Stokes' Theorem

- Manifolds with boundary
- Induced Orientation
- Stokes' theorem
- Applications

<u>References:</u>

A. Knapp, Stokes Theorem and Whitney Manifold, Ch 2
projecteuclid.org (free)

Schutz § 4.22 – 4.23

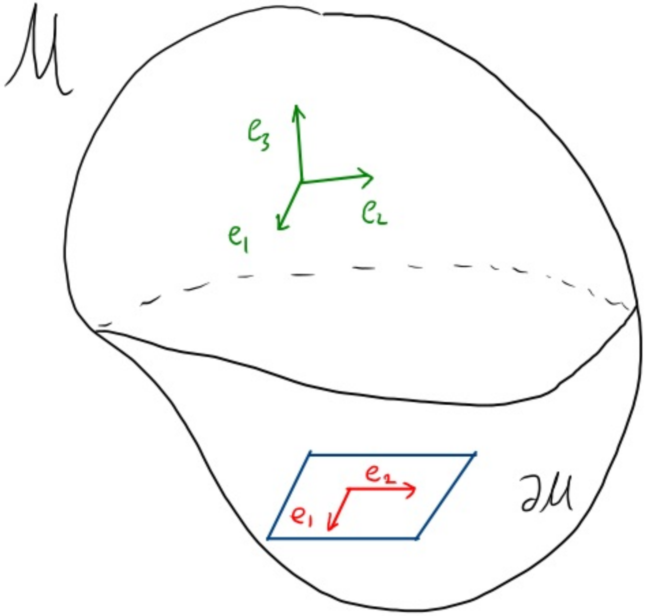
Stokes' theorem:

$$\int_M d\omega = \int_{\partial M} \omega$$

ω : $(n-1)$-form

$d\omega$: n-form

w/ compact support



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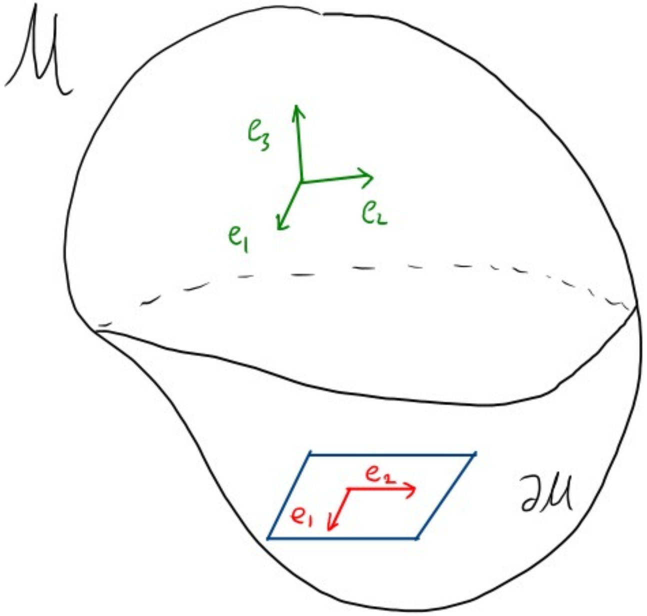


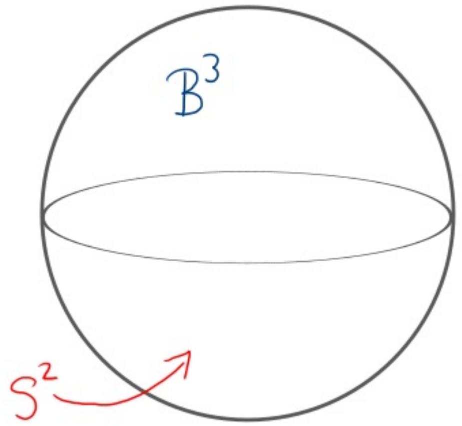
- What is a manifold with boundary?
- How is $\int_{\partial M} \omega$ defined?
  - orientation on ∂M
  - volume element on ∂M
  - restriction of ω on ∂M

Manifolds with boundary

examples:

$$B^3 = \{ x \in \mathbb{R}^3 \mid |x| \leq 1 \}$$

$$\partial B^3 = S^2 = \{ x \in \mathbb{R}^3 \mid |x| = 1 \}$$



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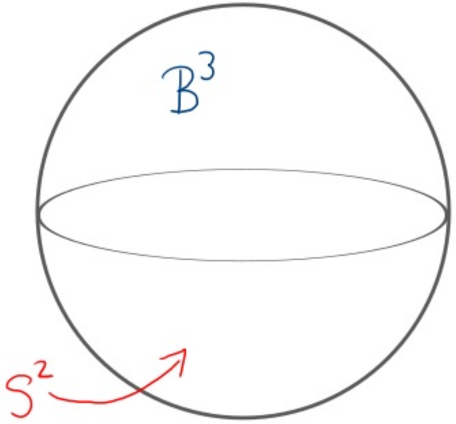


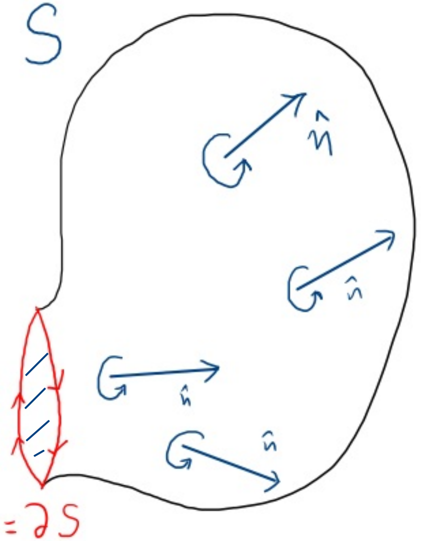
S : a 2-dim surface

$c = \partial S$: a closed curve

$S \setminus \partial S$: a differentiable manifold

S : a manifold w/boundary

∂S : the boundary



S: oriented

∂S: has an induced orientation

Manifolds with boundary

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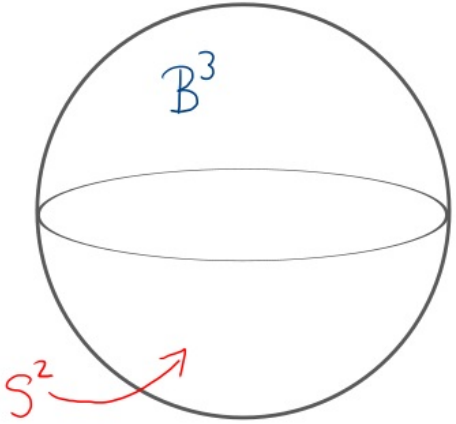

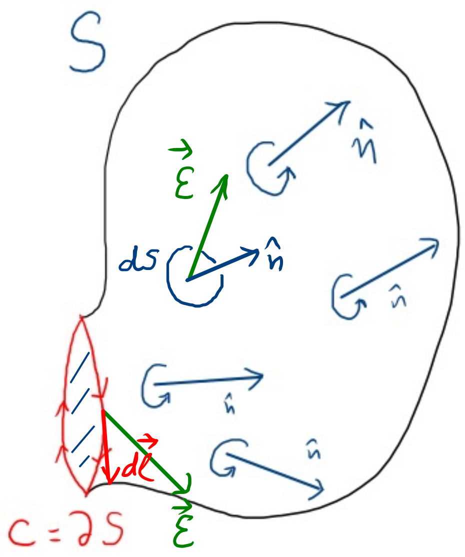

Stokes' Thm:

$$\int_S \vec{\nabla} \times \vec{\mathcal{E}} \cdot \hat{n} \, dS = \oint_C \vec{\mathcal{E}} \cdot \vec{dl}$$

relative orientation important

S: oriented

∂S: has an induced orientation, so that Stokes' thm has correct sign

Manifolds with boundary

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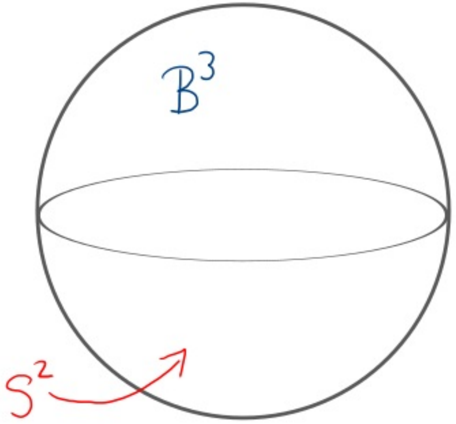

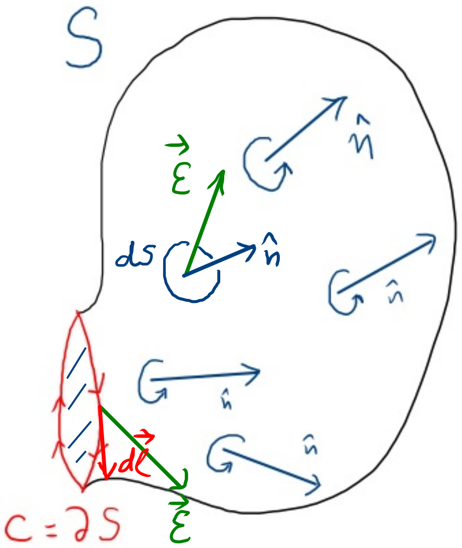

Boundaries have no boundaries!

Stokes' Thm:

$$\int_S \vec{\nabla} \times \vec{\varepsilon} \cdot \hat{n} \, dS = \oint_c \vec{\varepsilon} \cdot d\vec{l}$$

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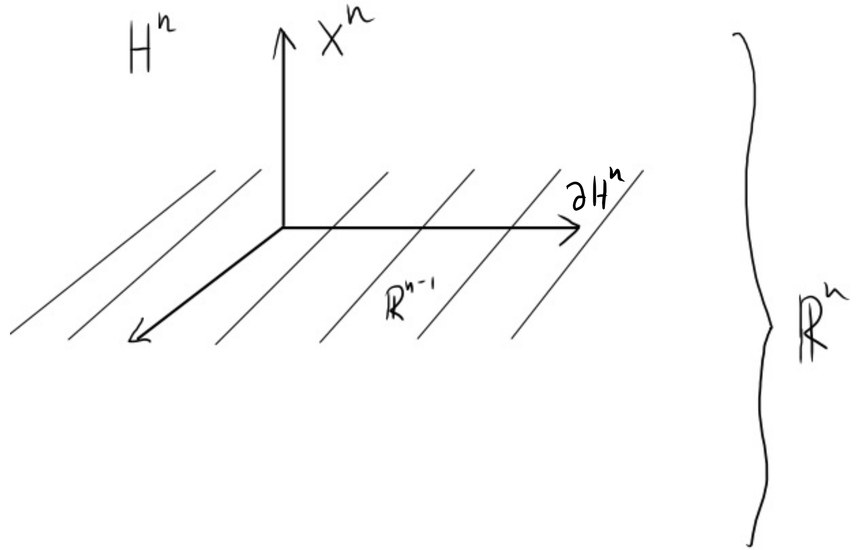
Preliminaries

$$H^n = \{ x \in \mathbb{R}^n \mid x^n \geq 0 \}$$

$$H^n_+ = \{ x \in \mathbb{R}^n \mid x^n > 0 \}$$

$$\partial H^n = \{ x \in \mathbb{R}^n \mid x^n = 0 \}$$



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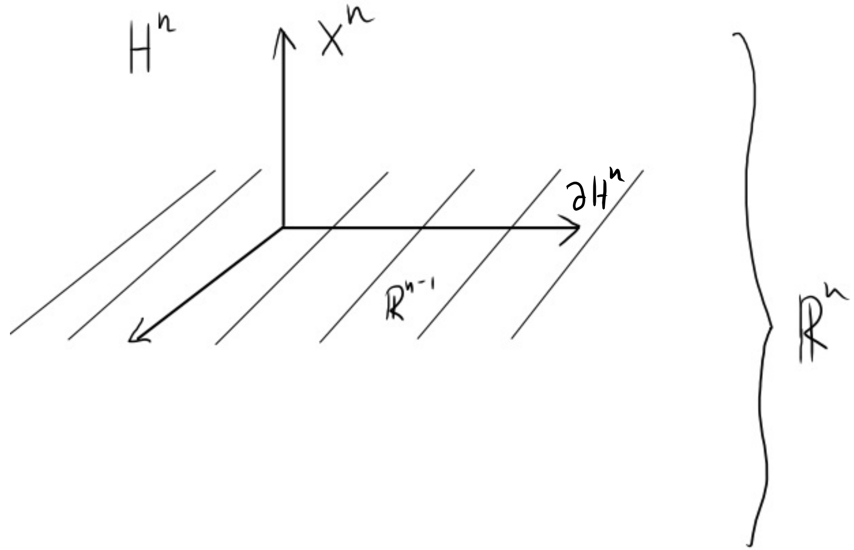
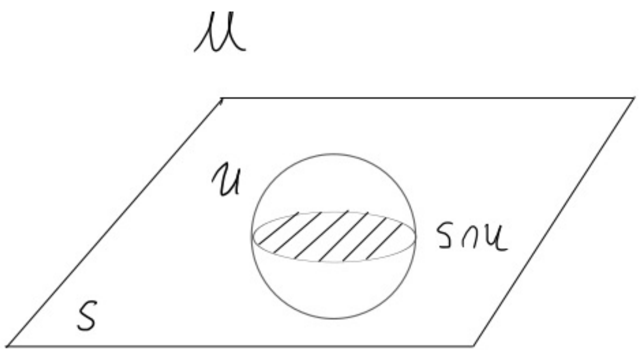


H^n is a topological space with the <u>relative topology</u> from $\mathbb{R}^n$

Preliminaries

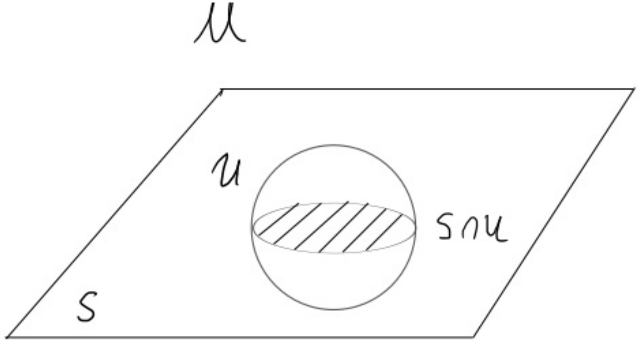


Relative or subspace topology:

Let $(\mathcal{M}, \tau)$ a topological space

$U \in \tau$ the open sets

Preliminaries



Relative or subspace topology:

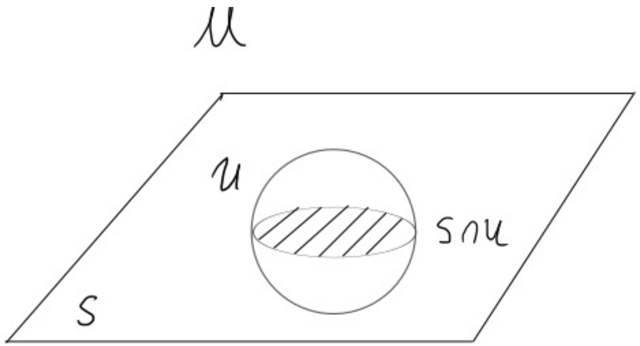
Let $(\mathcal{M}, \tau)$ a topological space

$\mathcal{U} \in \tau$ the open sets

If $S \subseteq \mathcal{M}$, the relative/subspace topology is

$$\tau_S = \{ S \cap \mathcal{U} \mid \mathcal{U} \in \tau \}$$

Preliminaries



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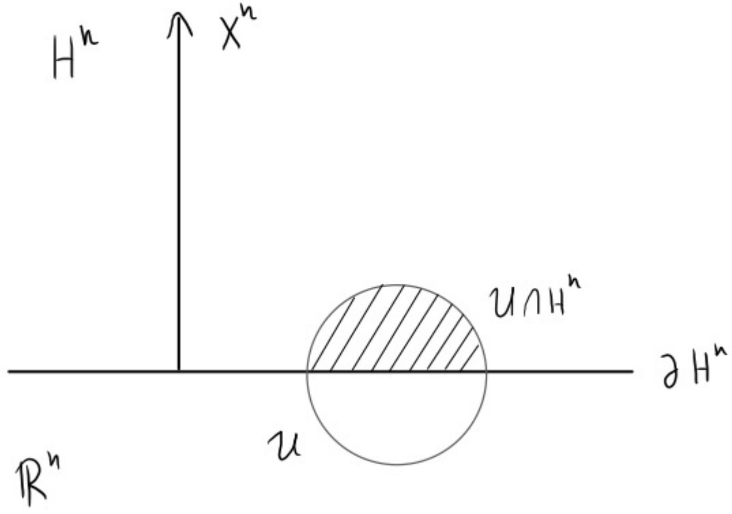
$\Rightarrow$ (S, τ_S) a topological space with $S \cap U$ its open sets

Preliminaries

- $U \subset \mathbb{R}^n$ an open set of $\mathbb{R}^n$



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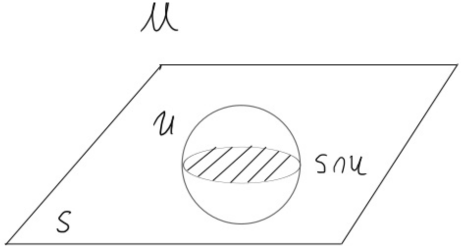

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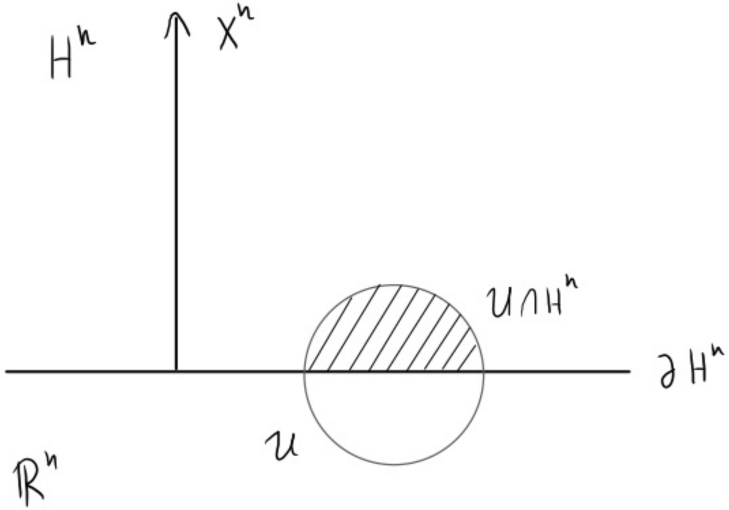

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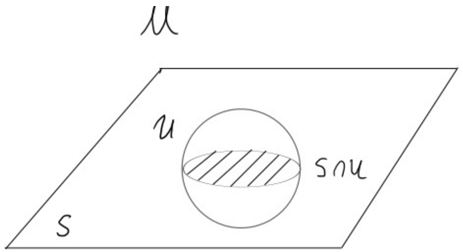

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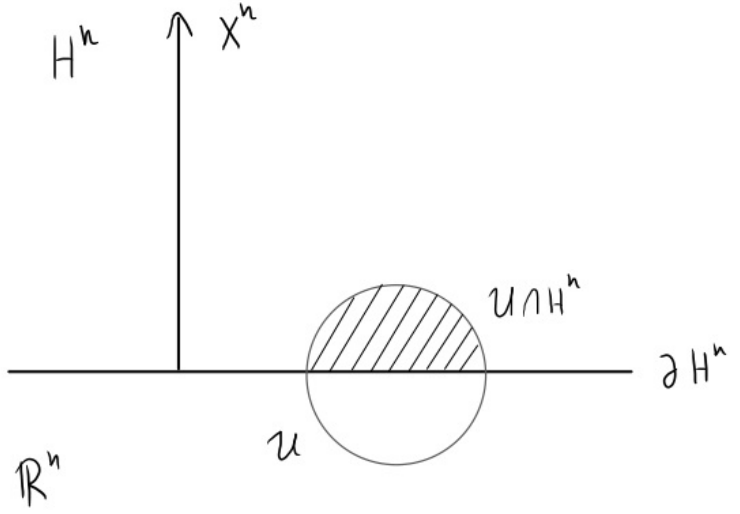

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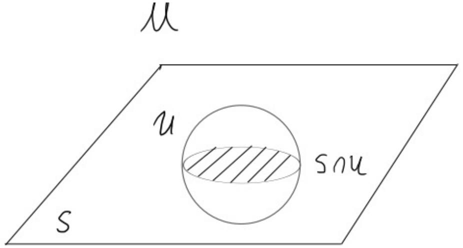

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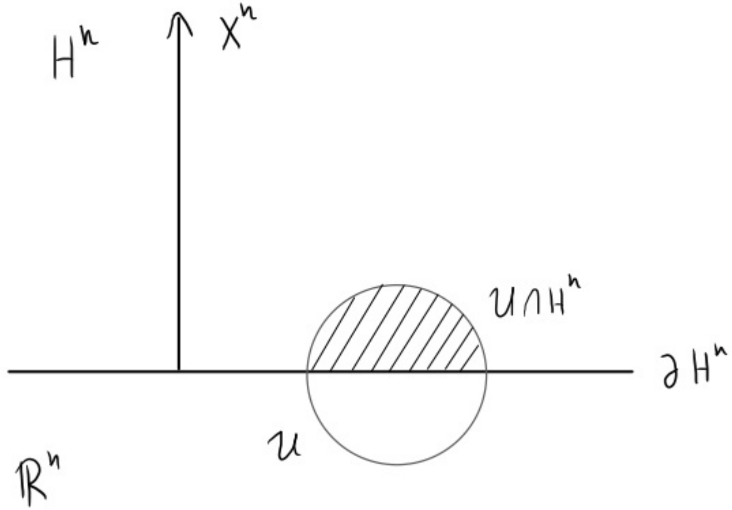

- smooth functions on $\mathbb{R}^n$, restricted to H^n, give smooth functions on H^n

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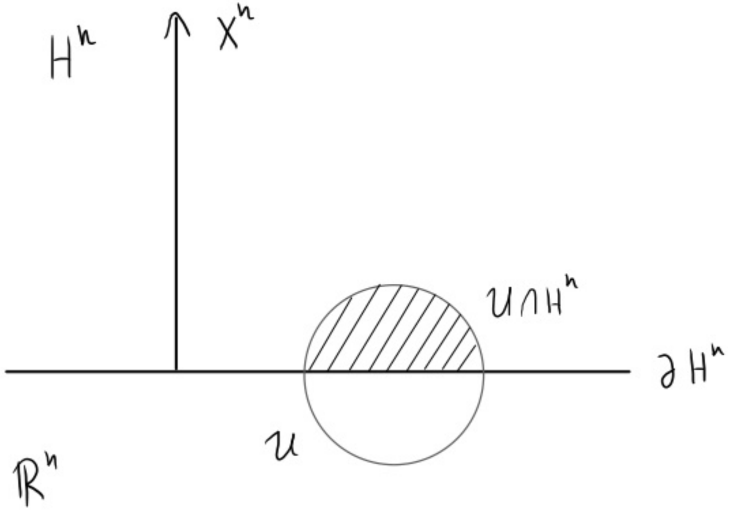

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- diffeomorphisms $\phi : U \to V$, $U, V \subset \mathbb{R}^n$, give diffeomorphisms $\phi : U \cap H^n \to V \cap H^n$

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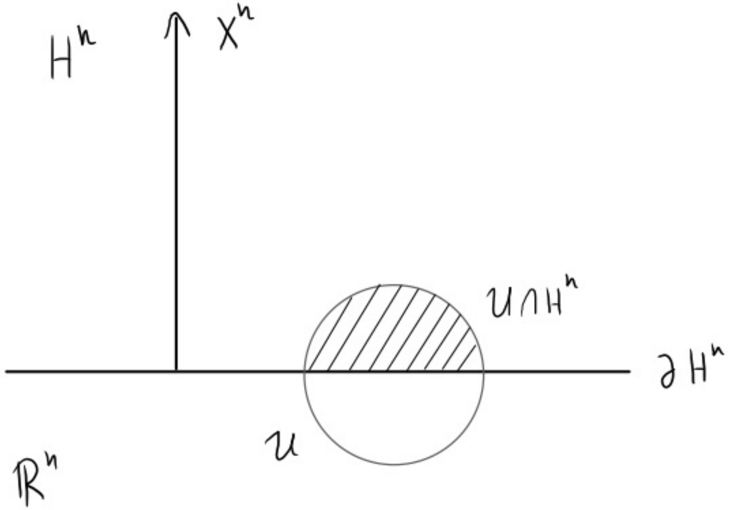

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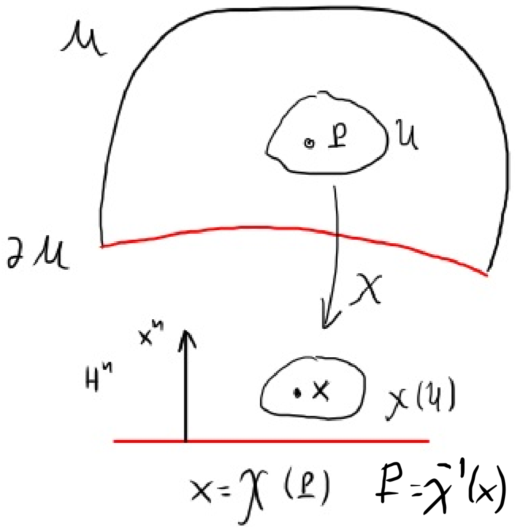
- diffeomorphisms $\phi : U \to V$, $U, V \subset \mathbb{R}^n$, give diffeomorphisms $\phi : U \cap H^n \to V \cap H^n$
- then, vectors, forms, tensors, etc can be defined on H^n

Manifolds with Boundary



- M a separable Hausdorff topological space
- a chart (U, χ):

  $U \subseteq M$, open in M

  $\chi : U \to \chi(U) \subseteq H^n$ is a homeomorphism (continuous, invertible)

Manifolds with Boundary

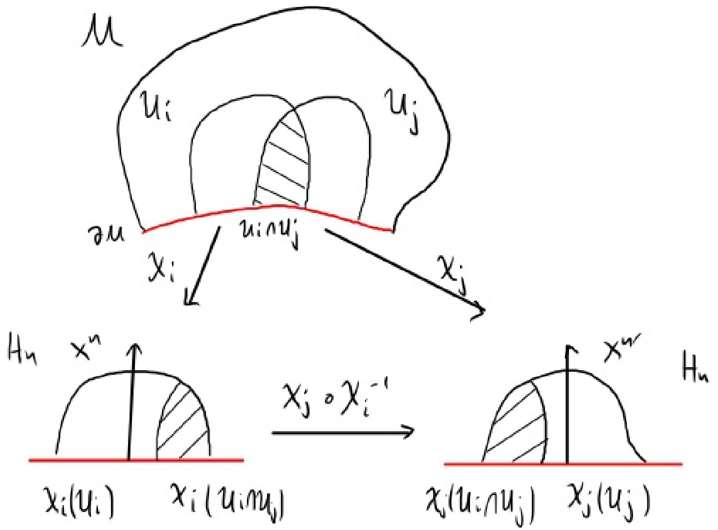
- $\mathcal{M}$ a separable Hausdorff topological space
- a chart $(\mathcal{U}, \chi)$:

  $\mathcal{U} \subseteq \mathcal{M}$, open in $\mathcal{M}$

  $\chi : \mathcal{U} \to \chi(\mathcal{U}) \subseteq H^n$ is a homeomorphism (continuous, invertible)

- an atlas of smoothly compatible charts:

$$\chi_j \circ \chi_i^{-1} : \chi_i(\mathcal{U}_i \cap \mathcal{U}_j) \to \chi_j(\mathcal{U}_i \cap \mathcal{U}_j)$$

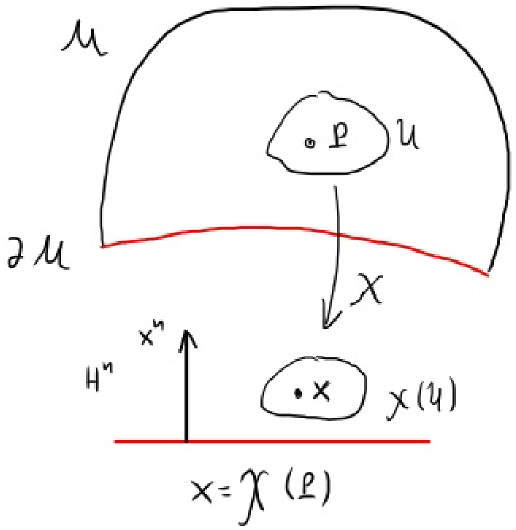


Manifolds with Boundary

- manifold points:

  $\underline{p} \in \mathcal{U}$, where $\chi(\mathcal{U}) \subseteq H^n_+$

  $$\mathcal{M}_+ = \{ \underline{p} \in \mathcal{M} \mid \underline{p} \text{ a manifold point} \}$$



Manifolds with Boundary

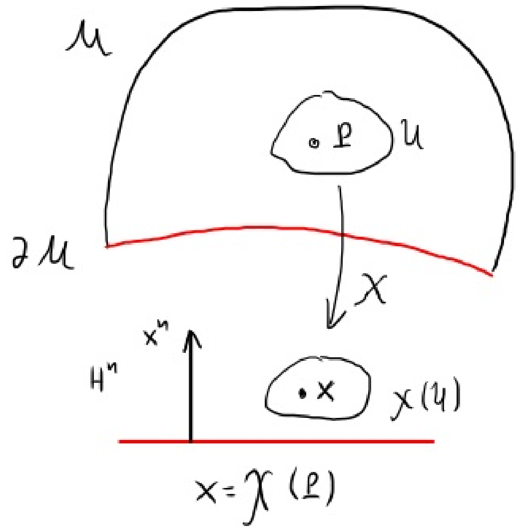
- manifold points:

  $p \in U$, where $\qquad x(U) \subseteq H^n_+$

  $$M_+ = \{ p \in M \mid p \text{ a manifold point} \}$$

- boundary points:

  $$p \in \partial M = M \setminus M_+$$



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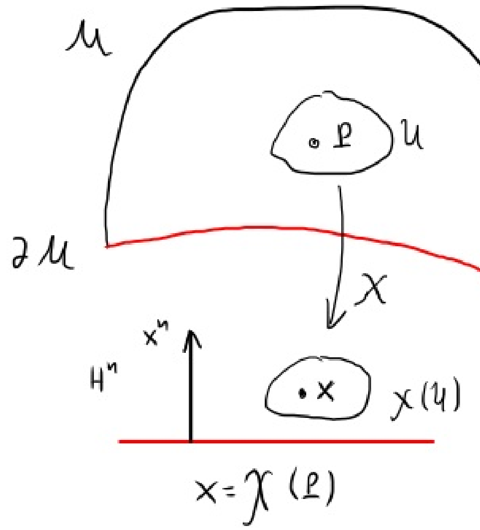

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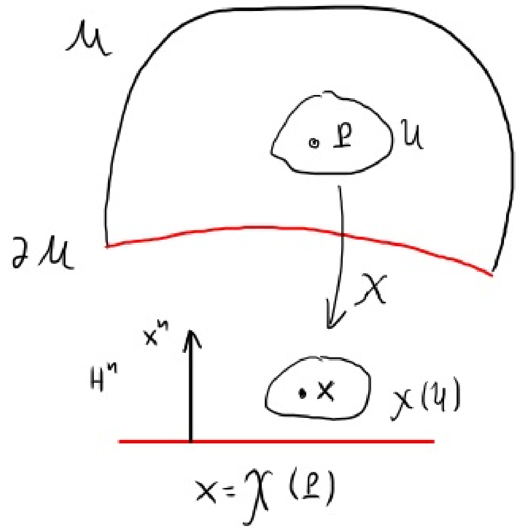
- if (U,x) a chart on M $\Rightarrow$ $(U \cap \partial M, x|_{U \cap \partial M})$ a $(n-1)$-dim chart on ∂M

Manifolds with Boundary

- ∂M is a $(n-1)$-dim manifold



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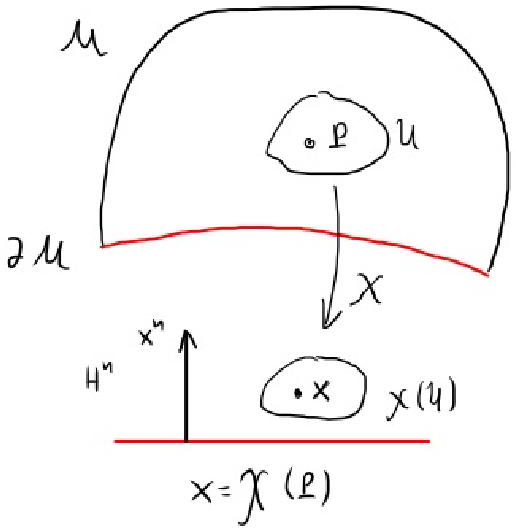
- $\partial\mathcal{M}$ is a $(n-1)$-dim manifold
- functions $f: \mathcal{M} \to \mathbb{R}$ are smooth if $\forall$ chart $f \circ \chi^{-1}$ smooth



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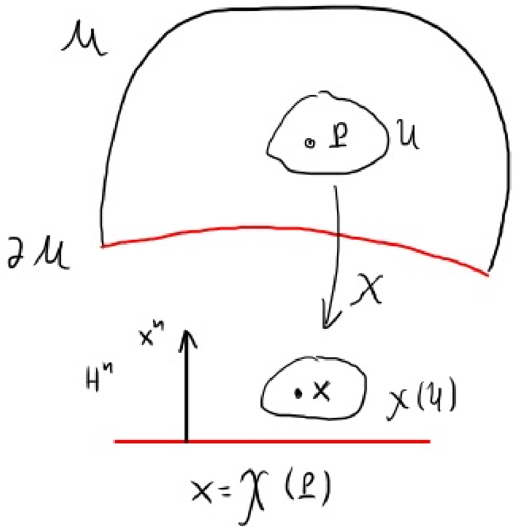
- if (U, χ) a chart on $\mathcal{M}$ $\Rightarrow$ $\left(U \cap \partial\mathcal{M}, \chi|_{U \cap \partial\mathcal{M}}\right)$ a $(n-1)$-dim chart on $\partial\mathcal{M}$

Manifolds with Boundary

- $\partial\mathcal{M}$ is a $(n-1)$-dim manifold
- functions $f: \mathcal{M} \to \mathbb{R}$ are smooth if $\forall$ chart $f \circ \mathcal{X}^{-1}$ smooth
- support of f:

  closure of set on which f is nonzero

  (a closed set)



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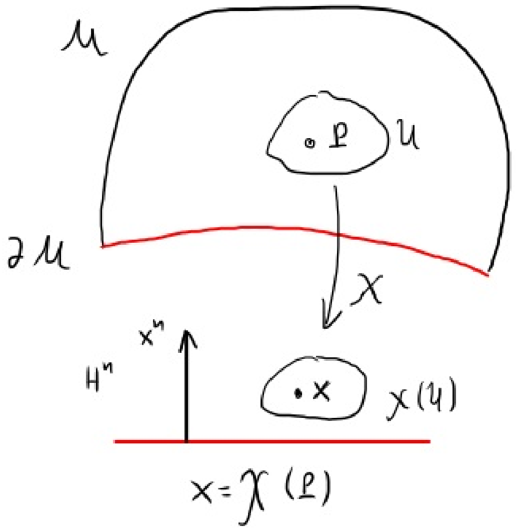

- functions with compact support $f \in F_c(\mathcal{M})$

  (e.g. functions that don't extend to "infinity")

  (remember later when we refer to n-forms with compact support ...)

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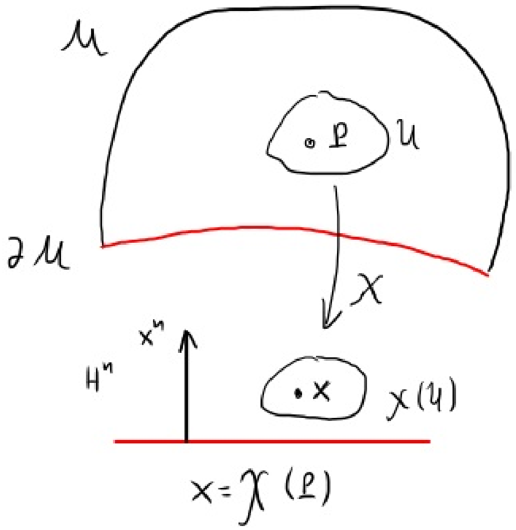
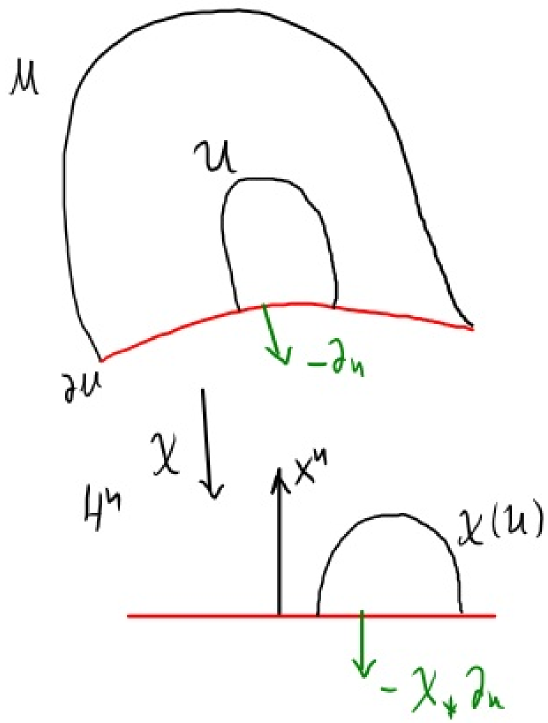


- functions with compact support $f \in \mathcal{F}_c(\mathcal{M})$
- vectors are tangent to curves in $\mathcal{M}$
  - curves of points in $\partial\mathcal{M}$, define vectors tangent to $\partial\mathcal{M}$

$\rightsquigarrow$ the rest of the construction of tensors etc., is straightforward

Induced Orientation on the Boundary

- Let $(\mathcal{U}, x)$ be a chart with $\mathcal{U} \cap \partial \mathcal{M} \neq \emptyset$

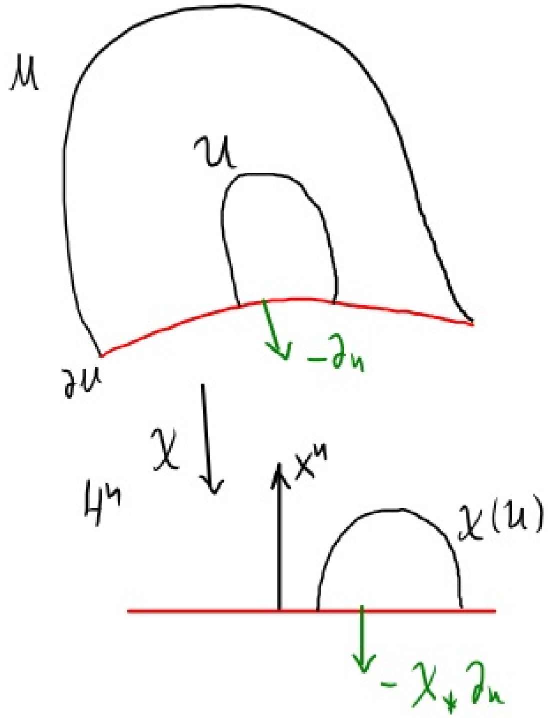


Induced Orientation on the Boundary

- Let (U, x) be a chart with $U \cap \partial M \neq \emptyset$
- The n-form

$$\omega = dx^1 \wedge \cdots \wedge dx^n$$

defines an orientation on M



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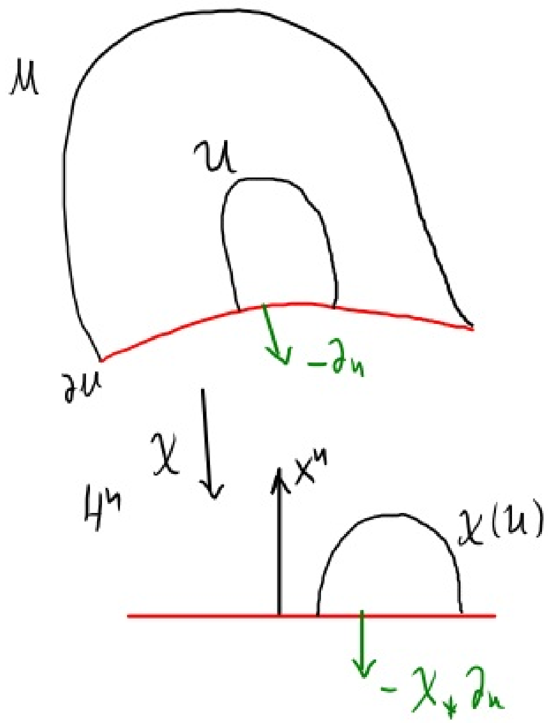

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- Consider the vector $-\partial_n$ ("outward" @ ∂M)

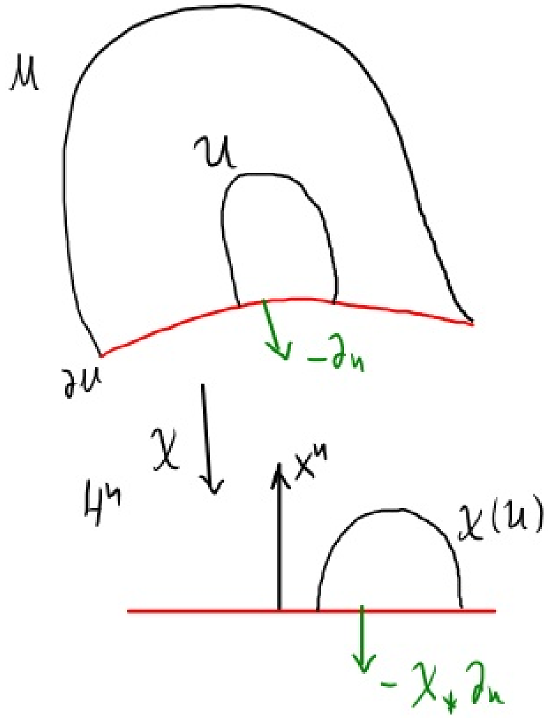
and the $(n-1)$-form

$$\omega' = i_{-\partial_n} \omega = \omega(-\partial_n, \dots)$$

Induced Orientation on the Boundary

- The action of ω' on $n-1$ vectors in $\mathcal{U}$ is

$$\omega'(V_1, \dots, V_{n-1}) = \omega(-\partial_n, V_1, \dots, V_{n-1})$$
$$= dx_1 \wedge dx_2 \wedge \cdots \wedge dx^n(-\partial_n, V_1, \dots, V_{n-1})$$



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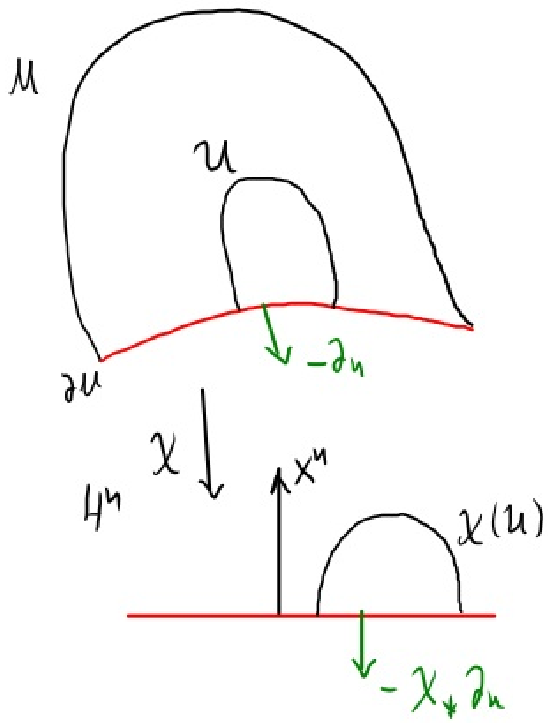
$$= \frac{1}{n!} \begin{vmatrix} dx^1(-\partial_n) & dx^1(V_1) & \cdots & dx^1(V_{n-1}) \\ \vdots & \vdots & & \vdots \\ dx^{n-1}(-\partial_n) & dx^{n-1}(V_1) & \cdots & dx^{n-1}(V_{n-1}) \\ dx^n(-\partial_n) & dx^n(V_1) & \cdots & dx^n(V_{n-1}) \end{vmatrix}$$



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$$\overset{,\rho}{=} dx_1\wedge dx_2\wedge\cdots\wedge dx^n(-\partial_n, V_1,\dots,V_{n-1})$$

$$= \frac{1}{n!}\begin{vmatrix} dx^1(-\partial_n) & dx^1(V_1) & \cdots & dx^1(V_{n-1}) \\ \vdots & \vdots & & \vdots \\ dx^{n-1}(-\partial_n) & dx^{n-1}(V_1) & \cdots & dx^{n-1}(V_{n-1}) \\ \underbrace{\boxed{dx^n(-\partial_n)}}_{=-1} & dx^n(V_1) & \cdots & dx^n(V_{n-1}) \end{vmatrix}$$

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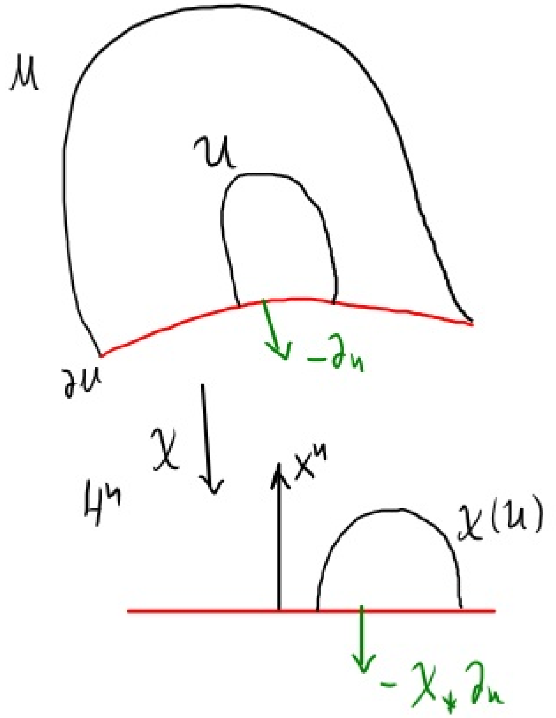

Induced Orientation on the Boundary

- The action of ω' on $n-1$ vectors

$$\omega'(V_1, \dots, V_{n-1}) = \omega(-\partial_n, V_1, \dots, V_{n-1})$$

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$$= \frac{(-1)^n}{n!} \begin{vmatrix} dx^1(V_1) & \cdots & dx^1(V_{n-1}) \\ \vdots & & \vdots \\ dx^{n-1}(V_1) & \cdots & dx^{n-1}(V_{n-1}) \end{vmatrix}$$

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$$= \frac{(-1)^n}{n} dx^1 \wedge \cdots \wedge dx^{n-1} (V_1, \dots, V_{n-1})$$

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Since this is true for any $(V_1,\dots,V_{n-1}) \Rightarrow$

$$\omega' = \frac{(-1)^n}{n}\, dx^1\wedge\cdots\wedge dx^{n-1}$$

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- ω' is well defined on ∂M (all vectors have no V^n component)

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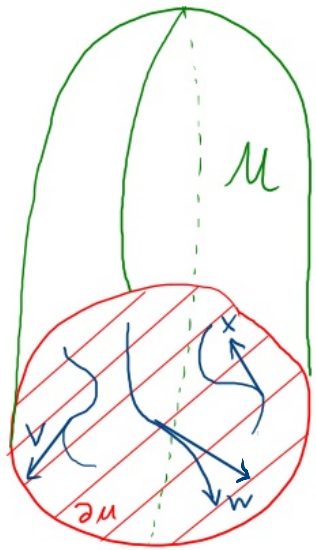
$$\omega' = \frac{(-1)^n}{n}\, dx^1\wedge\cdots\wedge dx^{n-1}$$

- ω' is well defined on ∂M
- is chosen as the volume element on ∂M giving ∂M, the induced orientation by ω

Restriction of $(n-1)$-forms on ∂M

- let ω be a $(n-1)$-form on M
- we define the restriction of ω on ∂M as a linear function on $(n-1)$ vectors on ∂M



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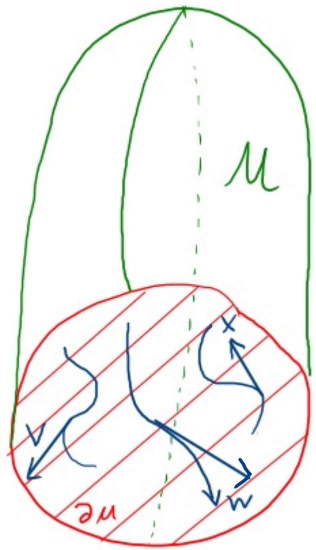

any $(n-1)$-form on M, in a coordinate system can be written as:

$$\omega = \sum_{\mu=1}^{n} \omega_\mu(x^1, \dots, x^{n-1}, x^n)\, dx^1 \wedge \dots \wedge dx^{\mu-1} \wedge \widehat{dx^\mu} \wedge dx^{\mu+1} \wedge \dots \wedge dx^n$$

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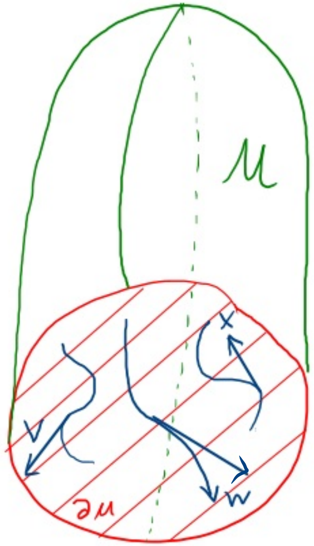

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Notation:

the caret means dx^μ is missing from the product

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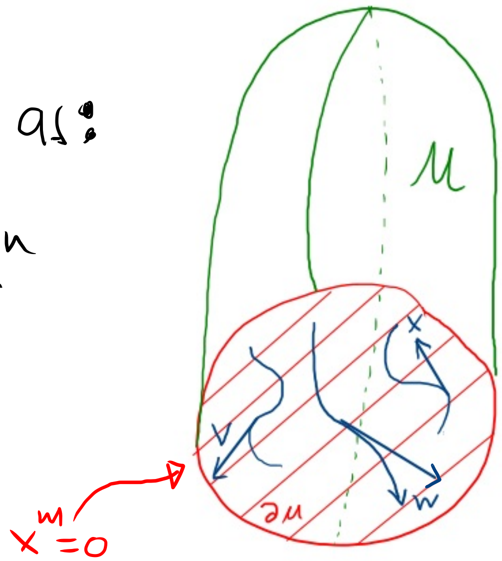
we assume that $x^n = 0$ give the points on ∂M



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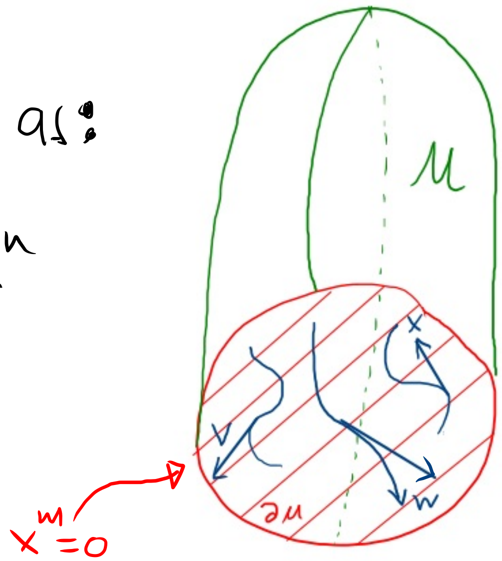

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If $V = V^1 \partial_1 + \dots + V^{n-1} \partial_{n-1}$ $(V^n = 0)$, then $dx^n(V) = 0$, and $V \in T\partial M$

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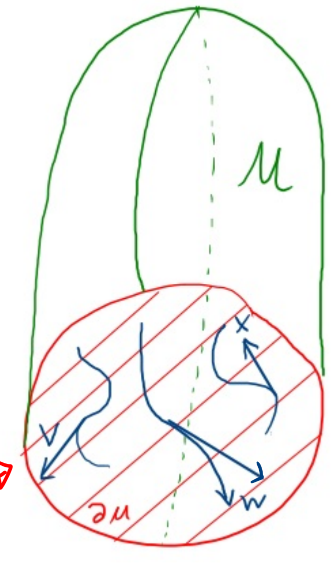

any $(n-1)$-form on M, in a coordinate system can be written as:

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all other terms are zero: $dx^n(v_k) = 0$ $\quad dx^1 \wedge \dots \wedge dx^{n-1} \wedge \widehat{dx^n}$

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If $V = V^1 \partial_1 + \dots + V^{n-1} \partial_{n-1}$ $\quad (V^n = 0)$, then $dx^n(V) = 0$, and $V \in T\partial M$

For $\{V_1, \dots, V_{n-1}\}$ vectors in $T\partial M$, $\omega(V_1, \dots, V_{n-1}) = \omega_n(x^1, \dots, x^{n-1}, 0)\, dx^1 \wedge \dots dx^{n-1}(V^1, \dots, V^{n-1})$

$\to x^n = 0$

Restriction of $(n-1)$-forms on ∂M

Therefore

$$\omega|_{\partial M} \equiv \omega_n(x^1,\dots,x^{n-1},0)\, dx^1\wedge\dots\wedge dx^{n-1}$$

a $(n-1)$-form on ∂M

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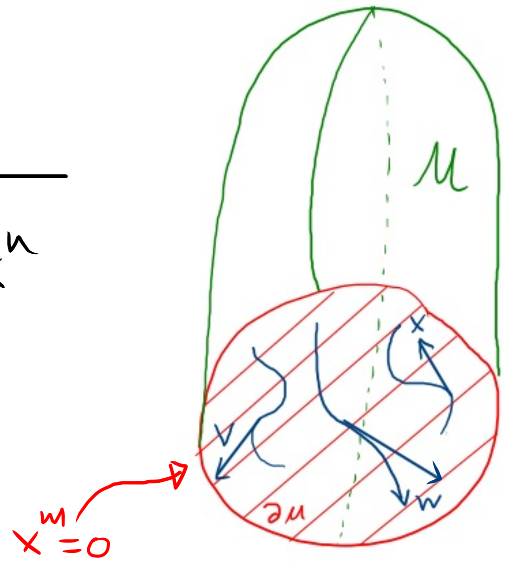

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Restriction of $(n-1)$-forms on ∂M

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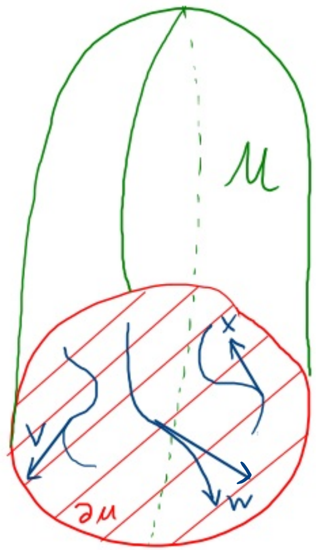
a $(n-1)$-form on ∂M



$$\omega = \sum_{\mu=1}^{n} \omega_\mu(x^1,\dots,x^{n-1},x^n)\, dx^1\wedge\dots\wedge dx^{\mu-1}\wedge \widehat{dx^\mu}\wedge dx^{\mu+1}\wedge\dots\wedge dx^n$$

Notice that:

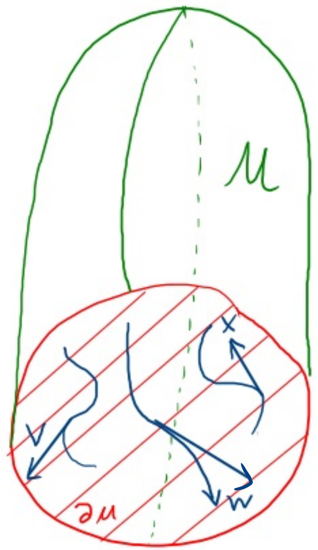
$$d\omega = \sum_{\nu=1}^{n}\sum_{\mu=1}^{n} \frac{\partial \omega_\mu}{\partial x^\nu}\, dx^\nu\wedge dx^1\wedge\dots\wedge\widehat{dx^\mu}\wedge\dots\wedge dx^n$$

Restriction of $(n-1)$-forms on $\partial\mathcal{U}$

Therefore

$$\omega|_{\partial\mathcal{U}} \equiv \omega_n(x^1,\dots,x^{n-1},0)\, dx^1\wedge\cdots\wedge dx^{n-1}$$

a $(n-1)$-form on $\partial\mathcal{U}$



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only $\mu=\nu$ term survives:
all others have $dx^\nu\wedge dx^\nu=0$ factors

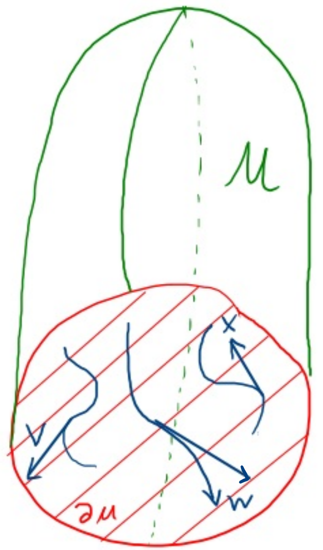
move to empty slot here, obtain $(-1)^{\mu-1}$

Restriction of $(n-1)$-forms on $\partial\mathcal{U}$

Therefore

$$\omega|_{\partial\mathcal{U}} \equiv \omega_n(x^1,\dots,x^{n-1},0)\, dx^1\wedge\dots\wedge dx^{n-1}$$

a $(n-1)$-form on $\partial\mathcal{U}$



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Stokes' Theorem

Let ω $(n-1)$-form on M with compact support, then

$$\int_M d\omega = \int_{\partial M} \omega$$

Stokes' Theorem

Let ω $(n-1)$-form on M with compact support, then

$$\int_M d\omega = \int_{\partial M} \omega$$

$\int_{\partial M}$: use induced orientation on ∂M

ω : use $\omega|_{\partial M}$ in $\int_{\partial M} \omega$

Stokes' Theorem

Let ω $(n-1)$-form on M with compact support, then

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Implications:

- $\int_M d\omega = 0$ on a manifold M, and ω compact support

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Let ω $(n-1)$-form on M with compact support, then

$$\int_M d\omega = \int_{\partial M} \omega$$

Implications:

- $\int_M d\omega = 0$ on a manifold M, and ω compact support
- ω closed $\Rightarrow d\omega = 0 \Rightarrow \int_{\partial M} \omega = 0$
- ω exact $\Rightarrow \omega = d\sigma \Rightarrow \int_M \omega = \int_M d\sigma = \int_{\partial M} \sigma$

if $\partial M = 0 \Rightarrow \int_M \omega = 0$

Stokes' Theorem

- if λ any $n-2$ form:

$$\int_{\partial(\partial M)} \lambda = \int_{\partial M} d\lambda = \int_M dd\lambda = 0 \quad \Rightarrow \quad \partial(\partial M) = \emptyset$$

the boundary of a boundary is $\emptyset$

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- If there is a metric on M, choose ϵ to be the volume form.

Any $(n-1)$-form $\omega = *V$, dual to a vector V. Then

$$d\omega = d*V = \nabla_\nu V^\nu \epsilon = \nabla_\nu V^\nu \sqrt{|g|}\, dx^0 \wedge \cdots \wedge dx^{n-1}$$

(see e.g. Carroll, Appendix E)

∇: covariant derivative from g

$\nabla_\nu V^\nu$: divergence of V

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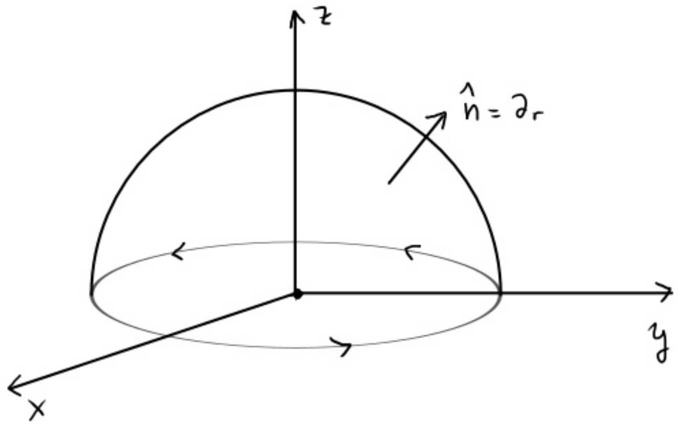
$$\int_M d\omega = \int_{\partial M} \omega \Leftrightarrow \int_M \nabla_\nu V^\nu \epsilon = \int_{\partial M} *V \Leftrightarrow \int_M \sqrt{|g|}\, \nabla_\nu V^\nu d^n x = \int_{\partial M} \sqrt{\gamma}\, n_\mu V^\mu d^{n-1} y$$

Example: Stokes' Theorem on the half sphere



S: $x = R\sin\theta\cos\varphi$ $\quad 0 \le \theta \le \frac{\pi}{2}$

$y = R\sin\theta\sin\varphi$ $\quad 0 \leq \varphi < 2\pi$

$z = R\cos\theta$

S: "northern hemisphere" $\quad c = \partial S =$ "equator"

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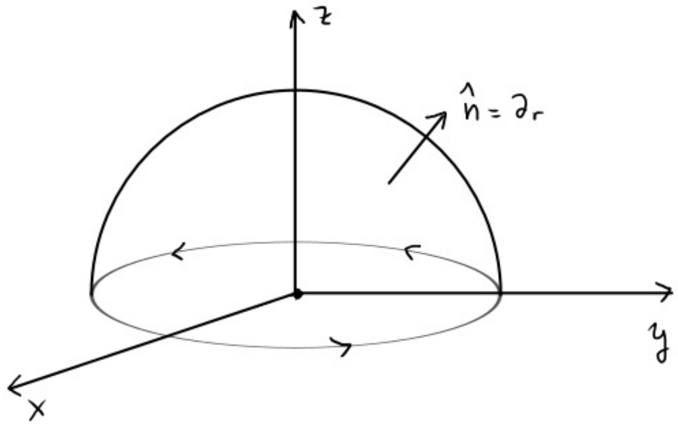

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Spherical coordinates on $\mathbb{R}^3$:

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

$$(\Lambda_{\mu'}{}^{\nu}) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \left(\frac{\partial x^{\nu}}{\partial x^{\mu'}} \right) = \begin{pmatrix} \cos\theta \sin\varphi & \sin\theta \sin\varphi & \cos\theta \\ r \cos\theta \cos\varphi & r \cos\theta \sin\varphi & -r \sin\theta \\ -r \sin\theta \sin\varphi & r \sin\theta \cos\varphi & 0 \end{pmatrix}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\theta = \operatorname{arccot} \frac{z}{\sqrt{x^2+y^2}}$$

$$\varphi = \arctan \frac{y}{x}$$

$$(\Lambda_{\mu}{}^{\mu'}) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial \theta}{\partial x} & \frac{\partial \varphi}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \theta}{\partial y} & \frac{\partial \varphi}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \theta}{\partial z} & \frac{\partial \varphi}{\partial z} \end{pmatrix} = \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right) = \begin{pmatrix} \sin\theta\cos\varphi & \frac{1}{r}\cos\theta\cos\varphi & -\frac{1}{r}\frac{\sin\varphi}{\sin\theta} \\ \sin\theta\sin\varphi & \frac{1}{r}\cos\theta\sin\varphi & \frac{1}{r}\frac{\cos\varphi}{\sin\theta} \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix}$$

Spherical coordinates on $\mathbb{R}^3$:

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

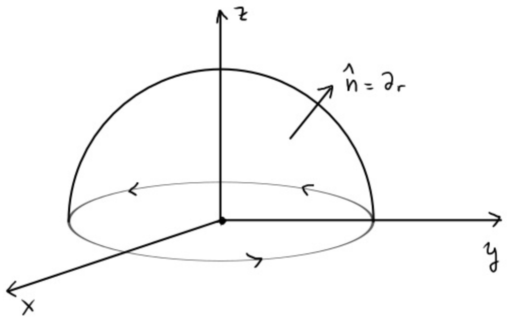
$$z = r\cos\theta$$

$$(\Lambda_{\mu'}{}^{\mu}) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) = \begin{pmatrix} \cos\theta\sin\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix}$$

$$\partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \cos\theta \sin\varphi \, \hat{x} + \sin\theta \sin\varphi \, \hat{y} + \cos\theta \, \hat{z}$$

$$|\partial_r| = (\partial_r \cdot \partial_r)^{1/2} = 1 \Rightarrow \quad \hat{n} = \partial_r \equiv \hat{r}$$



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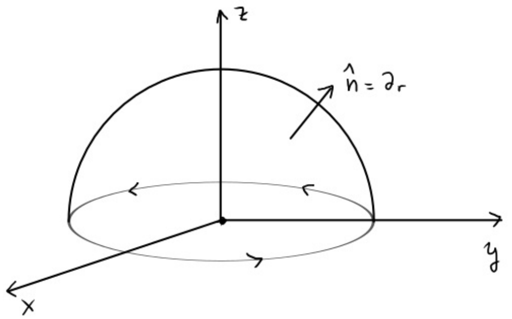
$$|\partial_\theta| = r \Rightarrow \hat{\theta} = \frac{1}{r}\partial_\theta$$

$$|\partial_\varphi| = r\sin\theta \Rightarrow \hat{\varphi} = \frac{1}{r\sin\theta}\partial_\varphi$$



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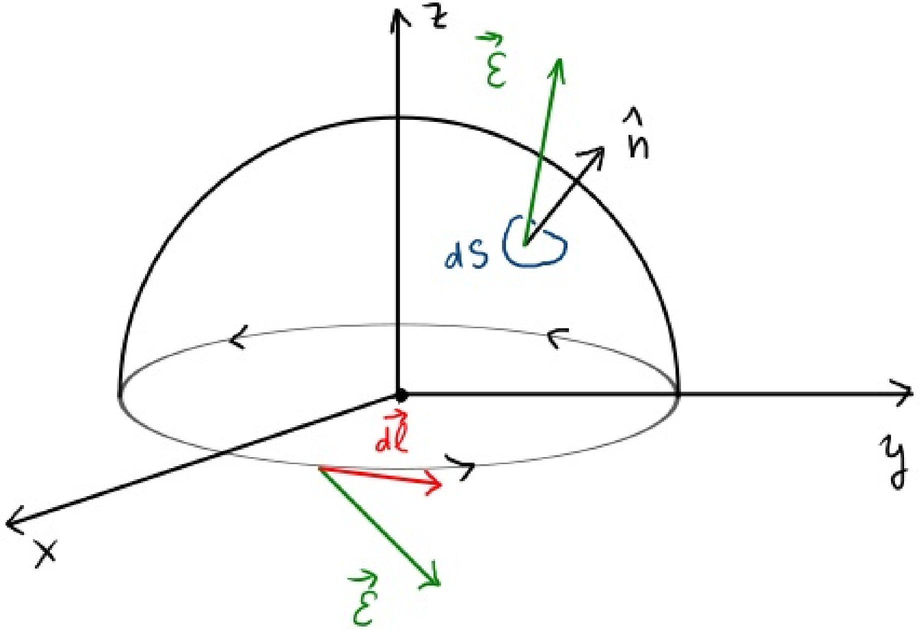
Stokes' Theorem:

$$\int_C \vec{\mathcal{E}} \cdot d\vec{\ell} = \int (\vec{\nabla} \times \vec{\mathcal{E}}) \cdot \hat{n} \, dS$$



Stokes' Theorem:

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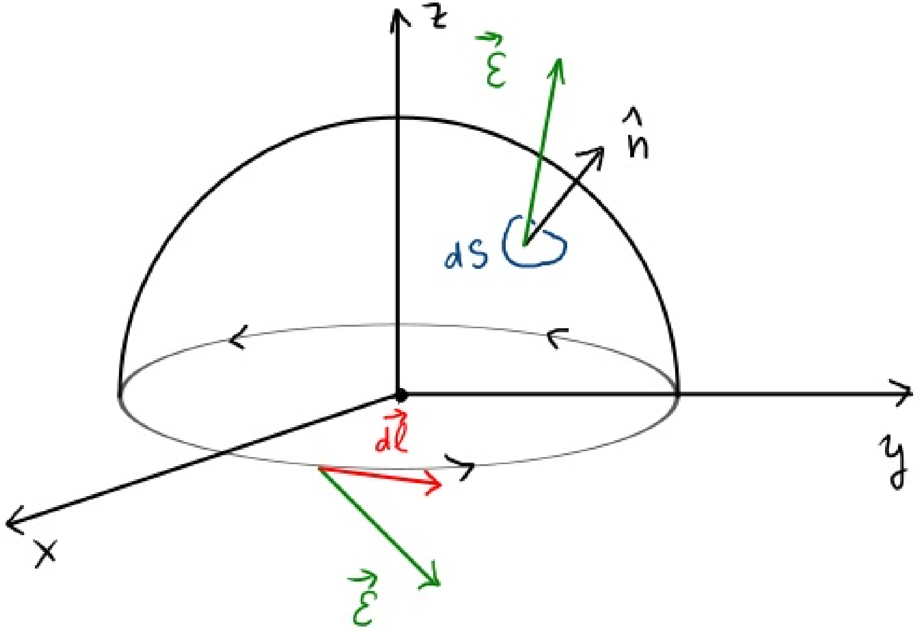

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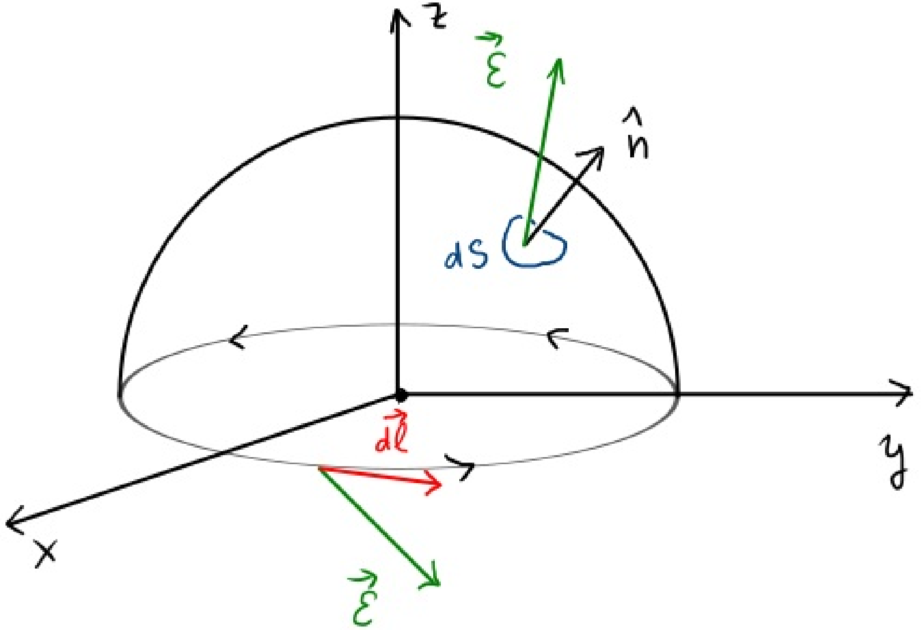
$$\vec{\nabla} \times \vec{\mathcal{E}} = \hat{x} \left(\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y \right) + \hat{y} \left(\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z \right) + \hat{z} \left(\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x \right)$$



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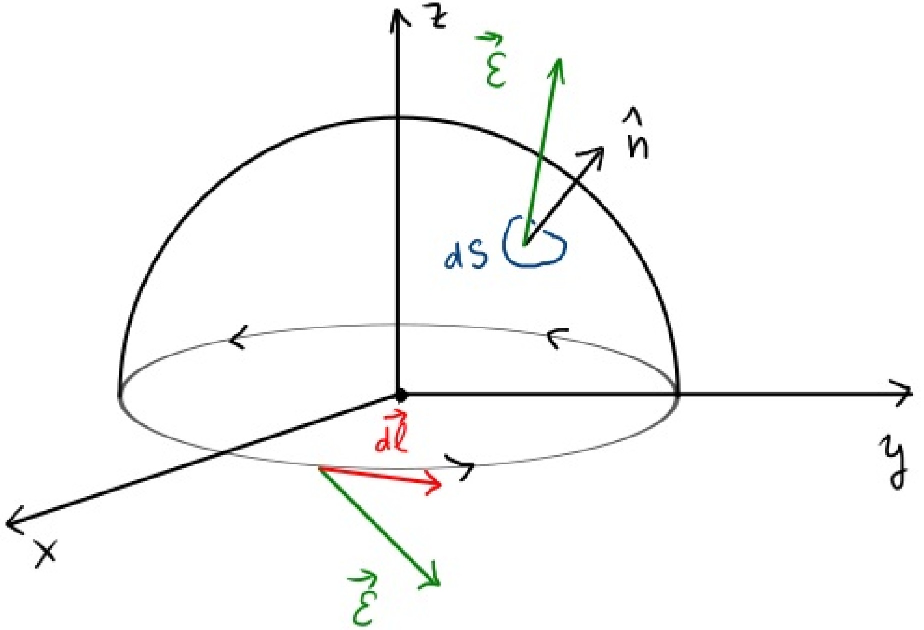

$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\begin{aligned} \vec{\nabla} \times \vec{\mathcal{E}} &= \hat{x} \left(\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y \right) \\ &+ \hat{y} \left(\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z \right) \\ &+ \hat{z} \left(\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x \right) \end{aligned}$$

$$\begin{aligned} (\vec{\nabla} \times \vec{\mathcal{E}}) \cdot \hat{n} \, ds &= n_x \left(\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y \right) R^2 \sin\theta \, d\theta \, d\varphi \\ &+ n_y \left(\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z \right) R^2 \sin\theta \, d\theta \, d\varphi \\ &+ n_z \left(\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x \right) R^2 \sin\theta \, d\theta \, d\varphi \end{aligned}$$

$$dS = R^2 \, d\Omega = R^2 \sin\theta \, d\theta \, d\varphi$$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla} \times \vec{\mathcal{E}} \cdot \hat{n}\, dS = \left\{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \right\} \times R^2 \sin\theta \, d\theta \, d\varphi$$

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Restrict $d\omega$ on the sphere so that $\int_S d\omega$ will be computed:

$$dy \wedge dz (\partial_\theta, \partial_\varphi) = dy(\partial_\theta)\, dz(\partial_\varphi) - dy(\partial_\varphi)\, dz(\partial_\theta)$$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \Rightarrow$$

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$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla} \times \vec{\mathcal{E}} \cdot \hat{n}\, dS = \{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \} \times R^2 \sin\theta \, d\theta \, d\varphi$$

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$$= (\partial_\theta)^y \, (\partial_\varphi)^z - (\partial_\varphi)^y \, (\partial_\theta)^z$$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

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$$d\omega = (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) dy \wedge dz + (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) dz \wedge dx + (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) dx \wedge dy$$

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$$= (\partial_\theta)^y (\partial_\varphi)^z - (\partial_\varphi)^y (\partial_\theta)^z = (\partial_\theta \times \partial_\varphi)^x \qquad \left((A \times B)^x = A^y B^z - B^z A^y \right)$$

Define the $(n-1)$-form:

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$$dy \wedge dz = (\partial_\theta \times \partial_\varphi)^x\, d\theta \wedge d\varphi$$

(any 2 form $\omega = \frac{1}{2!} \omega_{\mu\nu}\, dx^\mu \wedge dx^\nu = \omega_{\theta\varphi}\, d\theta \wedge d\varphi$)

$$\vec{\mathcal{E}}\cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla}\times\vec{\mathcal{E}}\cdot\hat{n}\,dS = \{n^x(\partial_y\mathcal{E}_z - \partial_z\mathcal{E}_y) + n^y(\partial_z\mathcal{E}_x - \partial_x\mathcal{E}_z) + n^z(\partial_x\mathcal{E}_y - \partial_y\mathcal{E}_x)\}\times R^2\sin\theta\,d\theta\,d\varphi$$

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$$dy\wedge dz = (\partial_\theta\times\partial_\varphi)^x\,d\theta\wedge d\varphi \text{, similarly}$$
$$dz\wedge dx = (\partial_\theta\times\partial_\varphi)^y\,d\theta\wedge d\varphi$$
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But: $\partial_\theta = r\hat{\theta}$

$\partial_\varphi = r \sin\theta \, \hat{\varphi}$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

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$$\left.\begin{aligned} \partial_\theta &= r\hat{\theta} \\ \partial_\varphi &= r\sin\theta\, \hat{\varphi} \end{aligned}\right\} \Rightarrow \partial_\theta \times \partial_\varphi = r^2 \sin\theta\, \hat{\theta} \times \hat{\varphi}$$

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But: $\left.\begin{array}{l} \partial_\theta = r\hat{\theta} \\ \partial_\varphi = r\sin\theta\, \hat{\varphi} \end{array}\right\} \Rightarrow \partial_\theta \times \partial_\varphi = r^2 \sin\theta \, \hat{\theta} \times \hat{\varphi} = r^2 \sin\theta \, \hat{r}$ $\quad (\hat{r} = \hat{\theta} \times \hat{\varphi})$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

$$d\omega = (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y)\, dy \wedge dz + (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z)\, dz \wedge dx + (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x)\, dx \wedge dy$$

$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla} \times \vec{\mathcal{E}} \cdot \hat{n}\, dS = \{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \} \times R^2 \sin\theta \, d\theta \, d\varphi$$

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But: $\left.\begin{aligned} \partial_\theta &= r\hat{\theta} \\ \partial_\varphi &= r\sin\theta\, \hat{\varphi} \end{aligned}\right\} \Rightarrow \partial_\theta \times \partial_\varphi = r^2 \sin\theta\, \hat{\theta} \times \hat{\varphi} = r^2 \sin\theta\, \hat{r}$

On the sphere: $r = R$, $\hat{n} = \hat{r}$, so $(\partial_\theta \times \partial_\varphi) = R^2 \sin\theta\, \hat{n}$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

$$\vec{\mathcal{E}}\cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla}\times\vec{\mathcal{E}}\cdot\hat{n}\,dS = \{n^x(\partial_y\mathcal{E}_z - \partial_z\mathcal{E}_y) + n^y(\partial_z\mathcal{E}_x - \partial_x\mathcal{E}_z) + n^z(\partial_x\mathcal{E}_y - \partial_y\mathcal{E}_x)\} \times R^2 \sin\theta\, d\theta\, d\varphi$$

$$d\omega = (\partial_y\mathcal{E}_z - \partial_z\mathcal{E}_y)\, dy\wedge dz + (\partial_z\mathcal{E}_x - \partial_x\mathcal{E}_z)\, dz\wedge dx + (\partial_x\mathcal{E}_y - \partial_y\mathcal{E}_x)\, dx\wedge dy$$

$$\left.\begin{aligned} d\omega = {} & (\partial_y\mathcal{E}_z - \partial_z\mathcal{E}_y)\, n^x R^2 \sin\theta\, d\theta\wedge d\varphi \\ & + (\partial_z\mathcal{E}_x - \partial_x\mathcal{E}_z)\, n^y R^2 \sin\theta\, d\theta\wedge d\varphi \\ & + (\partial_x\mathcal{E}_y - \partial_y\mathcal{E}_x)\, n^z R^2 \sin\theta\, d\theta\wedge d\varphi \end{aligned}\right\}\quad d\omega|_S \text{, restricted on } S$$

$dy\wedge dz = (\partial_\theta \times \partial_\varphi)^x\, d\theta\wedge d\varphi$, similarly

$$dz\wedge dx = (\partial_\theta \times \partial_\varphi)^y\, d\theta\wedge d\varphi$$

$$dx\wedge dy = (\partial_\theta \times \partial_\varphi)^z\, d\theta\wedge d\varphi$$

But:
$$\left.\begin{aligned} \partial_\theta &= r\hat{\theta} \\ \partial_\varphi &= r\sin\theta\,\hat{\varphi} \end{aligned}\right\} \Rightarrow \partial_\theta \times \partial_\varphi = r^2 \sin\theta\, \hat{\theta}\times\hat{\varphi} = r^2 \sin\theta\, \hat{r}$$

On the sphere: $r = R$, $\hat{n} = \hat{r}$, so $(\partial_\theta \times \partial_\varphi) = R^2 \sin\theta\, \hat{n}$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

$$\vec{\mathcal{E}} \cdot d\vec{l} = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz$$

$$\vec{\nabla} \times \vec{\mathcal{E}} \cdot \hat{n}\, dS = \left\{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \right\} \times R^2 \sin\theta \, d\theta \, d\varphi$$

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$$\begin{aligned} d\omega = {} & (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y)\, n^x R^2 \sin\theta \, d\theta \wedge d\varphi \\ & + (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z)\, n^y R^2 \sin\theta \, d\theta \wedge d\varphi \\ & + (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x)\, n^z R^2 \sin\theta \, d\theta \wedge d\varphi \end{aligned}$$

$$\begin{aligned} \int_S d\omega &= \int \left\{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \right\} R^2 \sin\theta \, d\theta \, d\varphi \\ &= \int \hat{n} \cdot \vec{\nabla} \times \vec{\mathcal{E}} \cdot dS \end{aligned}$$

$$\int_{\partial S} \omega = \int_{\partial S} (\mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz)\Big|_{\partial S} = \int_C \vec{\mathcal{E}} \cdot d\vec{l}$$

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By the way:
$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

Define the $(n-1)$-form:

$$\omega = \mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz \quad \Rightarrow$$

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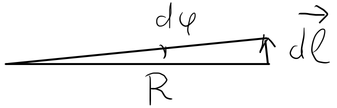
$$\int_S d\omega = \int \left\{ n^x (\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y) + n^y (\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z) + n^z (\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x) \right\} R^2 \sin\theta \, d\theta \, d\varphi$$

$$= \int \hat{n} \cdot \vec{\nabla} \times \vec{\mathcal{E}} \cdot dS$$

$$\int_{\partial S} \omega = \int_{\partial S} (\mathcal{E}_x dx + \mathcal{E}_y dy + \mathcal{E}_z dz)\Big|_{\partial S} = \int_C \vec{\mathcal{E}} \cdot d\vec{l}$$



By the way: on $C = \partial S$, $r = R$, $\theta = \pi/2$

$$\left.\begin{aligned} x &= R \cdot 1 \cdot \cos\varphi \\ y &= R \cdot 1 \cdot \sin\varphi \\ z &= R \cdot 0 = 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} dx &= -R \sin\varphi \, d\varphi \\ dy &= R \cos\varphi \, d\varphi \\ dz &= 0 \end{aligned}\right\} \Rightarrow \omega|_{\partial S} = -\mathcal{E}_x \sin\varphi \, R\, d\varphi + \mathcal{E}_y \cos\varphi \, R \, d\varphi = \mathcal{E}_\varphi R \, d\varphi = \mathcal{E}_\varphi \, dl$$

We also know that: $\displaystyle\int_{B^3} \vec{\nabla} \cdot \vec{\mathcal{E}} \, d^3x = \oint_{S^2} \vec{\mathcal{E}} \cdot \hat{n} \, dS$

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Consider $\sigma = \mathcal{E}_x\, dy \wedge dz + \mathcal{E}_y\, dz \wedge dx + \mathcal{E}_z\, dx \wedge dy$ 2-form in $B^3 \subset \mathbb{R}^3$

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As we computed before:

$\sigma|_{S^2} = \mathcal{E}_x\, n^x R^2 \sin\theta\, d\theta\wedge d\varphi + \mathcal{E}_y\, n^y R^2 \sin\theta\, d\theta\wedge d\varphi + \mathcal{E}_z\, n^z R^2 \sin\theta\, d\theta\wedge d\varphi$

now: $0 \leq \theta \leq \pi$

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$\Rightarrow \displaystyle\int_{S^2} \sigma = \int (\mathcal{E}_x n^x + \mathcal{E}_y n^y + \mathcal{E}_z n^z) R^2 \sin\theta\, d\theta\, d\varphi = \int \vec{\mathcal{E}}\cdot\hat{n}\, dS$

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$$\Rightarrow \int_{S^2}\sigma = \int \left(\mathcal{E}_x n^x + \mathcal{E}_y n^y + \mathcal{E}_z n^z\right) R^2 \sin\theta\, d\theta\, d\varphi = \int \vec{\mathcal{E}}\cdot\hat{n}\, dS$$

$$d\sigma = \underbrace{\partial_x \mathcal{E}_x\, dx\wedge dy}\wedge dz + \partial_y\mathcal{E}_y\, \underbrace{dy\wedge dz\wedge dx}_{= dx\wedge dy\wedge dz} + \partial_z\mathcal{E}_z\, \underbrace{dz\wedge dx\wedge dy}_{= dx\wedge dy\wedge dz}$$

all other $\partial_y\mathcal{E}_y\, dy\wedge dy\wedge dz$

$\partial_z\mathcal{E}_z\, dz\wedge dy\wedge dz$

terms vanish

We also know that: $\int_{B^3} \vec{\nabla}\cdot\vec{\mathcal{E}}\, d^3x = \oint_{S^2} \vec{\mathcal{E}}\cdot\hat{n}\, dS$

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$$d\sigma = \partial_x \mathcal{E}_x\, dx\wedge dy\wedge dz + \partial_y \mathcal{E}_y\, dy\wedge dz\wedge dx + \partial_z \mathcal{E}_z\, dz\wedge dx\wedge dy$$

$$= \left(\partial_x \mathcal{E}_x + \partial_y \mathcal{E}_y + \partial_z \mathcal{E}_z\right) dx\wedge dy\wedge dz$$

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$$d\sigma = \partial_x \mathcal{E}_x\, dx \wedge dy \wedge dz + \partial_y \mathcal{E}_y\, dy \wedge dz \wedge dx + \partial_z \mathcal{E}_z\, dz \wedge dx \wedge dy$$

$$= \left(\partial_x \mathcal{E}_x + \partial_y \mathcal{E}_y + \partial_z \mathcal{E}_z \right) dx \wedge dy \wedge dz = \left(\vec{\nabla}\cdot\vec{\mathcal{E}} \right) dx \wedge dy \wedge dz$$

$$\int_{B^3} d\sigma = \int \vec{\nabla}\cdot\vec{\mathcal{E}}\cdot d^3x$$