- One forms
  - Cotagent Space $T_p^* M$
  - Gradient df
  - Coordinate bases
  - Component Tranformations
- (k,l)-rank tensors
  - Vector space $T_p^{(k,l)} M$
  - tensor product
  - contractions, (anti) symmetrization

One forms

* Linear maps on T_pM

$$\omega : T_pM \longrightarrow \mathbb{R}$$
$$V \mapsto \omega(V)$$

* Linearity: for vectors V, W, one forms ω, σ, $\alpha, \beta \in \mathbb{R}$:

$$\omega(\alpha V + \beta W) = \alpha\,\omega(V) + \beta\,\omega(W)$$

* They form a vector space T^*_pM : cotagent space under: $(\alpha\omega + \beta\sigma)(V) = \alpha\,\omega(V) + \beta\,\sigma(V)$

One forms

* A basis on $T_p^* M$:

If $\{e_a\}$ is any basis on $T_p M$, its dual in $T_p^*(M)$ is $\{e^a\}$ s.t.:

$$e^a(e_b) = \delta^a{}_b$$

(not "orthogonal", we need inner product for that...)

---

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These are oneforms acting on any $V = V^a e_a \in T_pM$:

$$e^a(V) = e^a(V^b e_b) = V^b\, e^a(e_b) = V^b \delta^a{}_b = V^a$$

---

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$$e^a(V) = e^a(V^b e_b) = V^b e^a(e_b) = V^b \delta^a{}_b = V^a$$

Then, any one form ω can be written as:

$$\omega = \omega_a e^a \quad , \text{ where } \quad \omega_a = \omega(e_a)$$

$(\omega_a \in \mathbb{R})$

One forms

Indeed $\quad \omega = \omega_b e^b \Leftrightarrow \omega(e_a) = \omega_b \, e^b(e_a) = \omega_b \, \delta^b{}_a = \omega_a$

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$\Rightarrow \{e^a\}$ forms a basis of one-forms

$\Rightarrow \dim T_p^* M = n$

Then, any one form ω can be written as:

$\omega = \omega_a e^a$, where $\omega_a = \omega(e_a)$

$(\omega_a \in \mathbb{R})$

One forms

$$\omega = \omega_a e^a \qquad V = V^b e_b$$

The image of V under ω is the contraction of ω and V

$$\omega(V) = \omega(V^b e_b) = V^b \omega(e_b)$$

One forms

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The image of V under ω is the contraction of ω and V

$$\omega(V) = \omega(V^b e_b) = V^b \omega(e_b) = V^b \omega_a e^a(e_b)$$

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$$= \omega_a V^a$$

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$$= V^b \omega_a \delta^a{}_b$$

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notice the placement of indices

One forms

$\omega = \omega_a e^a$ $\qquad$ $V = V^b e_b$

* The spaces T_pM and T_p^*M are dual to each other:

a vector V can also be defined as a linear map on T_p^*M:

$$V(\omega) = \omega(V) = \omega_a V^a$$

One forms

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* It is possible to define one forms as the fundamental geometric objects and build vectors + tensors on top of them...

The gradient df

Given any smooth fuction on M $f \in \mathcal{F}(M)$, define the one form:

$$df(V) = V(f) = \frac{df}{dt}$$

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$$= \alpha\, df(V) + \beta\, df(W)$$

The gradient df

$$df(V) = V(f) = \frac{df}{dt}$$

* If $\{\partial_\mu\}$ a coordinate basis:

$$df(\partial_\mu) = \partial_\mu f$$

consistent with

$$df(V) = df(V^\mu \partial_\mu) = V^\mu df(\partial_\mu) = V^\mu \partial_\mu f = V(f)$$

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for $f = x^\mu$:

$$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta_\nu{}^\mu$$

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$$\Rightarrow \quad \omega(\partial_\mu) = \omega_\nu \, dx^\nu(\partial_\mu) = \omega_\nu \, \delta^\nu{}_\mu = \omega_\mu$$

$$\Rightarrow \quad \omega(V) = \omega_\nu \, dx^\nu(V^\mu \partial_\mu) = \omega_\nu V^\mu \, dx^\nu(\partial_\mu) = \omega_\nu V^\mu \delta^\nu{}_\mu = \omega_\mu V^\mu$$

$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta_\nu{}^\mu \Rightarrow \{dx^\mu\}$ dual to $\{\partial_\mu\}$

Change of coordinates: $x^\mu \to x^{\mu'}$

$$dx^{\mu'}(\partial_\nu) = \frac{\partial x^{\mu'}}{\partial x^\nu} \qquad \text{(from the definition of } dx^{\mu'}\text{)}$$

Change of coordinates: $x^\mu \to x^{\mu'}$

$$dx^{\mu'}(\partial_\nu) = \frac{\partial x^{\mu'}}{\partial x^\nu}$$ (gives ν component of $dx^{\mu'}$ in dx^ν basis)

$$\Rightarrow \quad dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu} dx^\nu$$

Change of coordinates: $x^\mu \to x^{\mu'}$

$$dx^{\mu'}(\partial_\nu) = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

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then:

$$\sigma = \sigma_\mu dx^\mu$$

$$\sigma = \sigma_{\mu'} dx^{\mu'}$$

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compare!

$$V^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} V^{\mu'}$$

Change of coordinates: $\quad x^{\mu} \to x^{\mu'}$

$$\sigma_{\mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \sigma_{\mu'}$$

$$\sigma_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \sigma_{\mu}$$

---

then:

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Geometric Interpretation of df

$$\Delta f = V(f)\,\Delta t$$

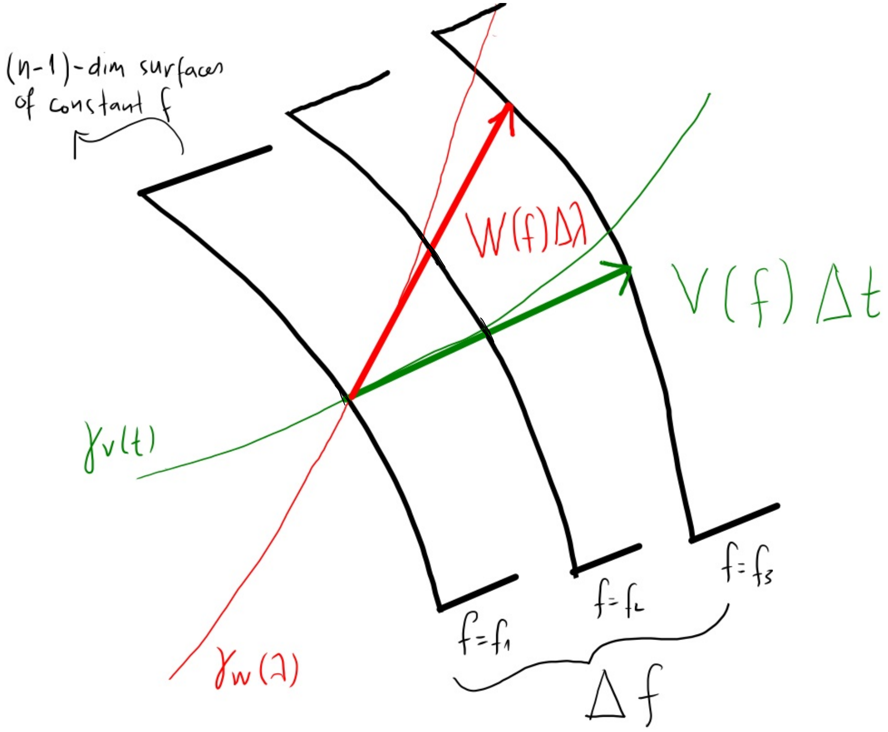
$$\| $$

$$df/dt$$

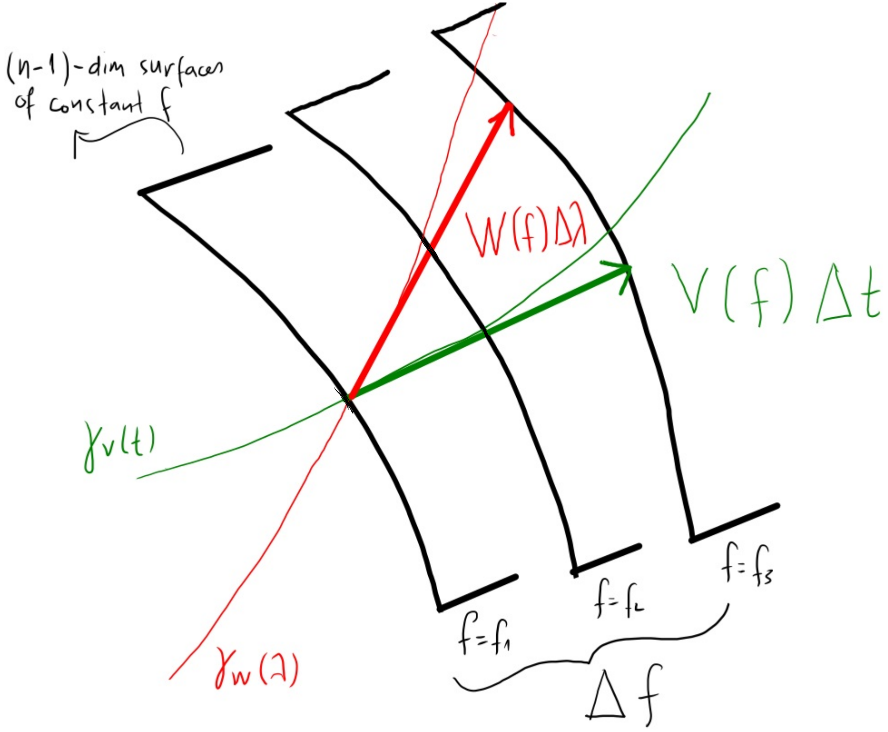
$$\Delta f = W(f)\,\Delta\lambda$$

$$\|$$

$$df/d\lambda$$



Geometric Interpretation of df



$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta \lambda$$

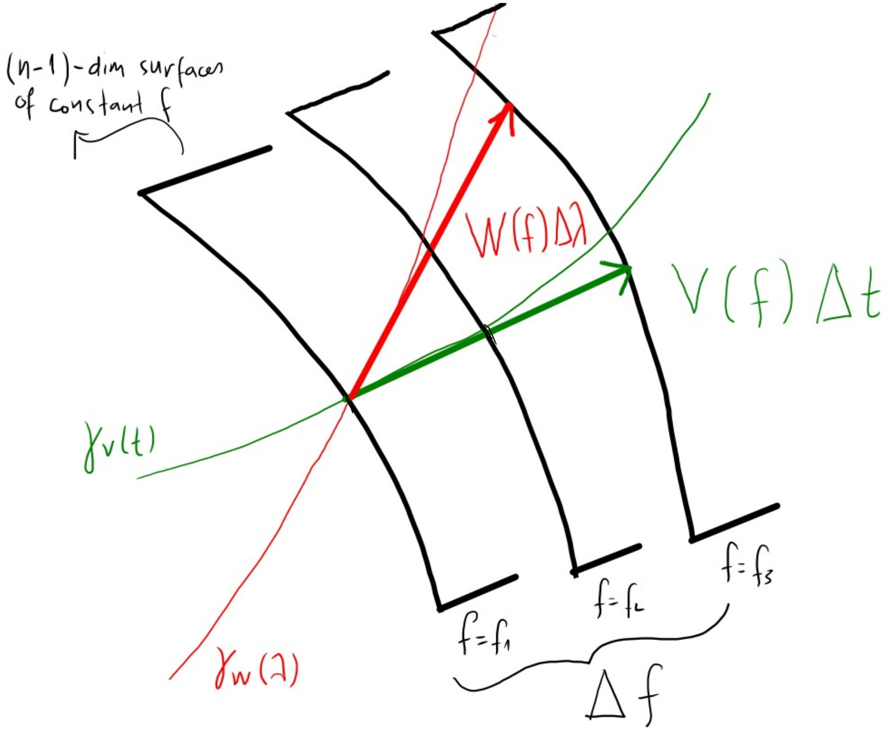
For all vectors:

$$\Delta f = df(V)\,\Delta t = \frac{\partial f}{\partial x^\mu} V^\mu \Delta t = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}\Delta t = \underbrace{\frac{\partial f}{\partial x^\mu}}_{\text{recall gradient of function}} \Delta x^\mu$$

$$\Delta f = \vec{\nabla} f \cdot d\vec{x}$$

Geometric Interpretation of df

$$\Delta f = V(f)\,\Delta t$$



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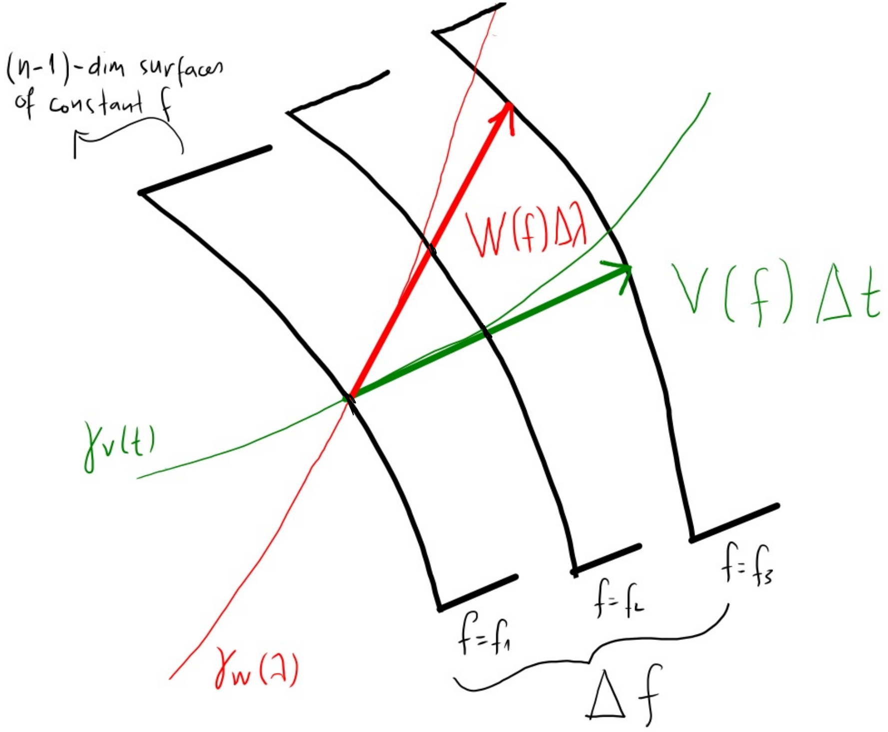
Geometric Interpretation of df



$$\Delta f = V(f)\,\Delta t$$

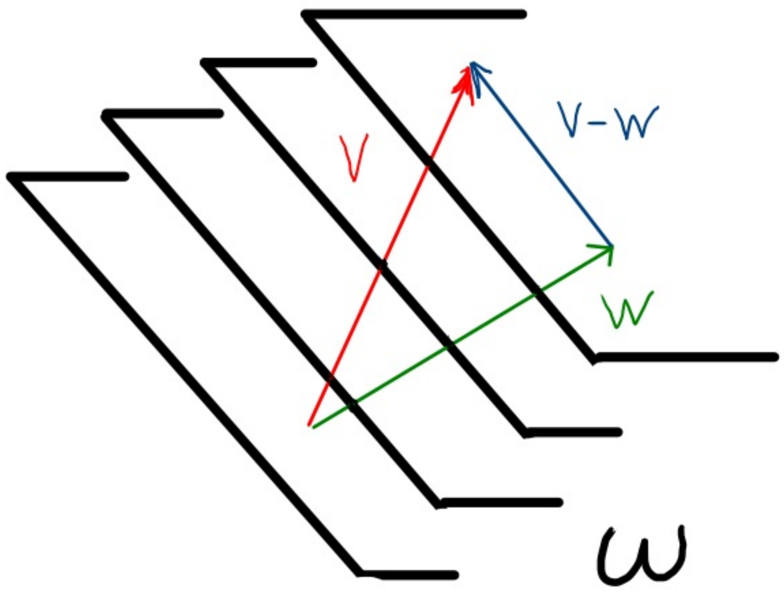
$$\Delta f = W(f)\,\Delta\lambda$$

For all vectors:

$$\Delta f = df(V)\,\Delta t = \begin{pmatrix}\text{\# of pierced surfaces by} \\ V \text{ per unit parameter}\end{pmatrix} \cdot \Delta t$$

Geometric Interpretation of df

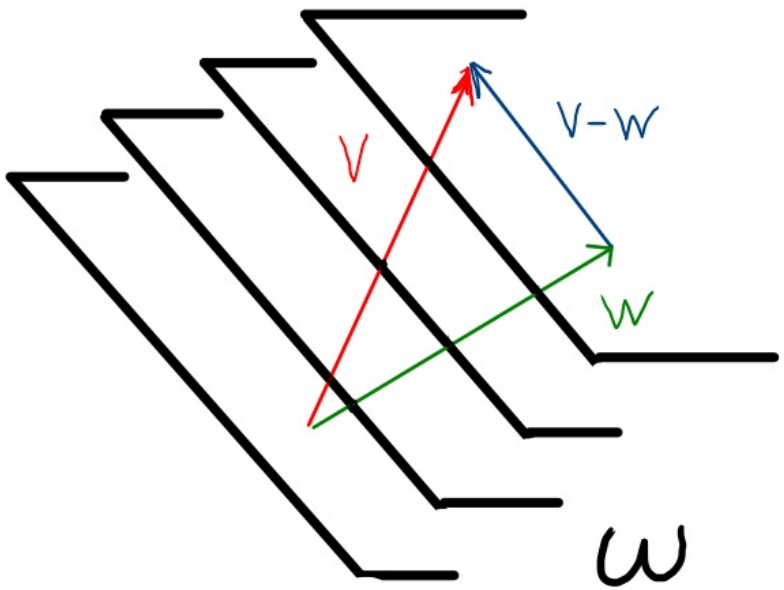
* We can picture a one form ω as a set of parallel hyperplanes:



Geometric Interpretation of df

* We can picture a one form ω as a set of parallel hyperplanes:

  - $\omega(V)$ is given by # of pierced planes by V



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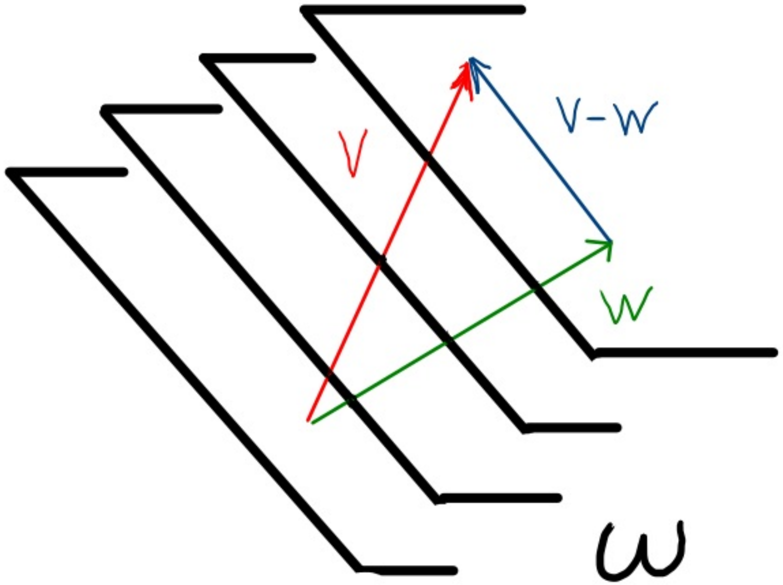

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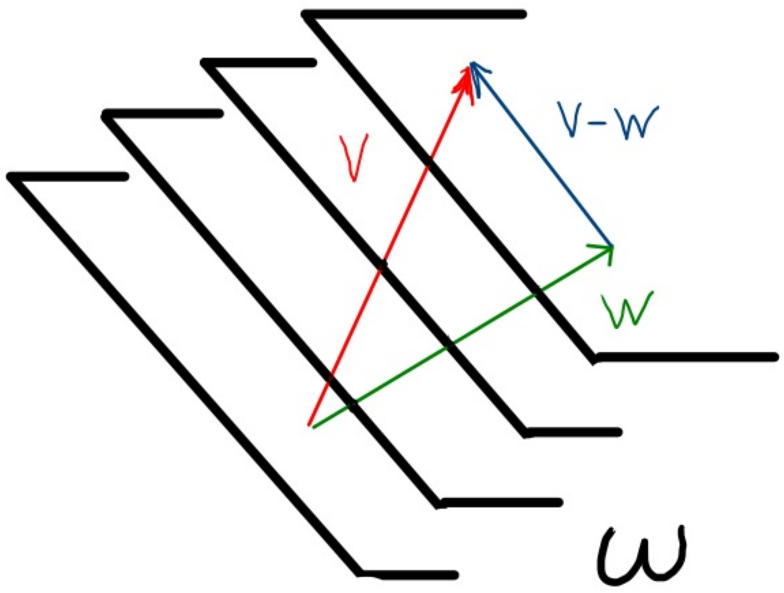
  - $\omega(V)$ is given by # of pierced planes by V
  - the denser they are, the larger $\omega(V)$ becomes
  - if $\omega(V) = \omega(W)$, then V, W differ by a vector "parallel" to the hyperplanes $(\omega(V-W) = 0)$

does not pierce ...



Geometric Interpretation of df

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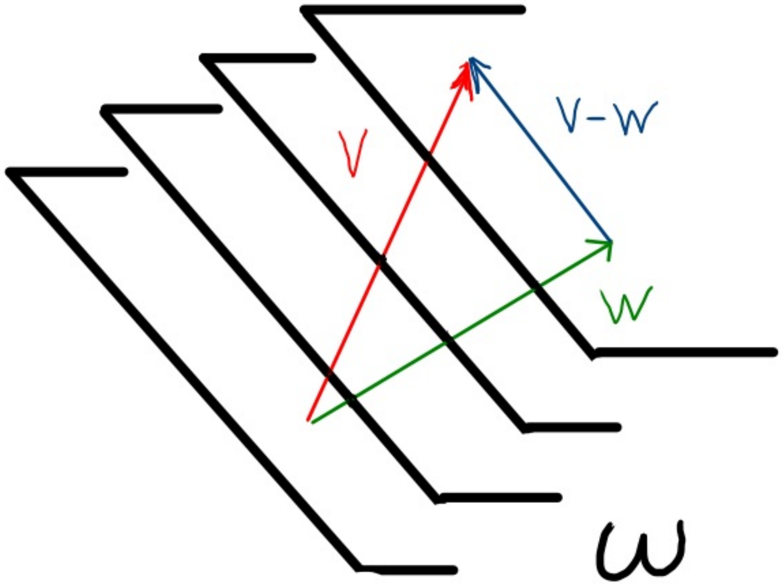
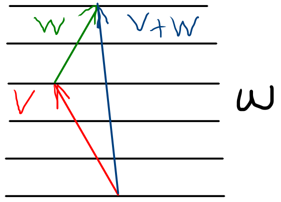


- depicts linearity of ω:



$$\omega(\alpha V) = \alpha\,\omega(V)$$
$$\omega(V+W) = \omega(V) + \omega(W)$$

- the denser they are, the larger $\omega(V)$ becomes

- if $\omega(V) = \omega(W)$, then V, W differ by a vector "parallel" to the hyperplanes ($\omega(V-W)=0$)

One form fields

* A map that assigns a 1-form at each point of M smoothly is a 1-form field on M

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If V is a smooth vector field, then a smooth ω gives

$$\omega(V)$$

a smooth function on M

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a smooth function on M

* For a smooth function f, df at each point of M is the gradient 1-form field of f s.t.

$$df(V) = V(f) = V^\mu \partial_\mu f$$

One form fields

* given x^μ, dx^μ is a smooth 1-form field on the chart, and

$$\omega = \omega_\mu \, dx^\mu$$

with $\omega_\mu(p)$ a smooth function on $\mathcal{M}$

---

* For a smooth function f, df at each point of $\mathcal{M}$ is the gradient 1-form field of f s.t.

$$df(V) = V(f) = V^\mu \partial_\mu f$$

One form fields

* given x^μ , dx^μ is a smooth 1-form field on the chart, and

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Example:

$$\omega([V,W]) = \omega([V,W]^\mu \partial_\mu)$$

Lie bracket of V and W

One form fields

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Example:

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$$= [V,W]^\mu \, \omega(\partial_\mu)$$

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$$\omega = \omega_\mu \, dx^\mu$$

with $\omega_\mu(p)$ a smooth function on M

Example:

$$\omega([V,W]) = \omega([V,W]^\mu \partial_\mu)$$
$$= [V,W]^\mu \omega(\partial_\mu)$$
$$= (V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu)\, \omega_\mu$$

Tensors

* A tensor T of type (k, l) is a linear map on

$$\underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-times}} \times \underbrace{T_pM \times \dots \times T_pM}_{l\text{-times}} \longrightarrow \mathbb{R}$$

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- form a vector space $T_p^{(k,l)}M$

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- type (k,l): or order, rank, valence, degree
- form a vector space $T_p^{(k,l)}M$
- vectors are type $(1,0)$ tensors

  1-forms $(0,1)$ tensors

Tensors

Example: $(1,1)$ tensor:

$$T: T_p^*M \times T_pM \to \mathbb{R}$$

$$(\omega, v) \mapsto T(\omega, v)$$

Tensors

Example: $(1,1)$ tensor:

$$T: T_p^*M \times T_pM \to \mathbb{R}$$

$$(\omega, V) \mapsto T(\omega; V)$$

Linearity: $T(\alpha\omega + \beta\sigma; V) = \alpha T(\omega; V) + \beta T(\sigma; V)$

$$T(\omega; \alpha V + \beta W) = \alpha T(\omega; V) + \beta T(\omega; W)$$

Tensors

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$$T(\omega; \alpha V + \beta W) = \alpha T(\omega; V) + \beta T(\omega; W)$$

Compute $T(\omega; V)$: coordinate bases $\{\partial_\mu\}$, $\{dx^\mu\}$

$$T(\omega; V) = T(\omega_\mu dx^\mu; V^\nu \partial_\nu)$$

Tensors

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Compute $T(\omega; V)$: coordinate bases $\{\partial_\mu\}$, $\{dx^\mu\}$

$$T(\omega; V) = T(\omega_\mu dx^\mu; V^\nu \partial_\nu) = \omega_\mu V^\nu T(dx^\mu; \partial_\nu) = T^\mu{}_\nu \, \omega_\mu V^\nu$$

$$T^\mu{}_\nu \equiv T(dx^\mu; \partial_\nu)$$

$$T(\omega, V) = T^{\mu}{}_{\nu}\, \omega_\mu V^\nu \quad , \qquad T^{\mu}{}_{\nu} = T(dx^\mu ; \partial_\nu)$$

Notice index placement!

For a (k,l) tensor:

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(dx^{\mu_1}, \dots, dx^{\mu_k}; \partial_{\nu_1}, \dots, \partial_{\nu_l})$$

$$T(\omega, V) = T^{\mu}{}_{\nu}\, \omega_{\mu} V^{\nu} \quad , \qquad T^{\mu}{}_{\nu} = T(dx^{\mu}; \partial_{\nu})$$

For a (k,l) tensor:

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = T(dx^{\mu_1}, \dots, dx^{\mu_k}; \partial_{\nu_1}, \dots, \partial_{\nu_l})$$

and

$$T(\omega, \dots ; V, \dots) = T^{\mu_1 \dots}{}_{\nu_1 \dots}\, \omega_{\mu_1} \dots \; V^{\nu_1} \dots$$

Tensor Product (or outer product)

- used to construct higher order tensors

S is of type (k_1, l_1)

T " " " (k_2, l_2)

Tensor Product

- used to construct higher order tensors

S is of type (k_1, l_1)

T " " " (k_2, l_2)

$S \otimes T$ " " " $(k_1 + k_2, l_1 + l_2)$ s.t.

$$S \otimes T(\sigma_1, \dots, \sigma_{k_1}, \tau_1, \dots, \tau_{k_2}; V_1, \dots, V_{l_1}, W_1, \dots, W_{l_2}) =$$

$$= S(\sigma_1, \dots, \sigma_{k_1}; V_1, \dots, V_{l_1}) \cdot T(\tau_1, \dots, \tau_{k_2}; W_1, \dots, W_{l_2})$$

Tensor Product

Note: • $S \otimes T \neq T \otimes S$

$$S \otimes T(\sigma_1, \dots, \sigma_{k_1}, \tau_1, \dots, \tau_{k_2}; V_1, \dots, V_{\ell_1}, W_1, \dots, W_{\ell_2}) =$$

$$= S(\sigma_1, \dots, \sigma_{k_1}; V_1, \dots, V_{\ell_1}) \cdot T(\tau_1, \dots, \tau_{k_2}; W_1, \dots, W_{\ell_2})$$

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$\{ dx^\mu \otimes dx^\nu \}$ a coordinate basis of $T_p^{(0,2)} \mathcal{M}$

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$\{ dx^\mu \otimes dx^\nu \}$ a coordinate basis of $T_p^{(0,2)} M$ $\Rightarrow$ $\dim T_p^{(0,2)} M = n^2$

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$S = S_{\mu\nu} \, dx^\mu \otimes dx^\nu$, because:

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$$\Updownarrow$$

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(notice index placement)

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$\{e_a\}$ in T_pM s.t $e^a(e_b) = \delta^a{}_b$ (dual)

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$T(\omega, \sigma; V) \in \mathbb{R}$

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$$= \underbrace{\left(\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\rho'}}{\partial x^\rho} T^{\mu'\nu'}{}_{\rho'} \right)}_{T^{\mu\nu}{}_\rho} \partial_\mu \otimes \partial_\nu \otimes dx^\rho$$

Example: $T \in T_p^{(2,1)}\mathcal{M}$, $R \in T_p^{(1,1)}\mathcal{M}$

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma$$

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- $T \otimes R$ is a $(2+1, 1+1) = (3,2)$ tensor
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- $T \otimes R = T^{\mu\nu}{}_{\rho}\ R^{\lambda}{}_{\sigma}\ \underset{\omega}{\partial_\mu} \otimes \underset{\sigma}{\partial_\nu} \otimes \underset{V}{dx^\rho} \otimes \underset{\rho}{\partial_\lambda} \otimes \underset{W}{dx^\sigma}$ $\qquad$ $T \otimes R(\omega, \sigma, V, \rho, W)$

Notice position: defines the "slot" to put 1-form ρ

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Notice position: defines the "slot" to put 1-form ρ

$$= T^{\mu\nu}{}_{\rho} R^{\lambda}{}_{\sigma}\, \underset{\omega}{\partial_\mu} \otimes \underset{\sigma}{\partial_\nu} \otimes \underset{\rho}{\partial_\lambda} \otimes \underset{V}{dx^\rho} \otimes \underset{W}{dx^\sigma}$$

← put in "canonical form"

Example: $T \in T_p^{(2,1)}\mathcal{M}$, $R \in T_p^{(1,1)}\mathcal{M}$

$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma$

- $T \otimes R$ is a $(2+1, 1+1) = (3,2)$ tensor
- $T \otimes R(\omega, \sigma, \rho; V, W) = T(\omega, \sigma; V) \cdot R(\rho; W)$
- $T \otimes R = T^{\mu\nu}{}_{\rho}\, R^{\lambda}{}_{\sigma}\, \underset{\omega}{\partial_\mu} \otimes \underset{\sigma}{\partial_\nu} \otimes \underset{V}{dx^\rho} \otimes \underset{\rho}{\partial_\lambda} \otimes \underset{W}{dx^\sigma}$

$= T^{\mu\nu}{}_{\rho}\, R^{\lambda}{}_{\sigma}\, \underset{\omega}{\partial_\mu} \otimes \underset{\sigma}{\partial_\nu} \otimes \underset{\rho}{\partial_\lambda} \otimes \underset{V}{dx^\rho} \otimes \underset{W}{dx^\sigma}$

Notation is unambiguous: All it matters is to contract the indices correctly and

$R(dx^\lambda; \partial_\sigma) = R^{\lambda}{}_{\sigma}$

$T(dx^\mu, dx^\nu; \partial_\rho) = T^{\mu\nu}{}_{\rho}$

Contractions:

Example: $T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

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Define the $(1,0)$ tensor:

$$T(\ldots, dx^\lambda ; \partial_\lambda)$$

$\hookrightarrow$ a slot for a 1-form

Contractions:

Example: $T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

free

Define the $(1,0)$ tensor:

$$T(\ldots, dx^\lambda ; \partial_\lambda) = T^{\mu\nu}{}_{\rho}\ \partial_\nu(dx^\lambda)\ dx^\rho(\partial_\lambda)\ \partial_\mu$$

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We say that the ν, ρ indices have been contracted

Contractions:

Contracting two indices transforms a $(k,l) \to (k-1,l-1)$ tensor with components

$$T^{\mu_1 \dots \lambda \dots \mu_k}{}_{\nu_1 \dots \lambda \dots \nu_l}$$

Contract indices between tensors:

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad (2,1) \text{ tensor}$$

$$R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma \qquad (1,1) \quad "$$

$$T \otimes R = T^{\mu\nu}{}_{\rho} R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\nu \otimes \partial_\lambda \otimes dx^\rho \otimes dx^\sigma \qquad (3,2) \quad "$$

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contract ρ, λ

$$S = T^{\mu\nu}{}_{\lambda} R^{\lambda}{}_{\sigma} \; \partial_\mu \otimes \partial_\nu \otimes dx^\sigma \qquad (2,1) \;\; "$$

Symmetric Tensors

$g_{\mu\nu} = g_{\nu\mu}$ totally symmetric in its 2 indices

$A_{\mu\nu\rho} = A_{\nu\mu\rho}$ symmetric in its first 2 indices

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$$g_{\mu\nu} = g_{\nu\mu} \iff g(\partial_\mu, \partial_\nu) = g(\partial_\nu, \partial_\mu)$$

$$\iff g(V, W) = g(W, V) \qquad \forall V, W$$

Symmetric Tensors

$g_{\mu\nu} = g_{\nu\mu}$ totally symmetric in its 2 indices $\Rightarrow g_{\mu'\nu'}$ also symmetric

$A_{\mu\nu\rho} = A_{\nu\mu\rho}$ symmetric in its first 2 indices $\Rightarrow \underbrace{A_{\mu'\nu'\rho'} \quad \text{"} \quad \text{"}}_{\text{prove!}}$

$\left.\begin{array}{c}\text{coordinate system}\\ \text{basis}\end{array}\right\}$ - independent property!

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$A_{\mu\nu\rho} = A_{\rho\mu\nu} = A_{\nu\rho\mu} =$ symmetric in all its 3 indices

$= A_{\rho\nu\mu} = A_{\mu\rho\nu} = A_{\nu\mu\rho}$ $3! = 6$ permutations

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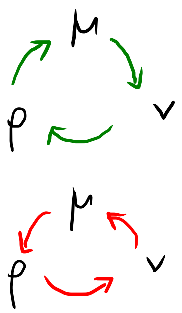

symmetric in all its 3 indices

$3! = 6$ permutations

Symmetrization

$$g_{(\mu\nu)} = \frac{1}{2}\left(g_{\mu\nu} + g_{\nu\mu} \right)$$

$$A_{(\mu\nu)\rho} = \frac{1}{2}\left(A_{\mu\nu\rho} + A_{\nu\mu\rho} \right)$$

$$A_{(\mu\nu\rho)} = \frac{1}{3!}\left(A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} + A_{\rho\nu\mu} + A_{\mu\rho\nu} + A_{\nu\mu\rho} \right)$$

$$S_{(\mu_1 \cdots \mu_k)} = \frac{1}{k!} \sum_{\sigma} S_{\sigma(\mu_1)\,\sigma(\mu_2) \cdots \sigma(\mu_k)}$$

$$\sigma = \begin{pmatrix} \mu_1 & \cdots & \mu_k \\ \sigma(\mu_1) & \cdots & \sigma(\mu_k) \end{pmatrix}$$

1-1 map of set of integers

Antisymmetric Tensors (skew-symmetric)

$$F^{\mu\nu} = -F^{\nu\mu}$$

$$A_{\mu\nu\rho} = -A_{\nu\mu\rho}$$

$$S_{\mu_1\cdots\mu_k} = \left\{ \begin{array}{ll} +S_{\sigma(\mu_1)\cdots}S_{\sigma(\mu_k)} & \sigma \text{ even} \\ -S_{\sigma(\mu_1)}\cdots S_{\sigma(\mu_k)} & \sigma \text{ odd} \end{array} \right\} \equiv \operatorname{sign}(\sigma)\, S_{\sigma(\mu_1)\cdots\sigma(\mu_k)}$$

σ - even: an even # of permutations, $\operatorname{sign}(\sigma) = +1$

odd: " odd " , $\operatorname{sign}(\sigma) = -1$

Antisymmetrization

$$F^{[\mu\nu]} = \frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu} \right)$$

$$A_{[\mu\nu]\rho} = \frac{1}{2}\left(A_{\mu\nu\rho} - A_{\nu\mu\rho} \right)$$

$$\begin{aligned} A_{[\mu\nu\rho]} = \frac{1}{3!}\Big(\; & A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} && \text{even} \\ & - A_{\rho\nu\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} && \text{odd} \end{aligned}$$

$$\begin{tikzcd} & \mu \arrow[dr] & \\ \rho \arrow[ur] & & \nu \arrow[ll] \end{tikzcd}$$

$$\begin{tikzcd} & \mu \arrow[dl] & \\ \rho \arrow[rr] & & \nu \arrow[ul] \end{tikzcd}$$

$$S_{[\mu_1 \cdots \mu_k]} = \frac{1}{k!} \sum_{\sigma} \operatorname{sign}(\sigma)\, S_{\sigma(\mu_1) \cdots \sigma(\mu_k)}$$