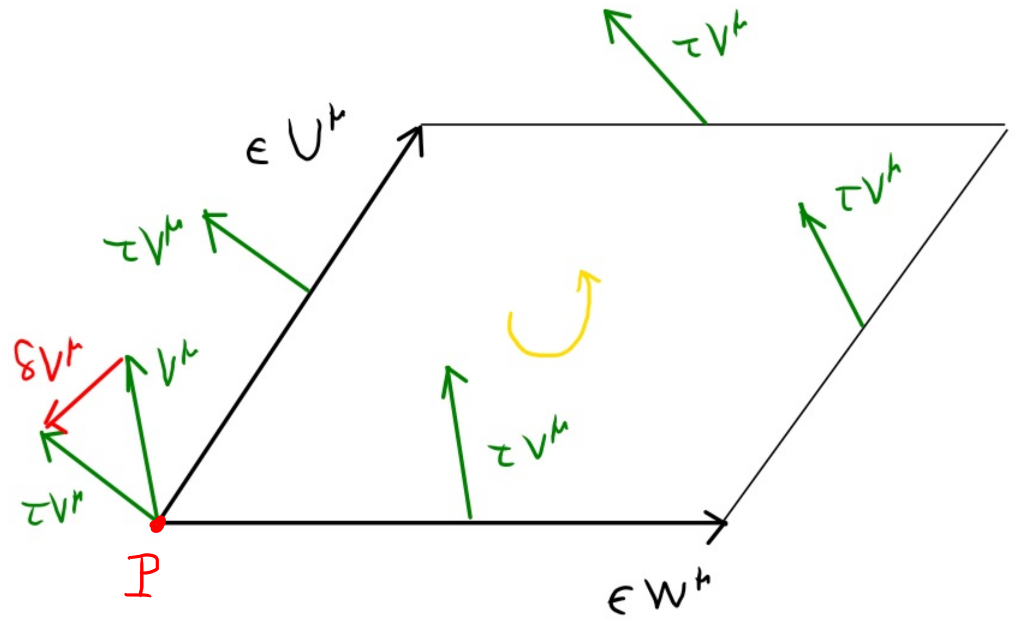
- shrink to infinitesimal curves
  - closed curve defined by ϵW^μ, ϵU^μ
  - parallel transport $P \to P$



$$V^\mu \to \tau V^\mu = V^\mu + \delta V^\mu$$

$$\tau V^\mu = \Theta^\mu{}_\nu V^\nu \Rightarrow \delta V^\rho = R^\rho{}_\sigma V^\sigma$$

- shrink to infinitesimal curves



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$$V^\mu \to \tau V^\mu = V^\mu + \delta V^\mu$$

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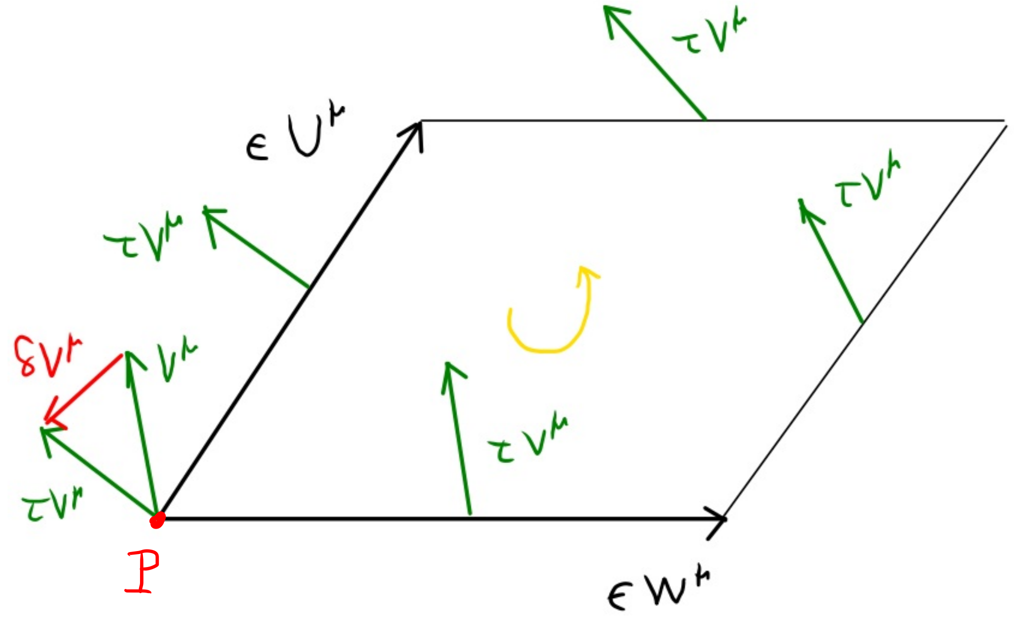
- depends on W^μ, U^μ linearly

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma W^\mu U^\nu$$

- shrink to infinitesimal curves
  - $W^\mu \leftrightarrow U^\mu$ reverses direction of motion on curve: $\delta V \to -\delta V$

$$\Rightarrow R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$



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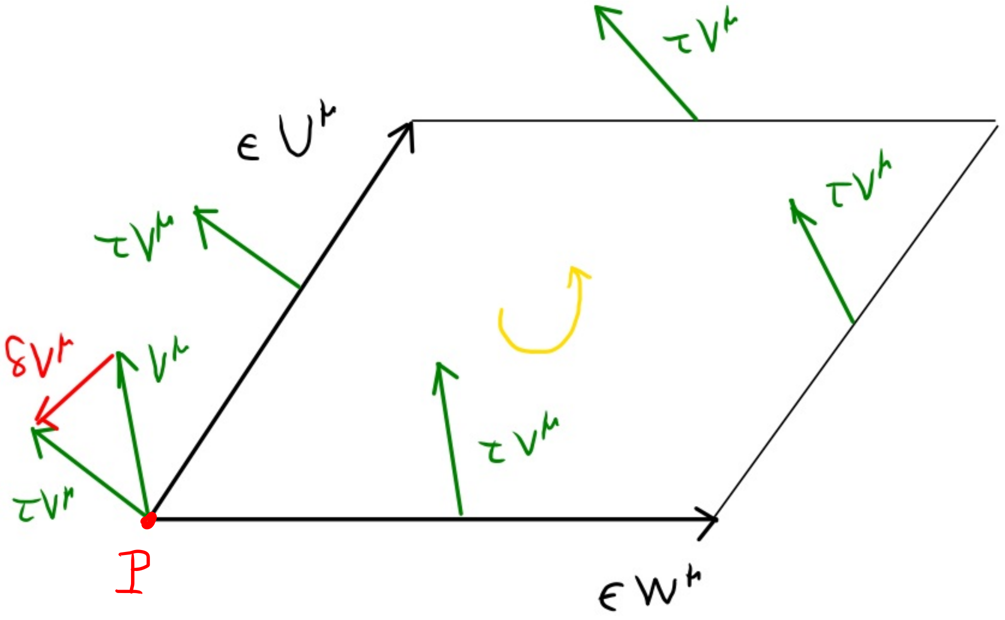

- $D_W V$: measures change of V along W relative to its parallel transport

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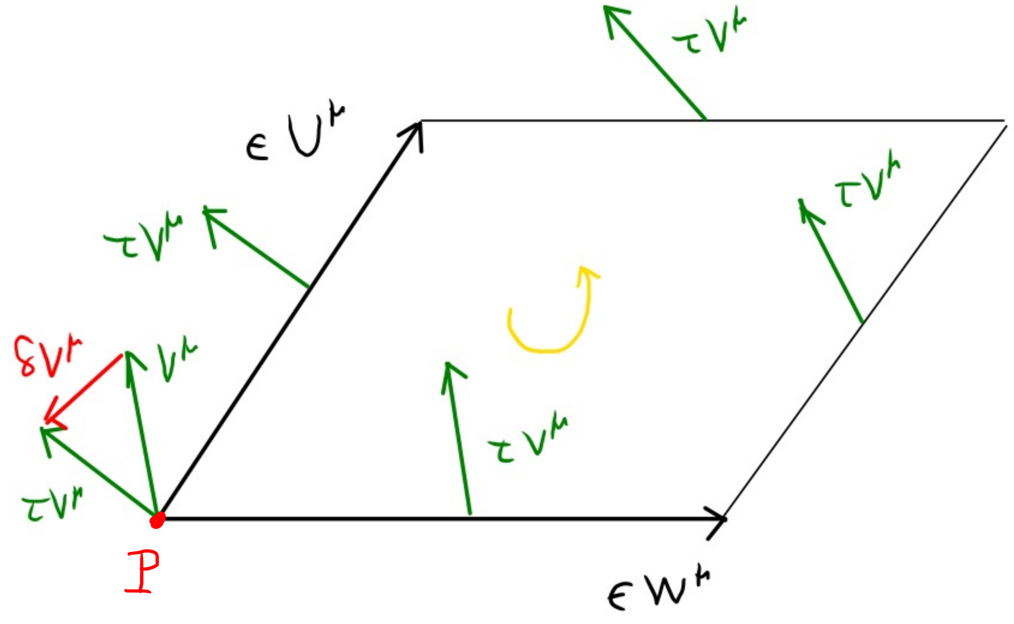

- $D_W V$: measures change of V along W relative to its parallel transport

  $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

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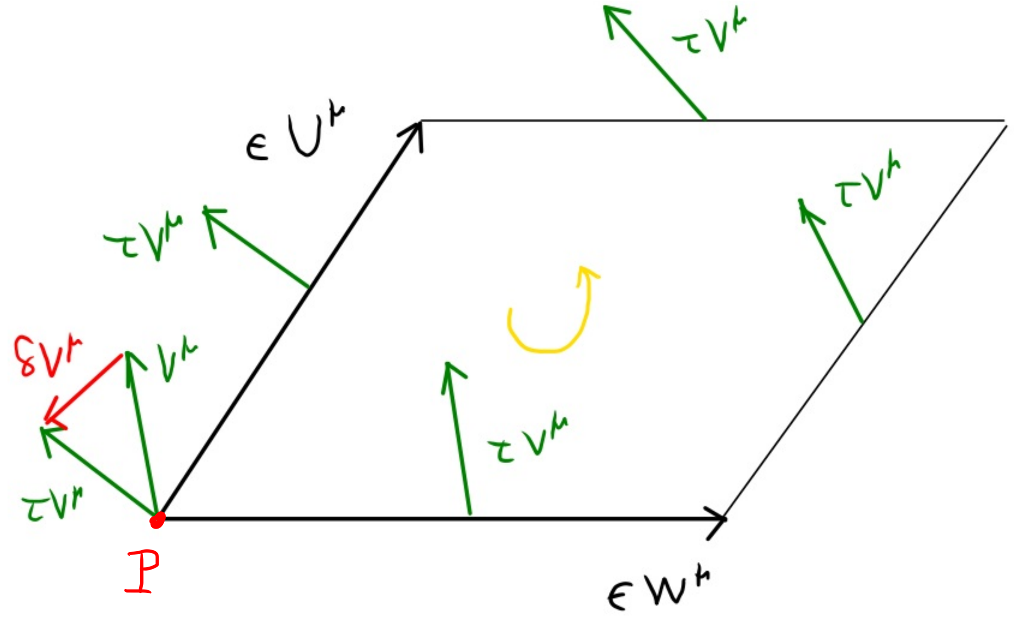


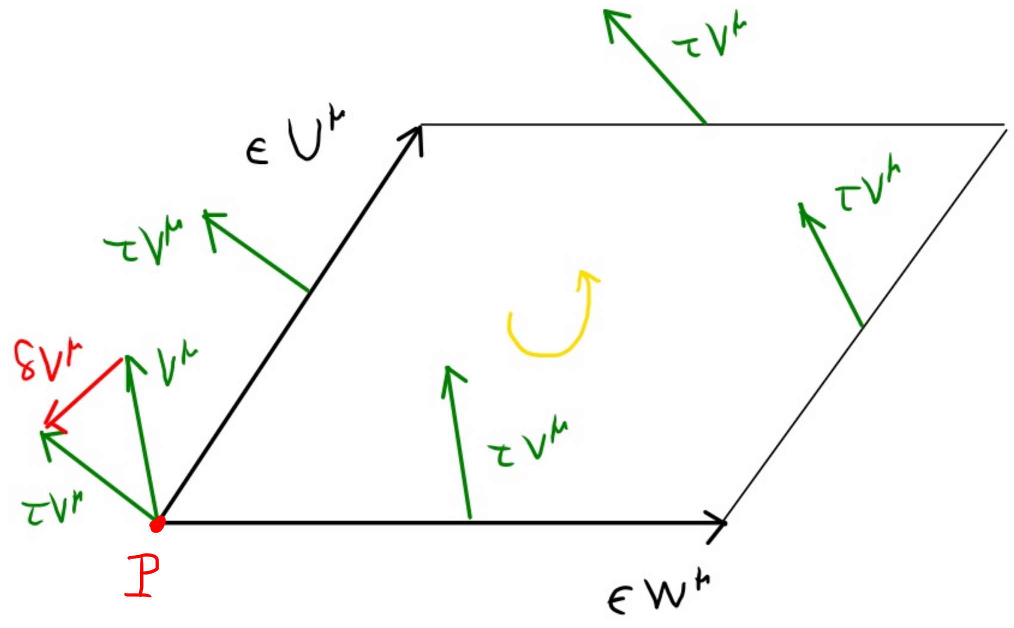
- $D_W V$: measures change of V along W relative to its parallel transport

  $\nabla_\nu V^\rho$: change of V^ρ along ∂_ν

$\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

- shrink to infinitesimal curves

$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho$:
change along loop $\partial_\mu, \partial_\nu$



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$\nabla_\mu \nabla_\nu V^\rho$: change along ∂_ν, then along ∂_μ

- ## Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

(torsion free)

$$[\nabla_\mu, \nabla_\nu] = \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$$

- ## Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = \underbrace{R^\rho{}_{\lambda\mu\nu}}_{} V^\lambda$$

component xfm indices

Careful: Placement of indices heavily author dependent!

- <u>Formal Definition</u>

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[\nabla_\mu, \nabla_\nu] V^\rho = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho$$

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$$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu \underbrace{(\nabla_\nu V^\rho)}_{\text{a (1,1) tensor}} - \Gamma^\lambda{}_{\mu\underline{\underline{\nu}}} \nabla_\lambda V^\rho + \Gamma^\rho{}_{\mu\lambda} \nabla_\nu V^{\underline{\underline{\lambda}}}$$

1st index   2nd index

- <u>Formal Definition</u>

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$$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda{}_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho{}_{\mu\lambda} \nabla_\nu V^\lambda$$

$$= \partial_\mu [\partial_\nu V^\rho + \Gamma^\rho{}_{\nu\lambda} V^\lambda]$$

$$- \Gamma^\lambda{}_{\mu\nu} [\partial_\lambda V^\rho + \Gamma^\rho{}_{\lambda\sigma} V^\sigma]$$

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$$= \partial_\mu \partial_\nu V^\rho + \partial_\mu \Gamma^\rho{}_{\nu\lambda} V^\lambda + \Gamma^\rho{}_{\nu\lambda} \partial_\mu V^\lambda$$

$$- \Gamma^\lambda{}_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda{}_{\mu\nu} \Gamma^\rho{}_{\lambda\sigma} V^\sigma$$

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Formal Definition

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$\mu \leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu V^\rho = \partial_\nu \partial_\mu V^\rho + \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda + \Gamma^\rho_{\mu\lambda} \partial_\nu V^\lambda$$
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- **Formal Definition**

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$\lambda \leftrightarrow \sigma$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\rho = \partial_\mu \Gamma^\rho{}_{\nu\lambda} V^\lambda + \Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} V^\lambda - (\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu})[\partial_\lambda V^\rho + \Gamma^\rho{}_{\lambda\sigma} V^\sigma]$$
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• Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

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$\mu \leftrightarrow \nu$

$$= (\partial_\mu \Gamma^\rho{}_{\nu\lambda} - \partial_\nu \Gamma^\rho{}_{\mu\lambda} + \Gamma^\rho{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\lambda} - \Gamma^\rho{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\lambda}) V^\lambda - 2 \underbrace{\Gamma^\lambda{}_{[\mu\nu]}}_{\text{torsion } T^\lambda{}_{\mu\nu} = 0\,!} \nabla_\lambda V^\rho$$

• Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$\mu \leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu V^\rho = \cancel{\partial_\nu \partial_\mu V^\rho} + \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda + \cancel{\Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda}$$

$$- \Gamma^\lambda_{\nu\mu}\partial_\lambda V^\rho - \Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma} V^\sigma$$

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$$- \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} V^\lambda$$

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$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu$$

- ## Formal Definition

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\lambda\mu\nu} V^\lambda$$

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$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu} \Gamma_{\nu\lambda} - \Gamma^\rho_{\nu} \Gamma_{\mu\lambda}$$

- Formal Definition

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$$- \partial_\nu \Gamma^\rho_{\mu\lambda} V^\lambda - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda$$

$\mu \leftrightarrow \nu$

$$= (\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}) V^\lambda$$

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda}$$

- $[\nabla_\mu, \nabla_\nu] V = R_{\mu\nu} V$

- $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$

  ↳ torsion free ⇒ depends on V, but not on ∂V

• $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$

– obvious that $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$ since $[\nabla_\mu, \nabla_\nu] \to [\nabla_\nu, \nabla_\mu] = -[\nabla_\mu, \nabla_\nu]$

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- expression $R = \partial\Gamma + \Gamma\Gamma$ valid for any torsion free $\tilde{\nabla}$:

$$R^\rho{}_{\sigma\mu\nu} = \tilde{\nabla}_\mu C^\rho{}_{\nu\sigma} - \tilde{\nabla}_\nu C^\rho{}_{\mu\sigma} + C^\rho{}_{\mu\lambda} C^\lambda{}_{\nu\sigma} - C^\rho{}_{\nu\lambda} C^\lambda{}_{\mu\sigma}$$

• Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:

The contraction $\omega_\mu V^\mu$ is a function, therefore

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, \omega_\lambda V^\lambda = 0$$

torsion free condition!

• Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:

The contraction $\omega_\mu V^\mu$ is a function, therefore

$$[\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = 0 \qquad \text{(torsion free condition)}$$

$$\nabla_\mu \nabla_\nu (\omega_\lambda V^\lambda) = \nabla_\mu \left[(\nabla_\nu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\nu V^\lambda) \right]$$

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$$= (\nabla_\mu \nabla_\nu \omega_\lambda) V^\lambda + (\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda) + (\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda) + \omega_\lambda \nabla_\mu \nabla_\nu V^\lambda$$

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$$= (\nabla_\mu \nabla_\nu \omega_\lambda) V^\lambda + (\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda) + (\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda) + \omega_\lambda \nabla_\mu \nabla_\nu V^\lambda$$

$\mu \leftrightarrow \nu$

$$\nabla_\nu \nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\nu \nabla_\mu \omega_\lambda) V^\lambda + (\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda) + (\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda) + \omega_\lambda \nabla_\nu \nabla_\mu V^\lambda$$

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$$\nabla_\nu \nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\nu \nabla_\mu \omega_\lambda) V^\lambda + \cancel{(\nabla_\mu \omega_\lambda)(\nabla_\nu V^\lambda)} + \cancel{(\nabla_\nu \omega_\lambda)(\nabla_\mu V^\lambda)} + \omega_\lambda \nabla_\nu \nabla_\mu V^\lambda$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu](\omega_\lambda V^\lambda) = \Big(\underbrace{[\nabla_\mu, \nabla_\nu]\omega_\lambda}_{\text{we want this}}\Big) V^\lambda + \omega_\lambda \Big(\underbrace{[\nabla_\mu, \nabla_\nu]V^\lambda}_{\text{we know that...}}\Big)$$

• Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:

$$\Rightarrow \quad 0 \quad = ([\nabla_\mu, \nabla_\nu]\omega_\lambda)V^\lambda + \omega_\lambda R^\lambda{}_{\sigma\mu\nu} V^\sigma$$

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$$= \underbrace{\{[\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\,\omega_\sigma\}}_{=\,0} V^\lambda$$

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• Action of $[\nabla_\mu, \nabla_\nu]$ on one forms:

$$\Rightarrow \quad 0 = ([\nabla_\mu, \nabla_\nu]\omega_\lambda)V^\lambda + \omega_\sigma R^\sigma{}_{\lambda\mu\nu} V^\lambda$$

$$= \left\{ [\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\omega_\sigma \right\} V^\lambda$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu]\omega_\lambda = - R^\sigma{}_{\lambda\mu\nu}\omega_\sigma$$

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$$= \{[\nabla_\mu, \nabla_\nu]\omega_\lambda + R^\sigma{}_{\lambda\mu\nu}\omega_\sigma\} V^\lambda$$

$$\Rightarrow [\nabla_\mu, \nabla_\nu]\omega_\lambda = - R^\sigma{}_{\lambda\mu\nu}\omega_\sigma$$

· Action of $[\nabla_\mu, \nabla_\nu]$ on higher rank tensors:

$$[\nabla_\mu, \nabla_\nu] S^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = R^{\mu_1}{}_{\lambda\mu\nu} S^{\lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} + \cdots + R^{\mu_k}{}_{\lambda\mu\nu} S^{\mu_1 \cdots \lambda}{}_{\nu_1 \cdots \nu_l}$$

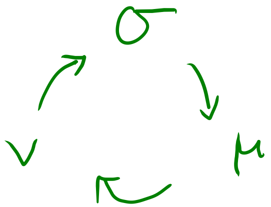
$$- R^\lambda{}_{\nu_1\mu\nu} S^{\mu_1 \cdots \mu_k}{}_{\lambda \cdots \nu_l} - \cdots + R^\lambda{}_{\nu_l\mu\nu} S^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \lambda}$$

Symmetries

- $$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$$

Symmetries

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \quad \Leftrightarrow \quad R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$



cyclic permutation

Symmetries

- $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$
- $R^\rho{}_{[\sigma\mu\nu]} = 0 \quad \Leftrightarrow \quad R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} = 0$

If $\exists\, g_{\mu\nu}$ and ∇_μ its Christoffel/Levi-Civita connection ($\nabla g = 0$ + torsion free), then

$$R_{\rho\sigma\mu\nu} \equiv g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$$

Symmetries

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- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

- $R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$

Symmetries

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \quad \Leftrightarrow \quad R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$

$$\Rightarrow \quad \frac{n^2(n^2-1)}{12}$$ independent components

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$\Rightarrow \dfrac{n^2(n^2-1)}{12}$ independent components

$n=2$	1	indep. component(s)
$n=3$	6	"
$n=4$	20	"

- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

- $R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$

Symmetries

1^{st} Bianchi identity

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \quad \Leftrightarrow \quad R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$

$\Rightarrow \dfrac{n^2(n^2-1)}{12}$ independent components

$+ \; 2^{nd}$ Bianchi identity:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad \Leftrightarrow$$

$$\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0$$

- $R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}$
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

- $R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$

Symmetries

constraints values at neighboring points!

- $R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu}$
- $R^{\rho}{}_{[\sigma\mu\nu]} = 0 \iff R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} = 0$

$\Rightarrow \dfrac{n^2(n^2-1)}{12}$ independent components

+ 2nd Bianchi identity:

$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \iff$

$\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0$

- $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
- $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

- $R_{\rho[\sigma\mu\nu]} = 0 \Rightarrow R_{[\rho\sigma\mu\nu]} = 0$

Independent contractions (assume Christoffel connections)

- Ricci tensor: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$

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- Weyl tensor: Riemann with all contractions removed

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- Weyl tensor: Riemann with all contractions removed

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) \qquad n > 2$$

$$+ \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

Independent contractions (assume Christoffel connections)

- Ricci tensor: $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \Rightarrow R_{\mu\nu} = R_{\nu\mu}$

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$$+ \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

- Symmetries remain: $C_{[\rho\sigma][\mu\nu]} = C_{\rho\sigma\mu\nu}$, $C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$, $C_{\rho[\sigma\mu\nu]} = 0$

Independent contractions

(assume Christoffel connections)

- trace free $C^{\lambda}{}_{\mu\lambda\nu} = 0$

(of course, we subtracted out $R_{\mu\nu}$ and R from $R^{\rho}{}_{\sigma\mu\nu}$)

- Weyl tensor: Riemann with all contractions removed

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right)$$

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Independent contractions (assume Christoffel connection)

- trace free $C^{\lambda}{}_{\mu\lambda\nu} = 0$
- independent components: $\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2}$

$n \leq 3 \quad C_{\rho\sigma\mu\nu} = 0$

$n = 4 \quad 10$ independent comp.

- Weyl tensor: Riemann with all contractions removed

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right)$$

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- If $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, C remains invariant (conformal xfms)

- Weyl tensor: Riemann with all contractions removed

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right)$$

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- Symmetries remain: $C_{[\rho\sigma][\mu\nu]} = C_{\rho\sigma\mu\nu}$, $C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}$, $C_{\rho[\sigma\mu\nu]} = 0$

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- If $g_{\mu\nu} \to \Omega^2(x)\, g_{\mu\nu}$, C remains invariant (conformal xfms)

- In the vacuum, Einstein equations $\Rightarrow R_{\mu\nu} = 0$
  ↳ e.g. gravitational waves

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- In the vacuum, Einstein equations $\Rightarrow R_{\mu\nu} = 0$
  $\Rightarrow C_{\rho\sigma\mu\nu}$ has all propagating degrees of freedom in vacuum

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- Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

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- If $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$, C remains invariant (conformal xfms)

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– Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

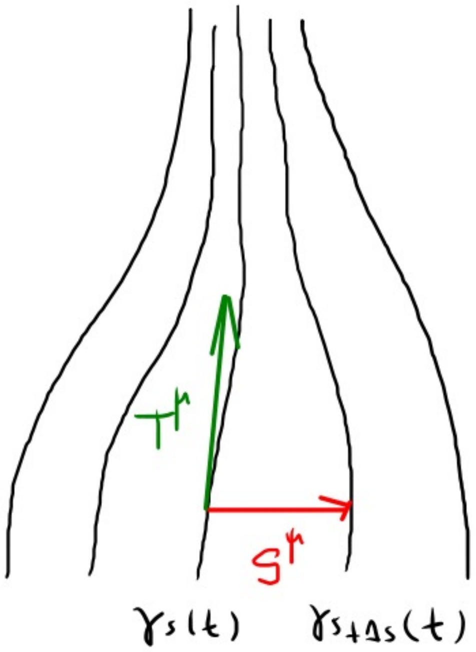
- has critical property $\nabla^{\mu} G_{\mu\nu} = 0$ $\quad \left(G_{\mu\nu} = 8\pi T_{\mu\nu} \ \& \ \nabla^{\mu} T_{\mu\nu} = 0 \right)$

Geodesic Deviation

- Consider a one-parameter family of geodesics $\gamma_s(t)$

  t: affine parameter

  $s \in \mathbb{R}$



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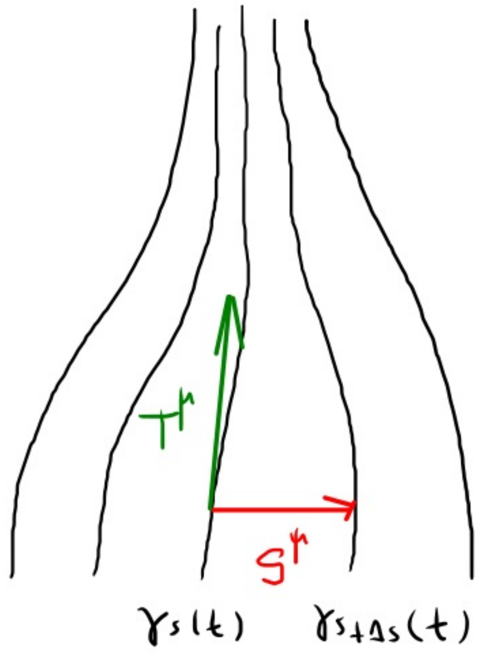

- Consider a small enough open set where they don't cross

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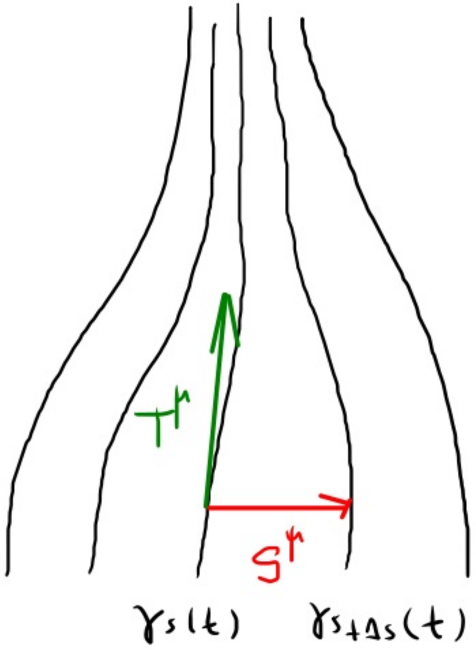
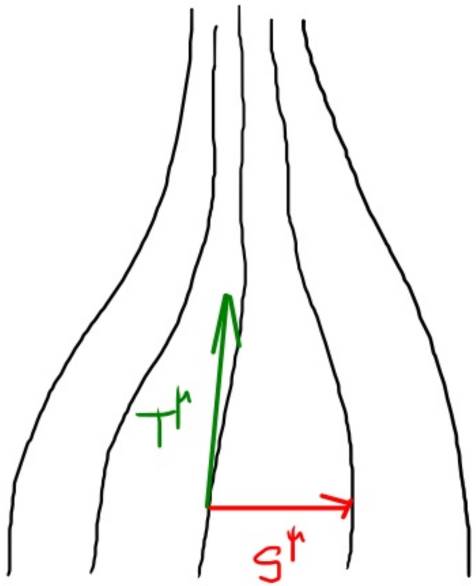
  t: affine parameter

  $s \in \mathbb{R}$



- consider a small enough open set where they don't cross
- assume they form a 2-d surface w/coordinates (s,t)

Geodesic Deviation



- Consider a one-parameter family of geodesics $\gamma_s(t)$

  t: affine parameter

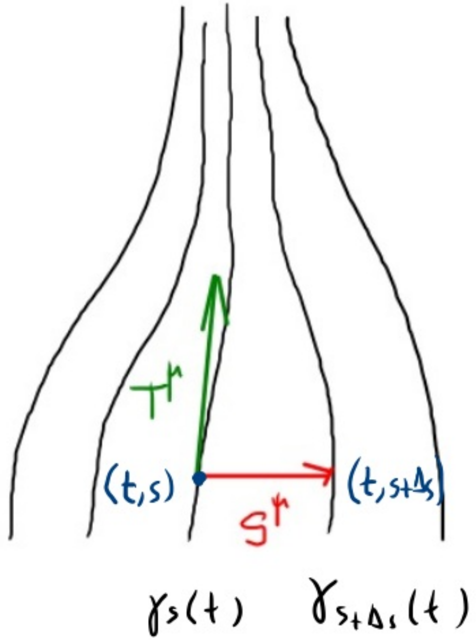
  $s \in \mathbb{R}$

- consider a small enough open set where they don't cross
- assume they form a 2-d surface w/coordinates (s,t)

$$\Rightarrow \quad T^\mu = \partial_t \quad \text{tangent vectors, s.t. } T^\nu \nabla_\nu T^\mu = 0$$

↳ because tangent to geodesics!

Geodesic Deviation



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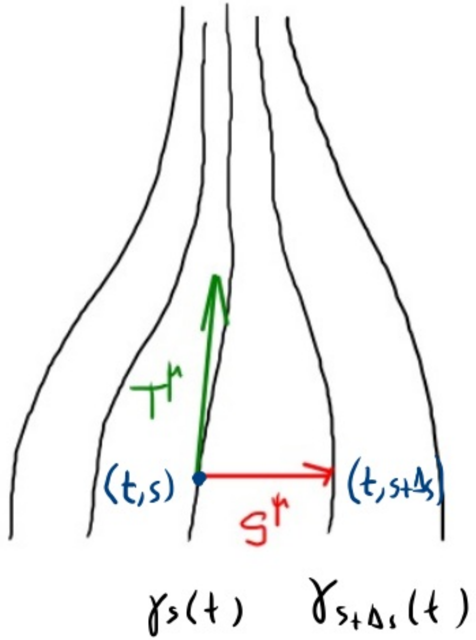
  $\Rightarrow$
  - $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$
  - $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at same t

Geodesic Deviation

- Consider a one-parameter family of geodesics $\gamma_s(t)$
  t: affine parameter
  $s \in \mathbb{R}$



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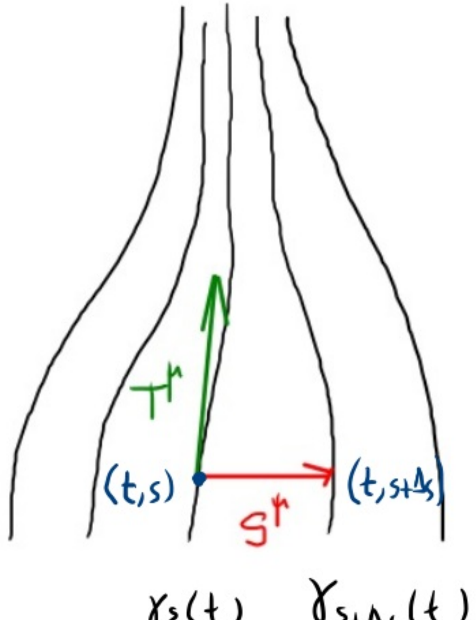
- $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ ← coordinate vectors condition

Geodesic Deviation

$D_T S^\mu$ = " relative velocity"



$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$

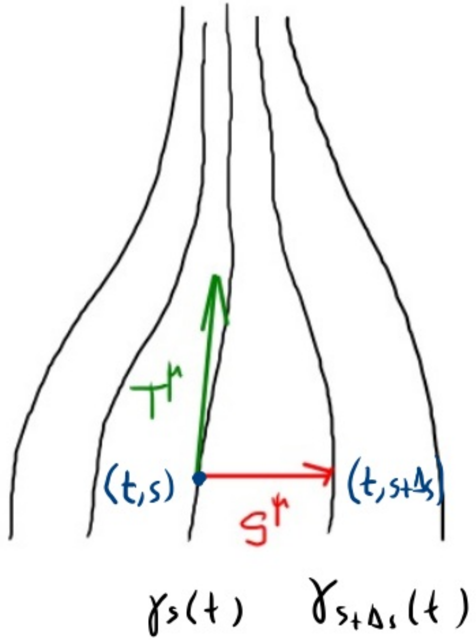
• $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at same t

• $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ $\leftarrow$ coordinate vectors condition

Geodesic Deviation

$$D_T S^\mu = \text{“ relative velocity”}$$

$$= T^\nu \nabla_\nu S^\mu \overset{(2)}{=} (\nabla_\nu T^\mu) S^\nu$$

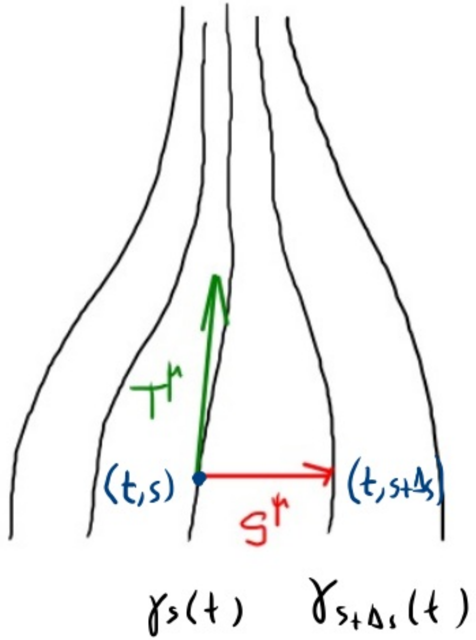


$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$ (1)

• $S^\mu = \partial_s$ deviation vectors: point to $\gamma_{s+\Delta s}(t)$ at same t

• $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ $\leftarrow$ coordinate vectors condition (2)

Geodesic Deviation



$$D_T S^\mu = \text{“ relative velocity”}$$

$$= T^\nu \nabla_\nu S^\mu \overset{(2)}{=} (\nabla_\nu T^\mu) S^\nu$$

$$\equiv B^\mu{}_\nu S^\nu , \quad B^\mu{}_\nu \equiv \nabla_\nu T^\mu$$

$\Rightarrow$ • $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$ (1)

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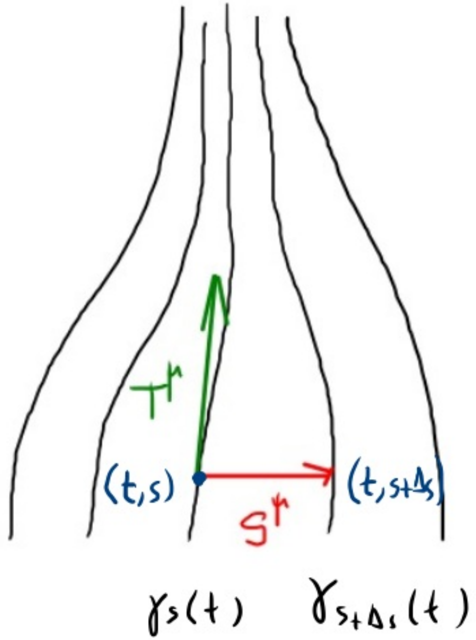
• $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ $\leftarrow$ coordinate condition vectors (2)

Geodesic Deviation

$$D_T S^\mu = \text{“ relative velocity”}$$

$$= T^\nu \nabla_\nu S^\mu \overset{(2)}{=} (\nabla_\nu T^\mu)\, S^\nu$$

$$\equiv B^\mu{}_\nu S^\nu \,, \quad B^\mu{}_\nu \equiv \nabla_\nu T^\mu$$



- linear xfm of S^μ
- failure of S^μ to be parallel transported $\Rightarrow$ (failure of neighboring geodesics to remain parallel)

$\Rightarrow$
- $T^\mu = \partial_t$ tangent vectors, s.t. $T^\nu \nabla_\nu T^\mu = 0$ (1)
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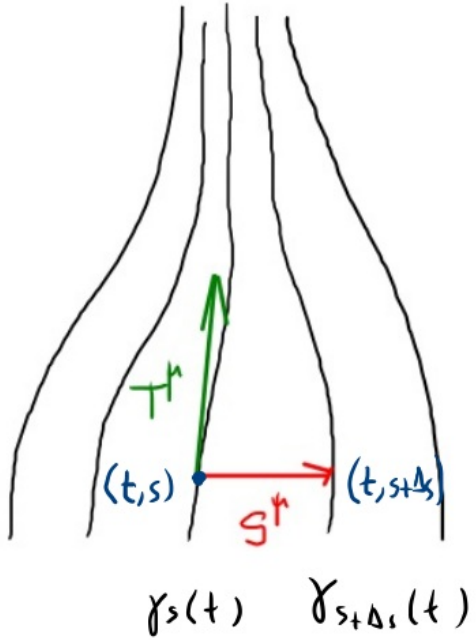
- $[S,T]^\mu = 0 \Leftrightarrow S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu$ $\Leftarrow$ coordinate vectors condition (2)

Geodesic Deviation

- relative velocity: $V^\mu \equiv D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu \equiv D_T V^\mu = T^\rho \nabla_\rho V^\mu$



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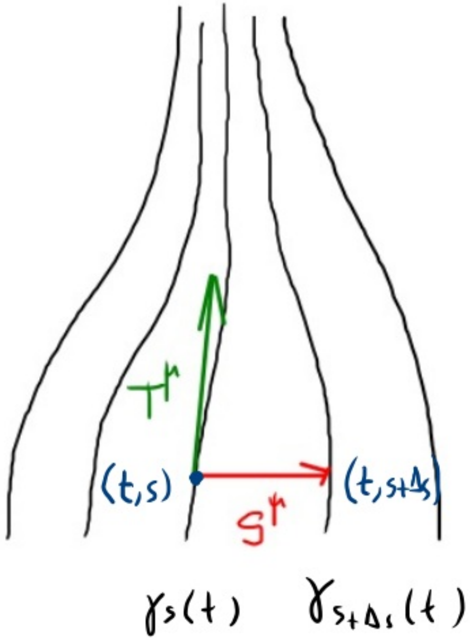
Geodesic Deviation

- relative velocity: $V^\mu \equiv D_T S^\mu = T^\rho \nabla_\rho S^\mu$
- relative acceleration: $A^\mu \equiv D_T V^\mu = T^\rho \nabla_\rho V^\mu$

$$A^\mu = T^\rho \nabla_\rho V^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu)$$



$$T^\nu \nabla_\nu T^\mu = 0 \quad (1)$$

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu \quad (2)$$

Geodesic Deviation

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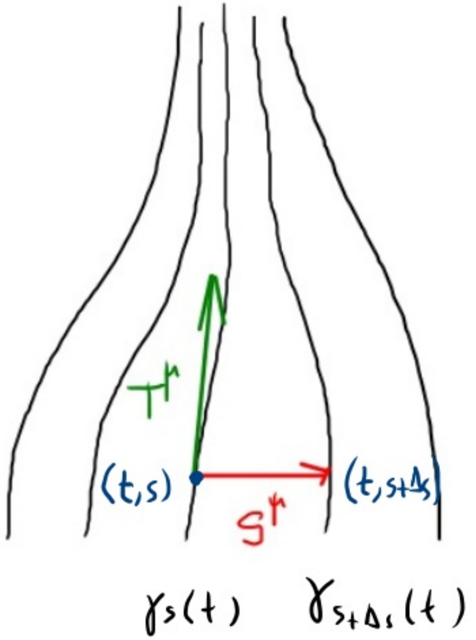
$$\overset{(2)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$



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Geodesic Deviation

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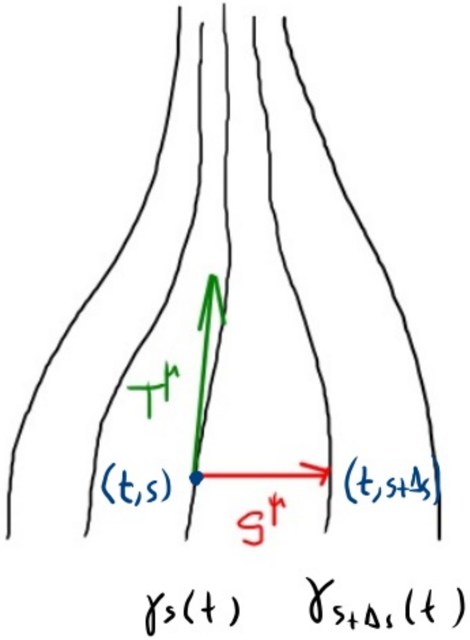
$$\overset{(2)}{=} T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu)$$

$$= (T^\rho \nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$



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Geodesic Deviation

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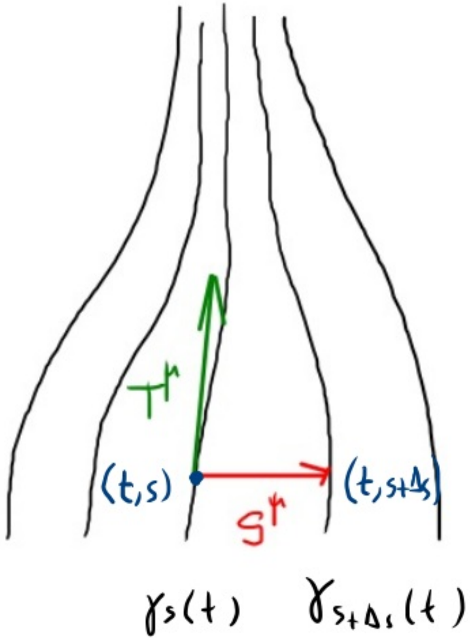

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$$= (T^\rho \nabla_\rho S^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu$$

$\Downarrow$ (2)

$$= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

Geodesic Deviation

- relative velocity: $V^\mu \equiv D_T S^\mu = T^\rho \nabla_\rho S^\mu$
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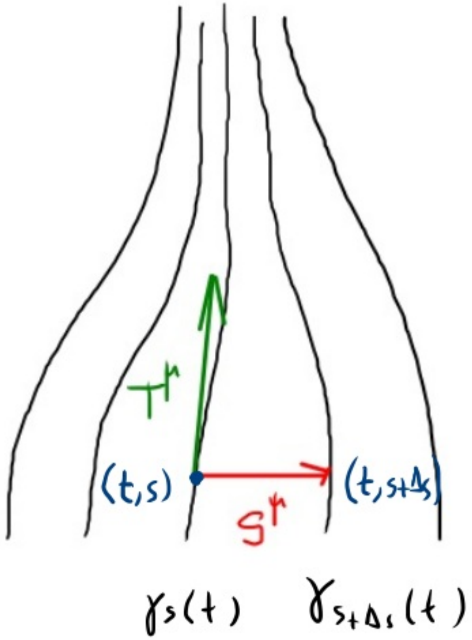

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$$\Downarrow (2)$$

$$= (S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{\nabla_\sigma [T^\rho \nabla_\rho T^\mu]\} - S^\sigma \{\nabla_\sigma T^\rho \nabla_\rho T^\mu\}} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

Geodesic Deviation

- relative velocity: $V^\mu \equiv D_T S^\mu = T^\rho \nabla_\rho S^\mu$
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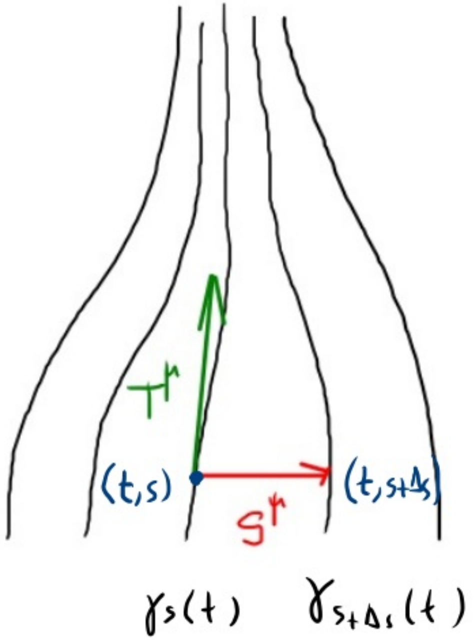

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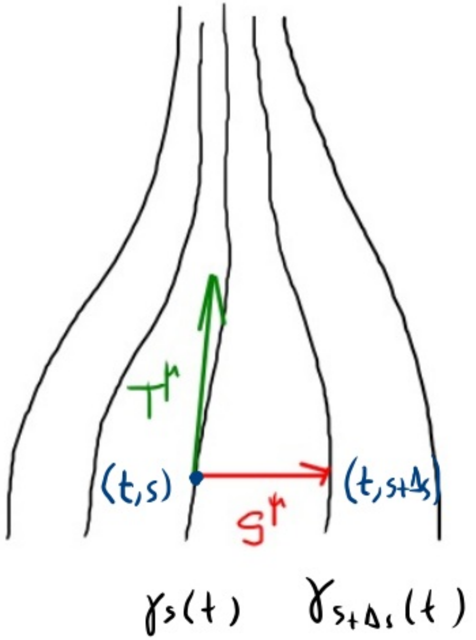
$$\Downarrow (2)$$

$$= \cancel{(S^\rho \nabla_\rho T^\sigma) \nabla_\sigma} T^\mu + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{ \nabla_\sigma [T^\rho \cancel{\nabla_\rho} T^\mu] \} - S^\rho \{ \nabla_\rho \cancel{T^\sigma} \nabla_\sigma T^\mu \}} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

parallel transported, eq. (1)

Geodesic Deviation

$$\Rightarrow A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$



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$$\Downarrow (2)$$

$$= \cancel{(S^\rho \nabla_\rho T^\sigma) \nabla_\sigma T^\mu} + \underbrace{T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu}_{S^\sigma \{\nabla_\sigma [T^\rho \cancel{\nabla_\rho T^\mu}]\} - S^\rho \{\nabla_\rho \cancel{T^\sigma} \nabla_\sigma T^\mu\}} + R^\mu{}_{\nu\rho\sigma} T^\nu)$$

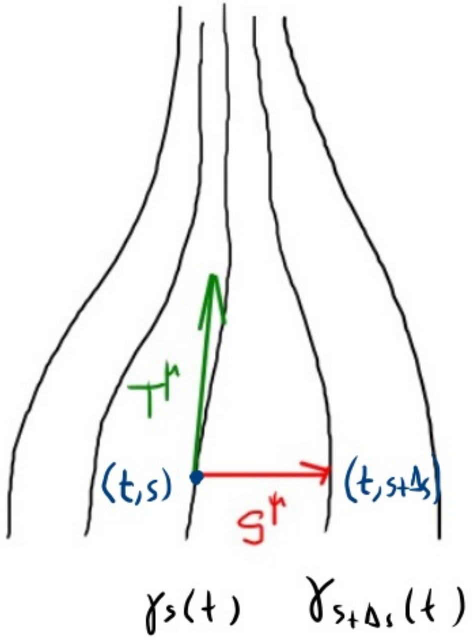
parallel transported, eq (1)

Geodesic Deviation

$$\Rightarrow A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

geodesic deviation equation

$$\begin{pmatrix} \text{relative} \\ \text{acceleration} \end{pmatrix} \propto R$$



$$T^\nu \nabla_\nu T^\mu = 0 \quad (1)$$

$$S^\rho \nabla_\rho T^\mu = T^\rho \nabla_\rho S^\mu \quad (2)$$

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$$S^\sigma \{ \nabla_\sigma [\cancel{T^\rho \nabla_\rho T^\mu}] \} - S^\rho \{ \nabla_\rho \cancel{T^\sigma} \nabla_\sigma T^\mu \}$$

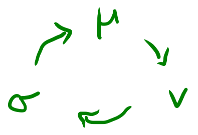
parallel transported, eq. (1)

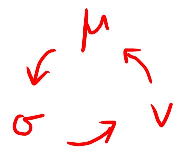
Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$ (torsion free)

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \frac{1}{3!}\Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma}$$

$$- R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma} \Big) = 0$$





Exercise:

Prove $R^\rho{}_{[\sigma\mu\nu]} = 0$

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \cancel{\frac{1}{3!}}\Big(R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} - R^\rho{}_{\sigma\nu\mu} - R^\rho{}_{\mu\sigma\nu} - R^\rho{}_{\nu\mu\sigma} \Big) = 0$$

(use $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}$)

$$\Leftrightarrow R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

<u>Exercise:</u> Prove $R^\rho{}_{[\sigma\mu\nu]} = 0$

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \left(R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} - R^\rho{}_{\sigma\nu\mu} - R^\rho{}_{\mu\sigma\nu} - R^\rho{}_{\nu\mu\sigma} \right) = 0$$

$$\Leftrightarrow R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Choose inertial frame: $\Gamma^\mu{}_{\nu\rho} = 0$ at P (torsion free)

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma}$$

$$R^\rho{}_{\nu\sigma\mu} = \partial_\sigma \Gamma^\rho{}_{\mu\nu} - \partial_\mu \Gamma^\rho{}_{\sigma\nu}$$

$$R^\rho{}_{\mu\nu\sigma} = \partial_\nu \Gamma^\rho{}_{\sigma\mu} - \partial_\sigma \Gamma^\rho{}_{\nu\mu}$$

Exercise: Prove $R^\rho{}_{[\sigma\mu\nu]} = 0$

$$R^\rho{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \left(R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma} - R^\rho{}_{\sigma\nu\mu} - R^\rho{}_{\mu\sigma\nu} - R^\rho{}_{\nu\mu\sigma} \right) = 0$$

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Choose inertial frame: $\Gamma^\mu{}_{\nu\rho} = 0$ at P (torsion free)

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \cancel{\Gamma^\rho{}_{\nu\sigma}} - \partial_\nu \cancel{\Gamma^\rho{}_{\mu\sigma}} +$$

$$R^\rho{}_{\nu\sigma\mu} = \partial_\sigma \cancel{\Gamma^\rho{}_{\mu\nu}} - \partial_\mu \cancel{\Gamma^\rho{}_{\sigma\nu}} +$$

$$R^\rho{}_{\mu\nu\sigma} = \partial_\nu \cancel{\Gamma^\rho{}_{\sigma\mu}} - \partial_\sigma \cancel{\Gamma^\rho{}_{\nu\mu}} +$$

Exercise: Prove $R^{\rho}{}_{[\sigma\mu\nu]} = 0$

$$R^{\rho}{}_{[\sigma\mu\nu]} = 0 \Leftrightarrow \Big(R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R^{\rho}{}_{\mu\nu\sigma} - R^{\rho}{}_{\sigma\nu\mu} - R^{\rho}{}_{\mu\sigma\nu} - R^{\rho}{}_{\nu\mu\sigma} \Big) = 0$$

$$\Leftrightarrow R^{\rho}{}_{\sigma\mu\nu} + R^{\rho}{}_{\nu\sigma\mu} + R_{\mu\nu\sigma} = 0$$

Choose inertial frame: $\Gamma^{\mu}{}_{\nu\rho} = 0$ at P (torsion free)

$$\left.\begin{aligned} R^{\rho}{}_{\sigma\mu\nu} &= \partial_{\mu}\cancel{\Gamma^{\rho}{}_{\nu\sigma}} - \partial_{\nu}\cancel{\Gamma^{\rho}{}_{\mu\sigma}} + \\ R^{\rho}{}_{\nu\sigma\mu} &= \partial_{\sigma}\cancel{\Gamma^{\rho}{}_{\mu\nu}} - \partial_{\mu}\cancel{\Gamma^{\rho}{}_{\sigma\nu}} + \\ R^{\rho}{}_{\mu\nu\sigma} &= \partial_{\nu}\cancel{\Gamma^{\rho}{}_{\sigma\mu}} - \partial_{\sigma}\cancel{\Gamma^{\rho}{}_{\nu\mu}} + \end{aligned}\right\} = 0$$

If a tensor is 0 at one frame, it is 0 at all frames!

Exercise: Prove $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection $\nabla g = 0$ + torsion free)

<u>Exercise:</u> Prove $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection $\nabla g = 0$ + torsion free)

$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, g_{\rho\sigma}$$ (metric compatibility)

<u>Exercise:</u> Prove $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ (Christoffel connection $\nabla g = 0$ + torsion free)

$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, g_{\rho\sigma}$$

$$= -R^\lambda{}_{\rho\mu\nu}\, g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}$ (Christoffel connection $\nabla g = 0$ + torsion free)

$$0 = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\, g_{\rho\sigma}$$

$$= - R^\lambda{}_{\rho\mu\nu}\, g_{\lambda\sigma} - R^\lambda{}_{\sigma\mu\nu}\, g_{\rho\lambda}$$

$$= - R_{\sigma\rho\mu\nu} - R_{\rho\sigma\mu\nu}$$

$$\Rightarrow \quad R_{\rho\sigma\mu\nu} = - R_{\sigma\rho\mu\nu}$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

<u>Exercise:</u> Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \quad +$$

$$R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \quad \oplus$$

$$R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow R_{\mu\nu\sigma\rho} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + R_{\nu\mu\sigma\rho} + R_{\nu\rho\mu\sigma} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + \cancel{R_{\rho\sigma\mu\nu}} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow \cancel{R_{\sigma\rho\mu\nu}} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \quad \oplus$$

$$R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} + R_{\nu\sigma\rho\mu} + R_{\nu\rho\mu\sigma} = 0$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$R^{\rho}{}_{[\mu\nu\sigma]} = 0 \Rightarrow R_{\rho\mu\nu\sigma} + R_{\rho\sigma\mu\nu} + R_{\rho\nu\sigma\mu} = 0$$

$$R^{\sigma}{}_{[\rho\mu\nu]} = 0 \Rightarrow R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0 \quad \oplus$$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \quad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow R_{\mu\nu\sigma\rho} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + R_{\nu\mu\sigma\rho} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \quad (2)$$

Exercise: Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$(1)+(2) \Rightarrow \quad 2R_{\rho\nu\sigma\mu} + 2R_{\sigma\mu\nu\rho} = 0$$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \quad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \quad \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow \quad R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \quad (2)$$

<u>Exercise:</u> Prove $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

$$(1)+(2) \Rightarrow \cancel{2}\, R_{\rho\nu\sigma\mu} + \cancel{2}\, R_{\sigma\mu\nu\rho} = 0 \Rightarrow R_{\rho\nu\sigma\mu} = R_{\sigma\mu\rho\nu}$$

$$\cancel{R_{\rho\mu\nu\sigma}} + R_{\rho\nu\sigma\mu} + \cancel{R_{\sigma\nu\rho\mu}} + R_{\sigma\mu\nu\rho} = 0 \quad (1)$$

$$R^{\mu}{}_{[\nu\sigma\rho]} = 0 \Rightarrow \cancel{R_{\mu\nu\sigma\rho}} + R_{\mu\rho\nu\sigma} + R_{\mu\sigma\rho\nu} = 0$$

$$R^{\nu}{}_{[\sigma\rho\mu]} = 0 \Rightarrow R_{\nu\sigma\rho\mu} + \cancel{R_{\nu\mu\sigma\rho}} + R_{\nu\rho\mu\sigma} = 0 \quad \oplus$$

$$\cancel{R_{\mu\rho\nu\sigma}} + R_{\mu\sigma\rho\nu} + \cancel{R_{\nu\sigma\rho\mu}} + R_{\nu\rho\mu\sigma} = 0 \quad (2)$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \iff \frac{1}{3!}\Big(\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu}$$

$$- \nabla_{\lambda} R_{\sigma\rho\mu\nu} - \nabla_{\rho} R_{\lambda\sigma\mu\nu} - \nabla_{\sigma} R_{\rho\lambda\mu\nu} \Big) = 0$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \Leftrightarrow \cancel{\frac{1}{3!}}\Big(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}$$

$$- \nabla_\lambda R_{\sigma\rho\mu\nu} - \nabla_\rho R_{\lambda\sigma\mu\nu} - \nabla_\sigma R_{\rho\lambda\mu\nu} \Big) = 0$$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$\begin{tikzcd} & \rho \arrow[dr] & \\ \lambda \arrow[ur] & & \sigma \arrow[ll] \end{tikzcd}$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Longleftrightarrow \quad \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$\nabla_\mu \quad \nabla_\nu \quad \nabla_\sigma$

$\nabla_\sigma \quad \nabla_\mu \quad \nabla_\nu$

$\nabla_\nu \quad \nabla_\sigma \quad \nabla_\mu$

$$\begin{tikzcd} & \nu \arrow[dr] & \\ \mu \arrow[ur] & & \sigma \arrow[ll] \end{tikzcd}$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$\iff \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$[\nabla_\mu, \nabla_\nu] \nabla_\sigma$

$[\nabla_\sigma, \nabla_\mu] \nabla_\nu$

$[\nabla_\nu, \nabla_\sigma] \nabla_\mu$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma]$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Longleftrightarrow \quad \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu] \nabla_\sigma - \nabla_\sigma [\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

__Exercise:__ Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu]$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\iff \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu \nabla_\nu \nabla_\sigma - \nabla_\nu \nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu \nabla_\nu + \nabla_\sigma \nabla_\nu \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma \nabla_\mu \nabla_\nu - \nabla_\mu \nabla_\sigma \nabla_\nu - \nabla_\nu \nabla_\sigma \nabla_\mu + \nabla_\nu \nabla_\mu \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu[\nabla_\nu, \nabla_\sigma]$$

<u>Exercise:</u> Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] = [\nabla_\mu, \nabla_\nu]\nabla_\sigma - \nabla_\sigma[\nabla_\mu, \nabla_\nu]$$

$$= \nabla_\mu\nabla_\nu\nabla_\sigma - \nabla_\nu\nabla_\mu\nabla_\sigma - \nabla_\sigma\nabla_\mu\nabla_\nu + \nabla_\sigma\nabla_\nu\nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] = [\nabla_\sigma, \nabla_\mu]\nabla_\nu - \nabla_\nu[\nabla_\sigma, \nabla_\mu]$$

$$= \nabla_\sigma\nabla_\mu\nabla_\nu - \nabla_\mu\nabla_\sigma\nabla_\nu - \nabla_\nu\nabla_\sigma\nabla_\mu + \nabla_\nu\nabla_\mu\nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = [\nabla_\nu, \nabla_\sigma]\nabla_\mu - \nabla_\mu[\nabla_\nu, \nabla_\sigma]$$

$$= \nabla_\nu\nabla_\sigma\nabla_\mu - \nabla_\sigma\nabla_\nu\nabla_\mu - \nabla_\mu\nabla_\nu\nabla_\sigma + \nabla_\mu\nabla_\sigma\nabla_\nu$$

Exercise: Prove $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

$$\Leftrightarrow \quad \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] =$$

$$= \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma - \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma - \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu + \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] =$$

$$= \nabla_\sigma \cancel{\nabla_\mu} \nabla_\nu - \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu - \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu + \nabla_\nu \cancel{\nabla_\mu} \nabla_\sigma$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] =$$

$$= \nabla_\nu \cancel{\nabla_\sigma} \nabla_\mu - \nabla_\sigma \cancel{\nabla_\nu} \nabla_\mu - \nabla_\mu \cancel{\nabla_\nu} \nabla_\sigma + \nabla_\mu \cancel{\nabla_\sigma} \nabla_\nu$$

$+$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] + [[\nabla_\sigma, \nabla_\mu], \nabla_\nu] + [[\nabla_\nu, \nabla_\sigma], \nabla_\mu] = 0$$

Jacobi Identity

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu,\nabla_\nu],\nabla_\sigma]V^\rho = [\nabla_\mu,\nabla_\nu]\underbrace{\nabla_\sigma V^\rho} - \nabla_\sigma \underbrace{[\nabla_\mu,\nabla_\nu]V^\rho}$$

$$= \overbrace{-R^\lambda{}_{\sigma\mu\nu}\nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu}\nabla_\sigma V^\lambda}^{(1,1)\text{ tensor}} - \nabla_\sigma\left(R^\rho{}_{\lambda\mu\nu}V^\lambda\right)$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= - R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

$$= - R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - (\nabla_\sigma R^\rho{}_{\lambda\mu\nu}) V^\lambda - R^\rho{}_{\lambda\mu\nu} (\nabla_\sigma V^\lambda)$$

<u>Exercise:</u> Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = [\nabla_\mu, \nabla_\nu] \nabla_\sigma V^\rho - \nabla_\sigma [\nabla_\mu, \nabla_\nu] V^\rho$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda - \nabla_\sigma (R^\rho{}_{\lambda\mu\nu} V^\lambda)$$

$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho + \cancel{R^\rho{}_{\lambda\mu\nu} \nabla_\sigma V^\lambda} - (\nabla_\sigma R^\rho{}_{\lambda\mu\nu}) V^\lambda - \cancel{R^\rho{}_{\lambda\mu\nu} (\nabla_\sigma V^\lambda)}$$

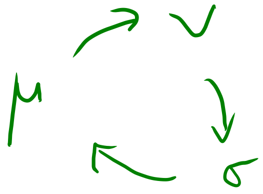
$$= -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_{\ }, \nabla_{\ }], \nabla_{\ }] V^\rho = -R^\lambda{}_{\ } \nabla_\lambda V^\rho - \nabla_{\ } R^\rho{}_{\lambda} V^\lambda$$

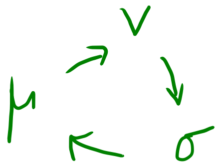
$$[[\nabla_{\ }, \nabla_{\ }], \nabla_{\ }] V^\rho = -R^\lambda{}_{\ } \nabla_\lambda V^\rho - \nabla_{\ } R^\rho{}_{\lambda} V^\lambda$$



__Exercise:__ Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

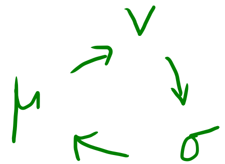


$$[[\nabla, \nabla], \nabla] V^\rho = -R^\lambda \quad \nabla_\lambda V^\rho - \nabla \, R^\rho{}_\lambda \quad V^\lambda$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$



$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

use $R^\lambda{}_{[\mu\nu\sigma]} = 0$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0\, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma}\right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \nabla_\sigma R_{\mu\nu\rho\lambda} + \nabla_\nu R_{\sigma\mu\rho\lambda} + \nabla_\mu R_{\nu\sigma\rho\lambda} = 0$$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = - R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = - R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = - R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda \qquad +$$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \underline{\nabla_\sigma R_{\mu\nu\rho\lambda}} + \underline{\nabla_\nu R_{\sigma\mu\rho\lambda}} + \underline{\nabla_\mu R_{\nu\sigma\rho\lambda}} = 0$$

$\sigma \to \mu \to \nu \to \sigma$

Exercise: Prove $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$

$$[[\nabla_\mu, \nabla_\nu], \nabla_\sigma] V^\rho = -R^\lambda{}_{\sigma\mu\nu} \nabla_\lambda V^\rho - \nabla_\sigma R^\rho{}_{\lambda\mu\nu} V^\lambda$$

$$[[\nabla_\sigma, \nabla_\mu], \nabla_\nu] V^\rho = -R^\lambda{}_{\nu\sigma\mu} \nabla_\lambda V^\rho - \nabla_\nu R^\rho{}_{\lambda\sigma\mu} V^\lambda$$

$$[[\nabla_\nu, \nabla_\sigma], \nabla_\mu] V^\rho = -R^\lambda{}_{\mu\nu\sigma} \nabla_\lambda V^\rho - \nabla_\mu R^\rho{}_{\lambda\nu\sigma} V^\lambda$$

$+$

$$0 = -0 \, \nabla_\lambda V^\rho - \left(\nabla_\sigma R^\rho{}_{\lambda\mu\nu} + \nabla_\nu R^\rho{}_{\lambda\sigma\mu} + \nabla_\mu R^\rho{}_{\lambda\nu\sigma} \right) V^\lambda$$

$$\Rightarrow \nabla_\sigma R_{\rho\lambda\mu\nu} + \nabla_\nu R_{\rho\lambda\sigma\mu} + \nabla_\mu R_{\rho\lambda\nu\sigma} = 0$$

$$\Rightarrow \underline{\nabla_\sigma R_{\mu\nu\rho\lambda}} + \underline{\nabla_\nu R_{\sigma\mu\rho\lambda}} + \underline{\nabla_\mu R_{\nu\sigma\rho\lambda}} = 0 \Rightarrow \nabla_{[\sigma} R_{\mu\nu]\rho\lambda} = 0$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma}$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2}$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \to$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \to \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

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$$R_{\mu[\nu\rho\sigma]} = 0$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

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$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!}$$

3-combination of n objects

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \to$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \to \frac{n(n-1)}{2} \cdot \frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \to \frac{n^2(n-1)(n-2)}{6} \quad \text{conditions}$$

<u>Exercise</u>: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \to$$

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$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \quad \frac{n(n-1)(n-2)}{3!} \to \frac{n^2(n-1)(n-2)}{6} \quad \text{conditions}$$

$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\frac{(n+1)}{6} = \frac{n^2(n^2-1)}{12}$$

Exercise: Count independent components of Riemann (Levi-Civita connection)

$$R_{[\mu\nu][\rho\sigma]} = R_{\mu\nu\rho\sigma} \rightarrow$$

$$\frac{n(n-1)}{2} \qquad \frac{n(n-1)}{2} \rightarrow \frac{n(n-1)}{2}\cdot\frac{n(n-1)}{2} = \left[\frac{n(n-1)}{2}\right]^2$$

$$R_{\mu[\nu\rho\sigma]} = 0$$

$$n \qquad \frac{n(n-1)(n-2)}{3!} \rightarrow \frac{n^2(n-1)(n-2)}{6} \quad \text{conditions}$$

other symmetries

$$R_{[\mu\nu\rho\sigma]} = 0$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

not independent!

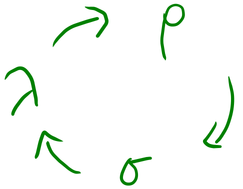
$$\frac{n^2(n-1)^2}{4} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n-1)}{2}\left[\frac{n-1}{2} - \frac{n-2}{3}\right] = \frac{n^2(n-1)}{2}\frac{(n+1)}{6} = \frac{n^2(n^2-1)}{12}$$

Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Corrolary:
- we can't have $R_{\mu\nu} = 8\pi T_{\mu\nu}$
- $\nabla^\mu G_{\rho\mu} = 0$ and we may have $G_{\mu\nu} = 8\pi T_{\mu\nu}$

Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0$



<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

Bianchi id: $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} \right)$$

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$$= g^{\mu\lambda} \nabla_\lambda \left(g^{\nu\sigma} R_{\rho\sigma\mu\nu} \right) + g^{\nu\sigma} \nabla_\sigma \left(g^{\mu\lambda} R_{\lambda\rho\mu\nu} \right) + \nabla_\rho \left(g^{\nu\sigma} g^{\mu\lambda} R_{\sigma\lambda\mu\nu} \right)$$

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$(-)\cdot(-) = (+)$

$\downarrow$

$g^{\nu\sigma} R_{\sigma\rho\nu\mu}$

$\downarrow$

$R^{\nu}{}_{\rho\nu\mu}$

$\downarrow$

$R_{\rho\mu}$

Exercise: Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$$g^{\nu\sigma} R_{\rho\sigma\mu\nu} \to g^{\nu\sigma} R_{\sigma\rho\nu\mu} \to R^\nu{}_{\rho\nu\mu} \to R_{\rho\mu}$$

$$g^{\mu\lambda} R_{\lambda\rho\mu\nu} \to R^\mu{}_{\rho\mu\nu} \to R_{\rho\nu}$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

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$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
g^{\nu\sigma} R_{\sigma\rho\nu\mu} & R^{\mu}{}_{\rho\mu\nu} & g^{\mu\lambda} R^{\nu}{}_{\lambda\mu\nu} \\
\downarrow & \downarrow & \downarrow \; (-) \\
R^{\nu}{}_{\rho\nu\mu} & R_{\rho\nu} & -g^{\mu\lambda} R^{\nu}{}_{\lambda\nu\mu} \\
\downarrow & & \downarrow \\
R_{\rho\mu} & & -g^{\mu\lambda} R_{\lambda\mu} \\
 & & \downarrow \\
 & & -R
\end{array}$$

Exercise: Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

Bianchi id: $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} \right)$$

$$= g^{\mu\lambda} \nabla_{\lambda} \left(g^{\nu\sigma} R_{\rho\sigma\mu\nu} \right) + g^{\nu\sigma} \nabla_{\sigma} \left(g^{\mu\lambda} R_{\lambda\rho\mu\nu} \right) + \nabla_{\rho} \left(g^{\nu\sigma} g^{\mu\lambda} R_{\sigma\lambda\mu\nu} \right)$$

$$= g^{\mu\lambda} \nabla_{\lambda} R_{\rho\mu} + g^{\nu\sigma} \nabla_{\sigma} R_{\rho\nu} - \nabla_{\rho} R$$

$\downarrow$ $R_{\rho\nu}$

$\downarrow$ $R_{\rho\mu}$

$\downarrow$ $-R$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

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$$= g^{\mu\lambda} \nabla_\lambda R_{\rho\mu} + g^{\nu\sigma} \nabla_\sigma R_{\rho\nu} - \nabla_\rho R$$

$$= \nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\rho\nu} - \nabla_\rho R$$

$$= 2 \nabla^\mu R_{\rho\mu} - \nabla_\rho R$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \neq 0$

Bianchi id: $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} = 0 \Rightarrow$

$$0 = g^{\nu\sigma} g^{\mu\lambda} \left(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} \right)$$

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$$= g^{\mu\lambda} \nabla_\lambda R_{\rho\mu} + g^{\nu\sigma} \nabla_\sigma R_{\rho\nu} - \nabla_\rho R$$

$$= \nabla^\mu R_{\rho\mu} + \nabla^\nu R_{\rho\nu} - \nabla_\rho R$$

$$= 2 \nabla^\mu R_{\rho\mu} - \nabla_\rho R \Rightarrow \nabla^\mu R_{\rho\mu} = \frac{1}{2} \nabla_\rho R$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

Then $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu}\left(\frac{1}{2} g_{\mu\nu} R\right)$$

Exercise: Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2} \nabla_{\rho} R \neq 0$

Then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu} \left(\frac{1}{2} g_{\mu\nu} R \right)$$

$$= \frac{1}{2} \nabla_{\nu} R - \frac{1}{2} g_{\mu\nu} \nabla^{\mu} R$$

<u>Exercise:</u> Prove that the Bianchi identities imply $\nabla^{\mu} R_{\rho\mu} = \frac{1}{2}\nabla_{\rho} R \neq 0$

Then $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow$$

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \nabla^{\mu}\left(\frac{1}{2} g_{\mu\nu} R\right)$$

$$= \frac{1}{2} \nabla_{\nu} R - \frac{1}{2} g_{\mu\nu} \nabla^{\mu} R$$

$$= \frac{1}{2} \nabla_{\nu} R - \frac{1}{2} \nabla_{\nu} R \quad = 0$$