- Vectors
  - definition as tangents to curves
  - proof that they form a vector space
- Tangent Space at a point T_pM
  - coordinate bases
  - component xfms
- Vector Fields
  - integral curves
  - Lie bracket, Lie derivative

Vectors

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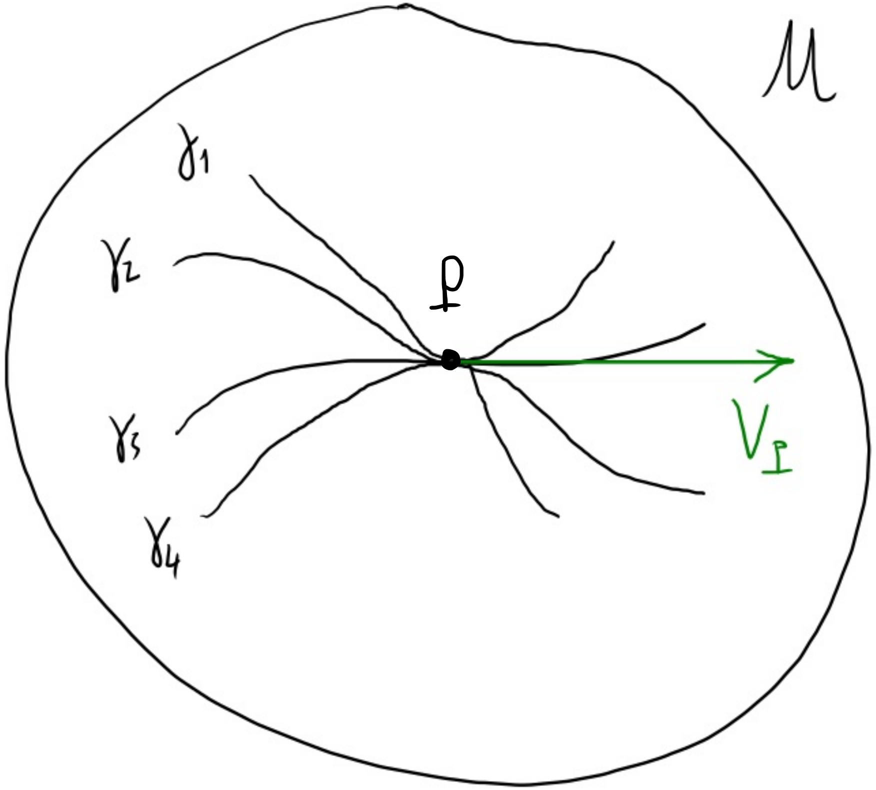
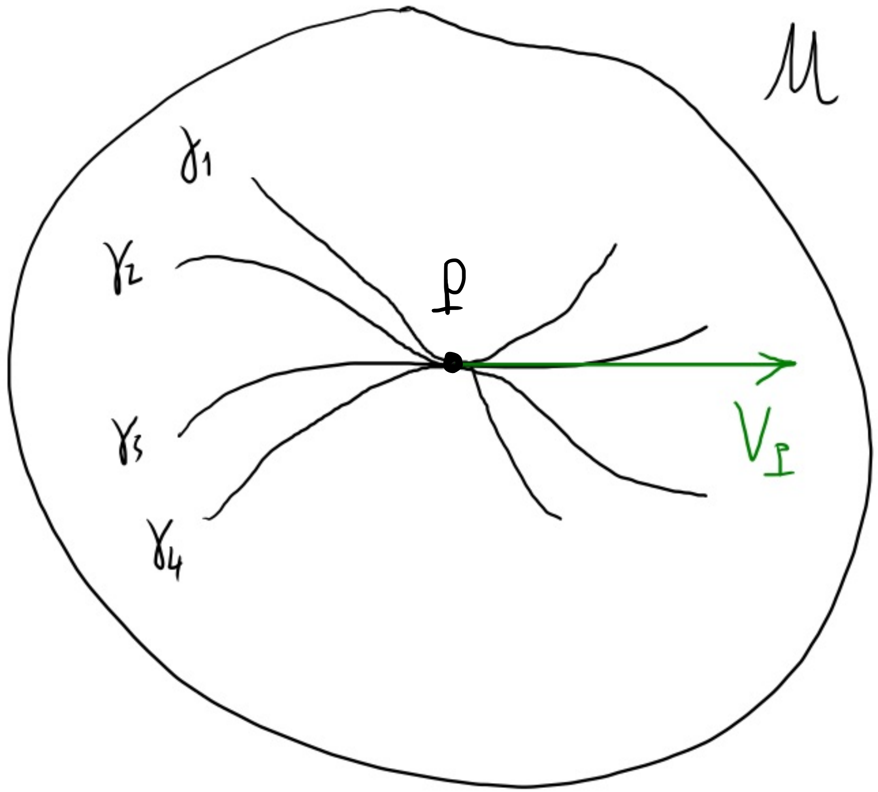
* define them as objects tangent to curves
  - they form a vector space $T_p\mathcal{M}$
* will use them as fundamental objects to define one-forms as linear functions on $T_p\mathcal{M}$
  - then we can define higher rank tensors as linear functions on vectors + one forms



* many curves passing through p have same tangent vector

$\Rightarrow$ equivalence class of

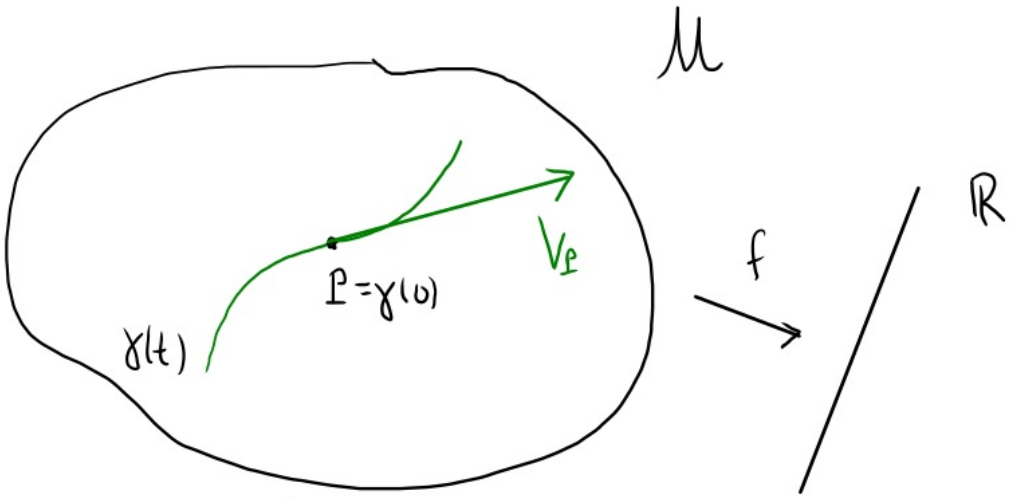
$$\gamma_i \sim \gamma_j$$

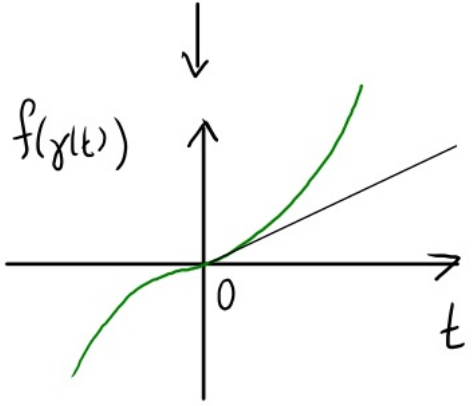


* many curves passing through p have same tangent vector

* the vector V_p ("velocity") depends on the rate of change of "things" while moving on each curve

Remark: The vector will depend also on choice of parameter on curve, not only on the points that belong to the curve

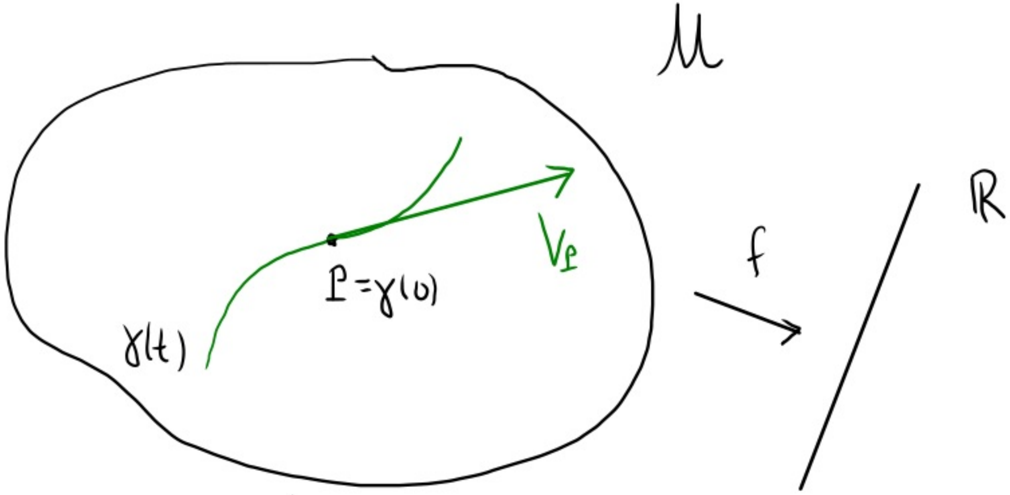


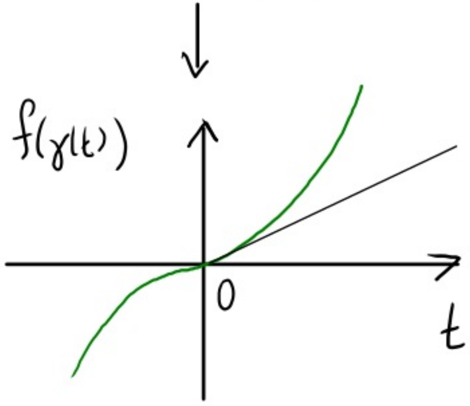


$$\frac{df(\gamma(0))}{dt} = \frac{d\, f\circ\gamma(t)}{dt}\Big|_0 \equiv \frac{df}{dt}\Big|_0$$

* "things" are functions on M

$$f : M \to \mathbb{R}$$





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composed with f, defines a real function on γ

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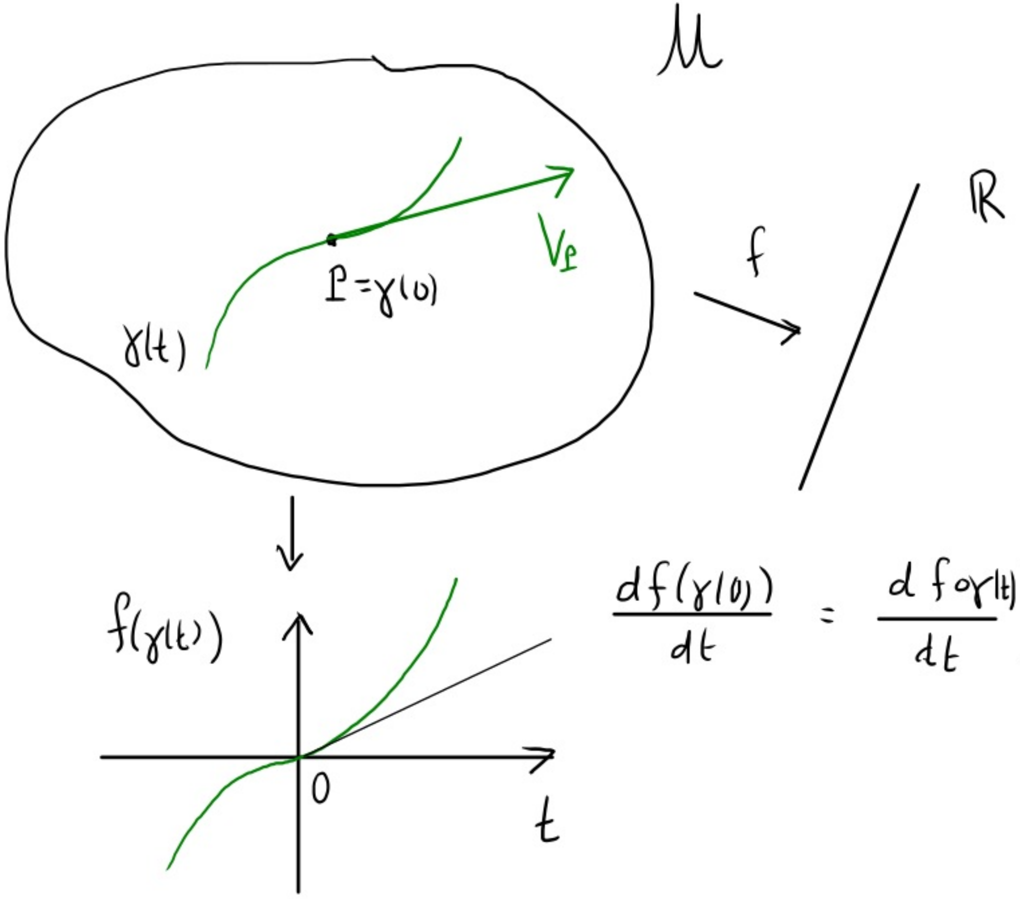
$$t \to f(\gamma(t))$$



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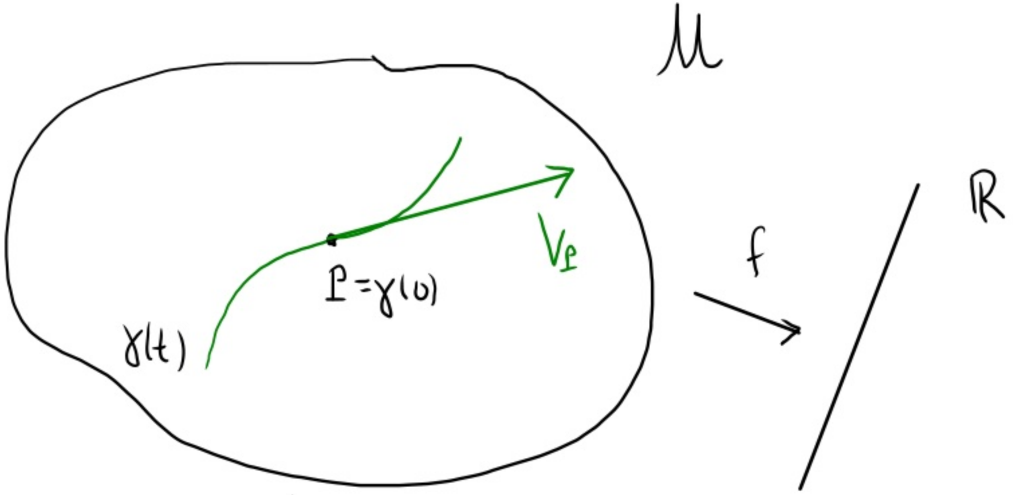
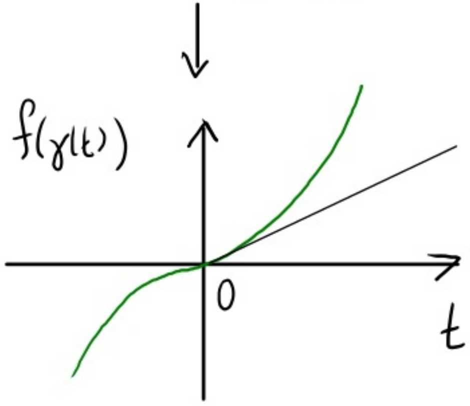
* the derivative at $P=\gamma(0)$: $\left.\frac{df}{dt}\right|_0 \equiv \frac{df(\gamma(0))}{dt}$

How fast f changes at P





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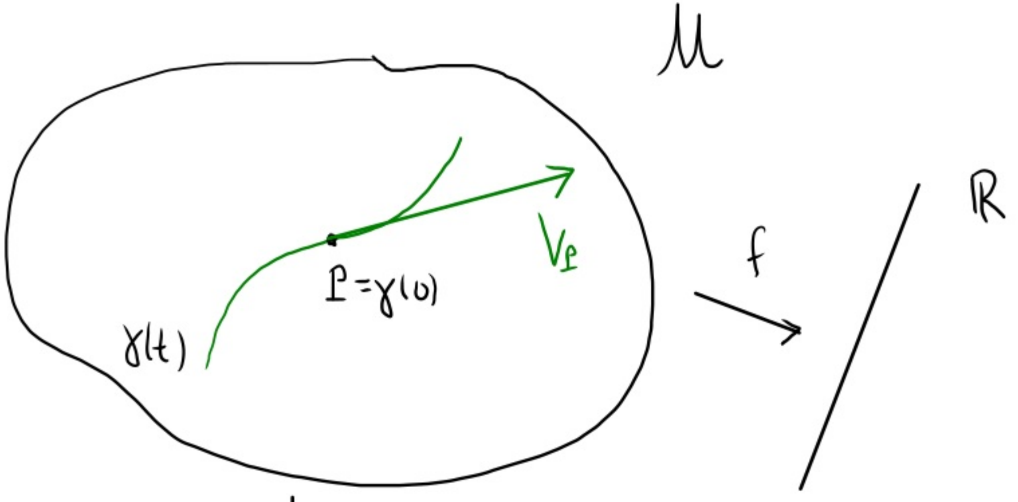

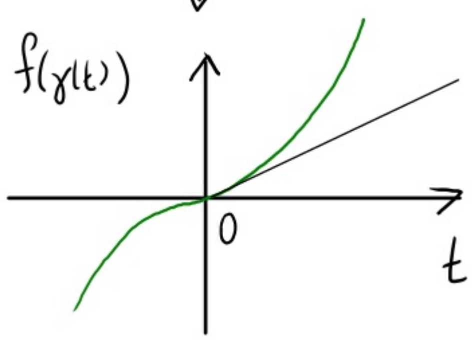

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* the derivatives $\left.\frac{df}{dt}\right|_0$ for <u>any</u> function f, a measure of how fast things change at p

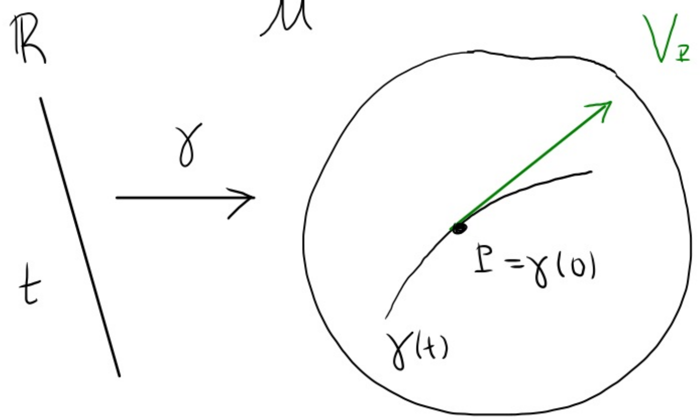
* we <u>define</u> V_p to be the operator $V_p = \left.\frac{d}{dt}\right|_0$

s.t. for any f $\quad V_p(f) = \left.\frac{df}{dt}\right|_0$

* $V_p(f)$ is the directional derivative of f at p

* can be thought of as acting linearly on functions:

$$V_p(\alpha f + \beta g) = \alpha V_p(f) + \beta V_p(g) \qquad \alpha, \beta \in \mathbb{R}$$



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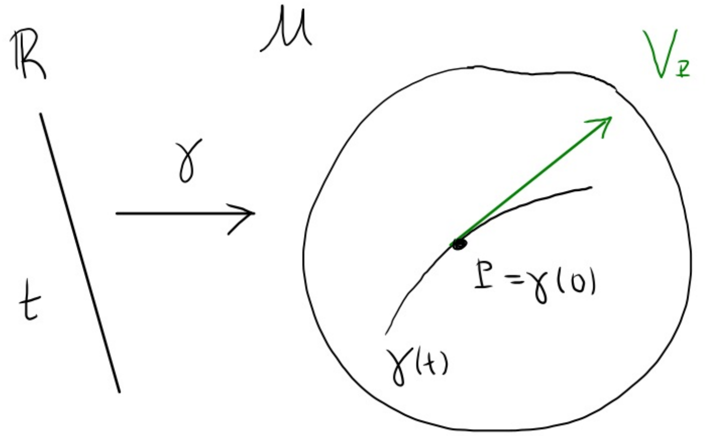

$$V_p(\alpha f + \beta g) = \alpha V_p(f) + \beta V_p(g) \qquad \alpha, \beta \in \mathbb{R}$$

~ obvious from the linear action of the derivative operator:

$$\frac{d}{dt}(\alpha f + \beta g)\Big|_0 = \alpha \frac{df}{dt}\Big|_0 + \beta \frac{dg}{dt}\Big|_0$$

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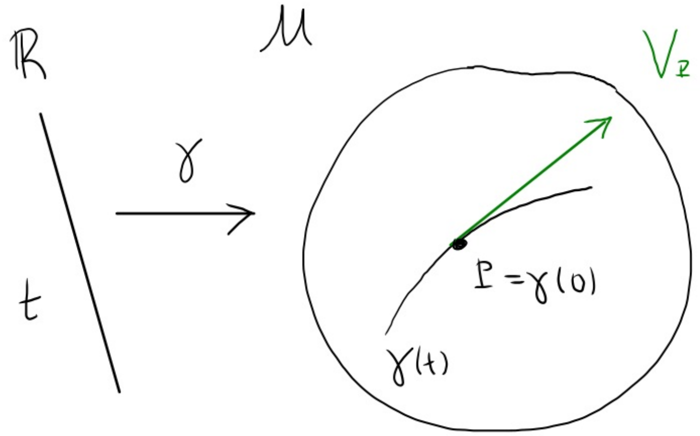

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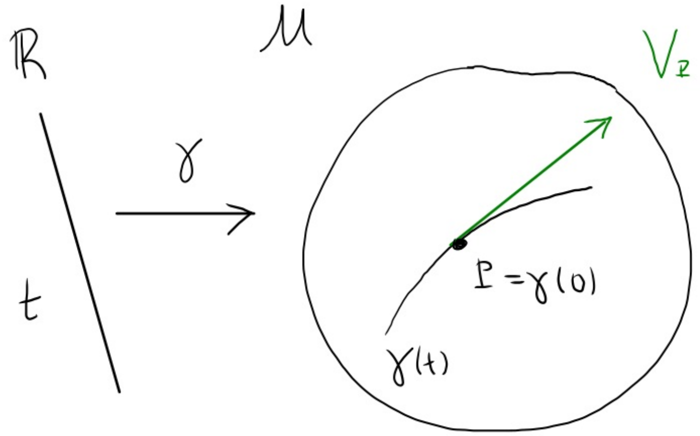

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$$V_p(f\cdot g) = V_p(f)\cdot g + f\, V_p(g)$$

Because it is a property of the derivative of the product $f\cdot g$:

$$\frac{d}{dt}(f\cdot g)\Big|_0 = \frac{df}{dt}\cdot g\Big|_0 + f\,\frac{dg}{dt}\Big|_0$$

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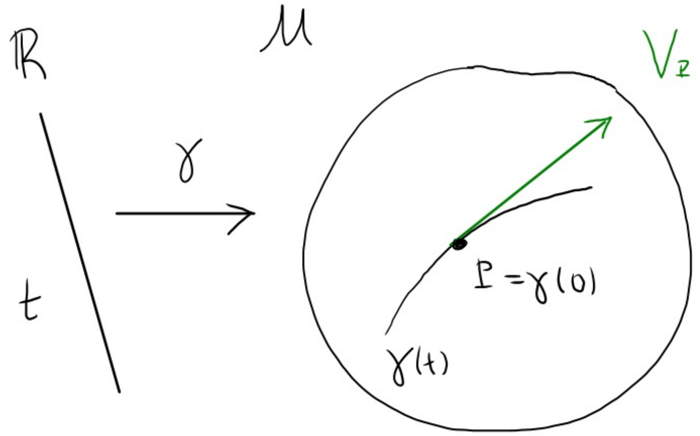

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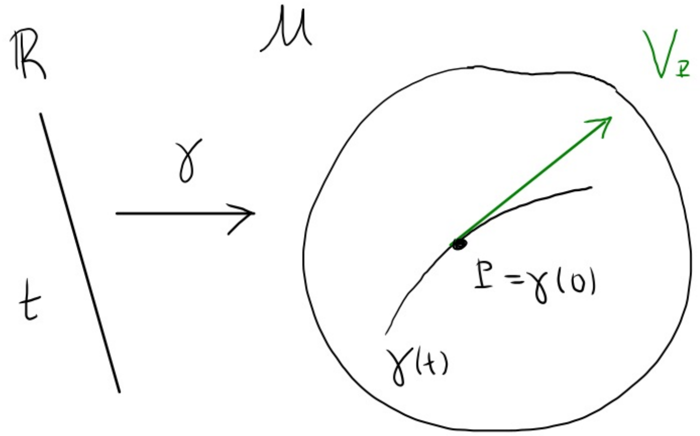
$\Rightarrow$ V_P a "derivation"

* $V_P(f)$ is the directional derivative of f at P



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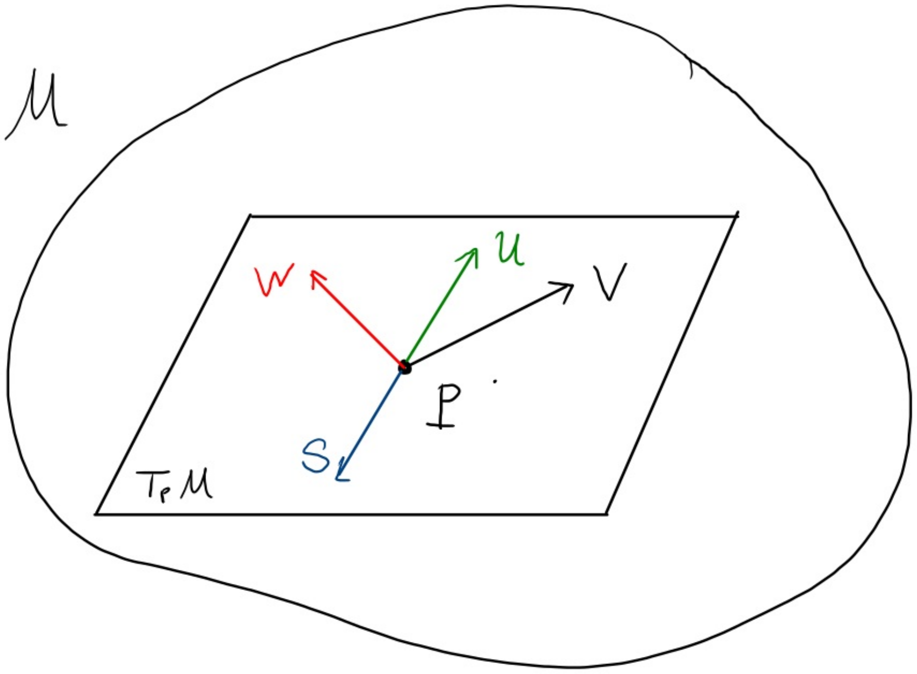
* vectors at P are identified with all possible derivations on functions on M

* Vectors at $\underline{P}$ form a vector space: M

$T_p M$

The tangent space of M at $\underline{P}$



* therefore, if $U, V \in T_p M$, then

$$\alpha V + \beta U \in T_p M$$

* Indeed, $W = \alpha V + \beta U$ is a derivation:

$\forall f, g \in \mathcal{F}(M)$

$$W(c_1 f + c_2 g) = c_1 W(f) + c_2 W(g)$$

$$W(f \cdot g) = W(f) \cdot g + f \cdot W(g)$$

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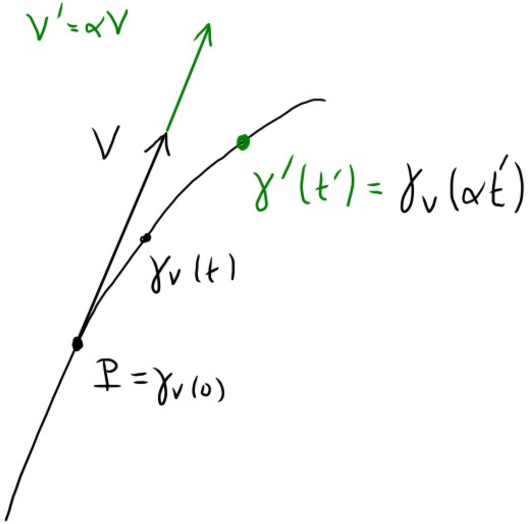
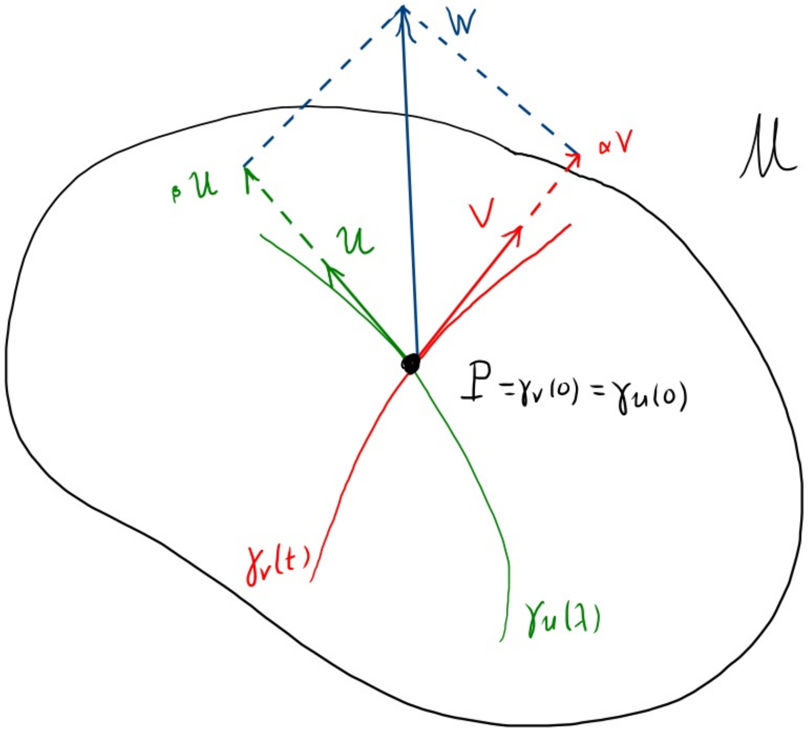
* But this does not give us a geometric understanding...

* The vector αV is easy to understand:

- consider a reparametrization of $\gamma_v(t)$:





- Define a new curve $\gamma'(t') = \gamma_v(\underbrace{\alpha t'}_{t = \alpha t'})$

$$V'_p(f) = \frac{d}{dt'} f \circ \gamma'(t') \Big|_0$$

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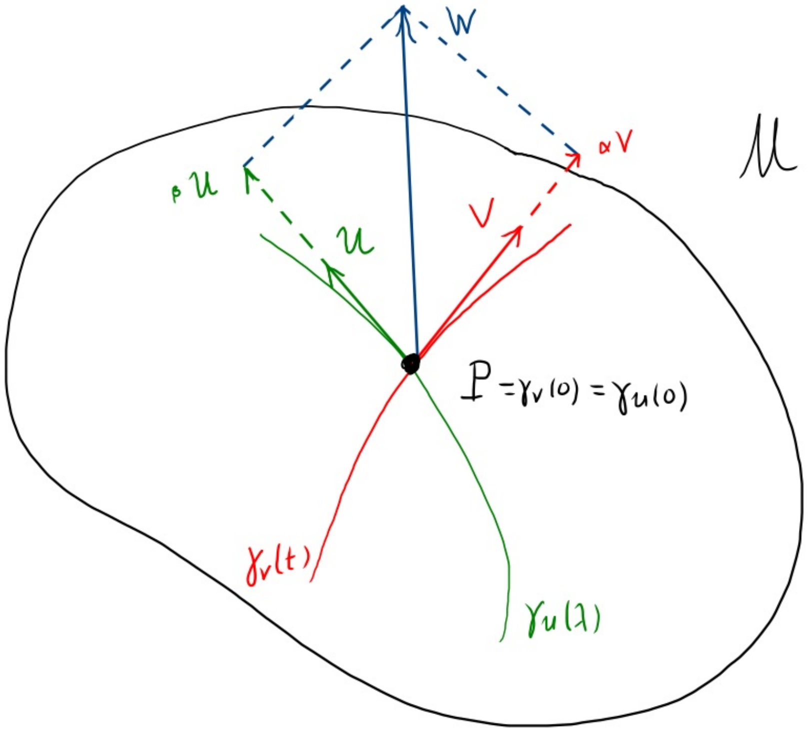

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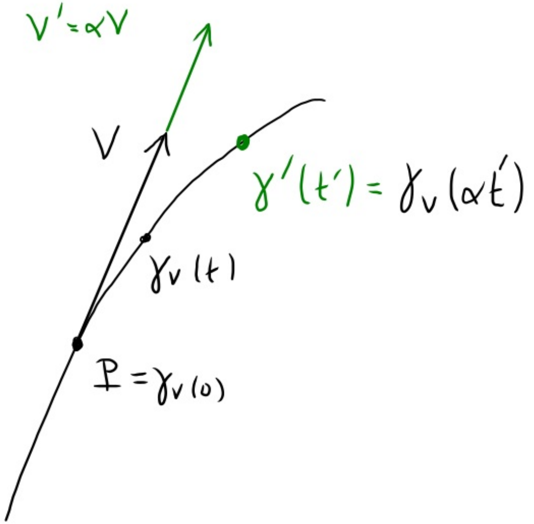

• Define a <u>new</u> curve $\gamma'(t') = \gamma_v(\alpha t')$

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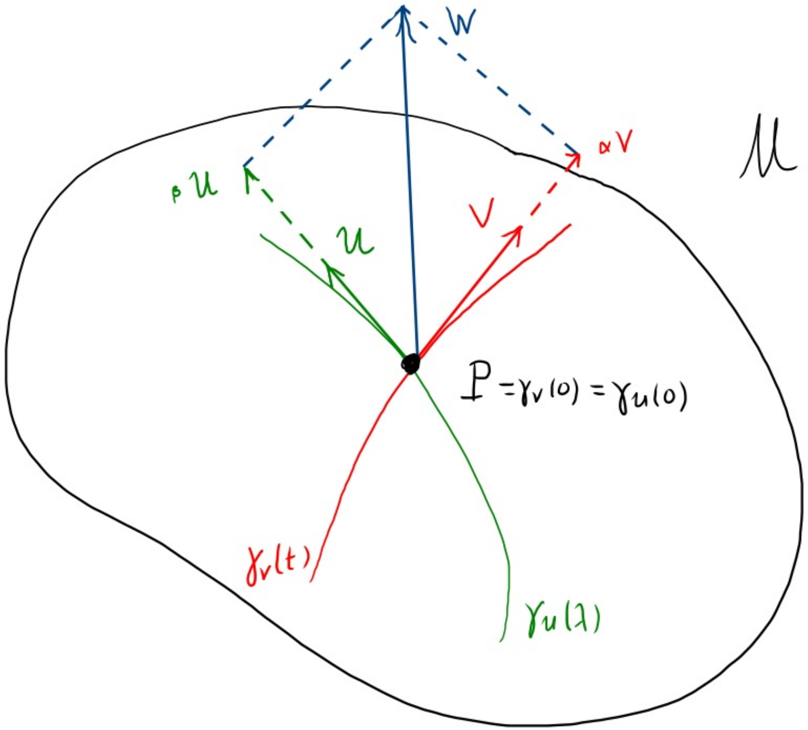

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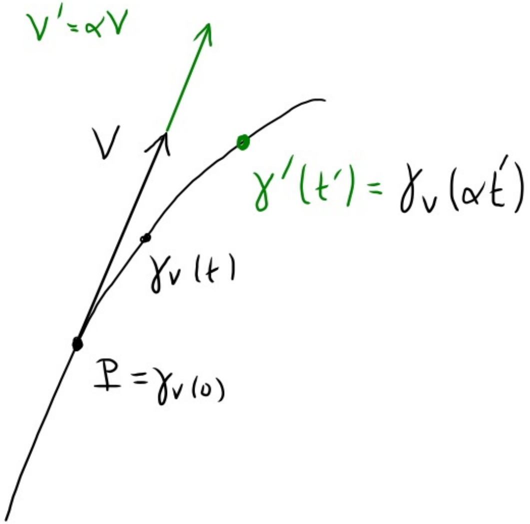

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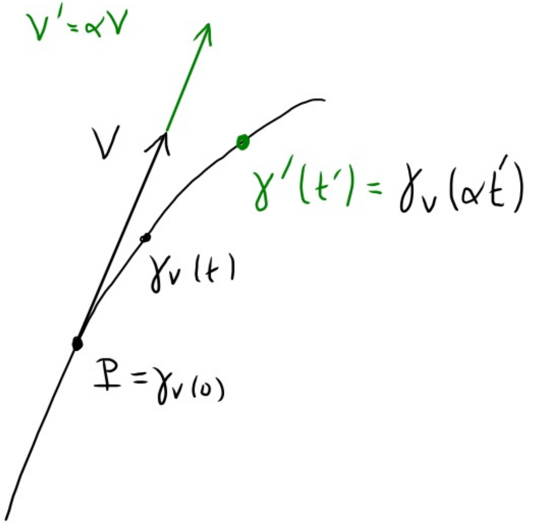

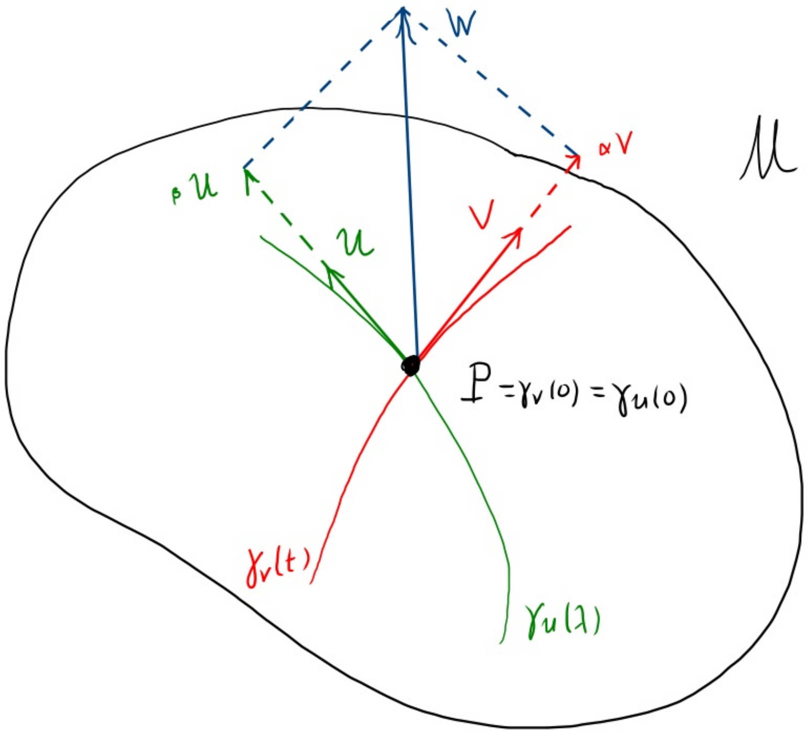

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* The vector $\alpha V + \beta U$ is given by

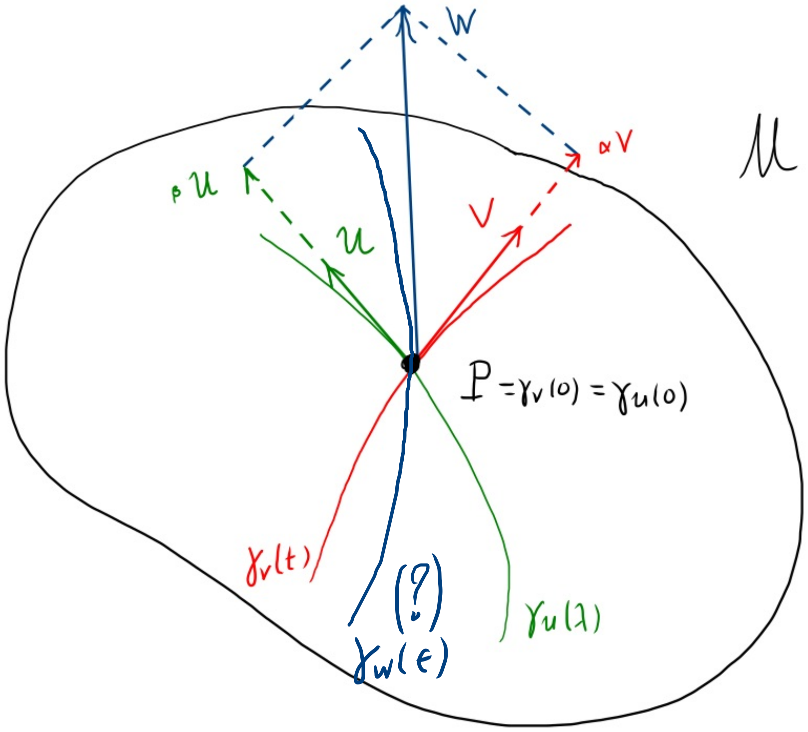
$$(\alpha V + \beta U)(f) = \alpha \left.\frac{df}{dt}\right|_0 + \beta \left.\frac{df}{d\lambda}\right|_0$$

But is there a class of curves such that

$$\alpha \left.\frac{df}{dt}\right|_0 + \beta \left.\frac{df}{d\lambda}\right|_0 = \left.\frac{df}{d\epsilon}\right|_0$$

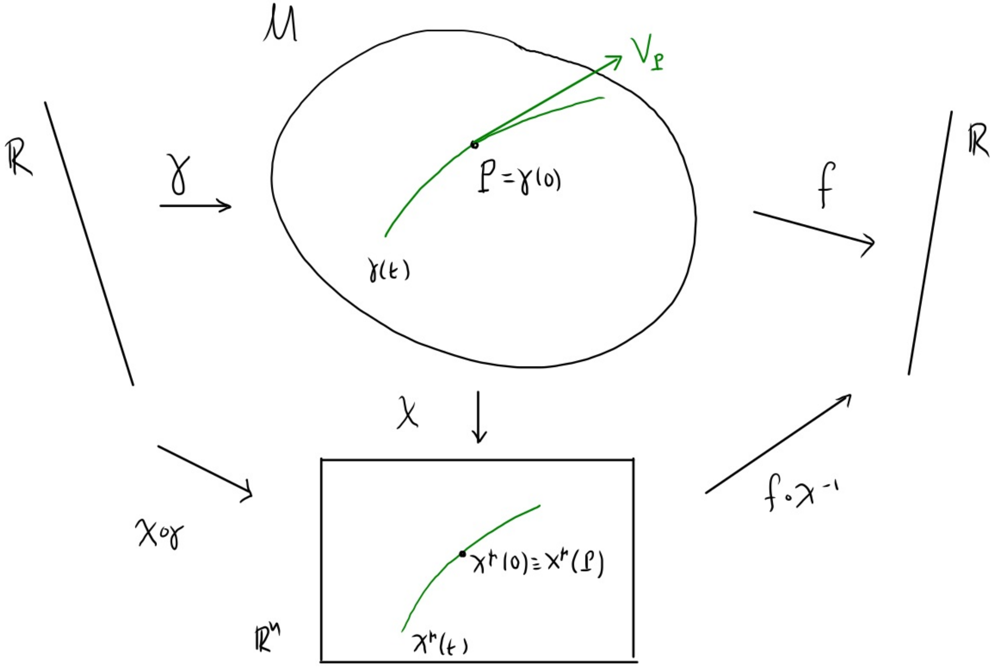
for all f?

yes... but we will see why...



* Consider first $V_p(f)$:

$$V_p(f) = \frac{df}{dt}\Big|_p$$



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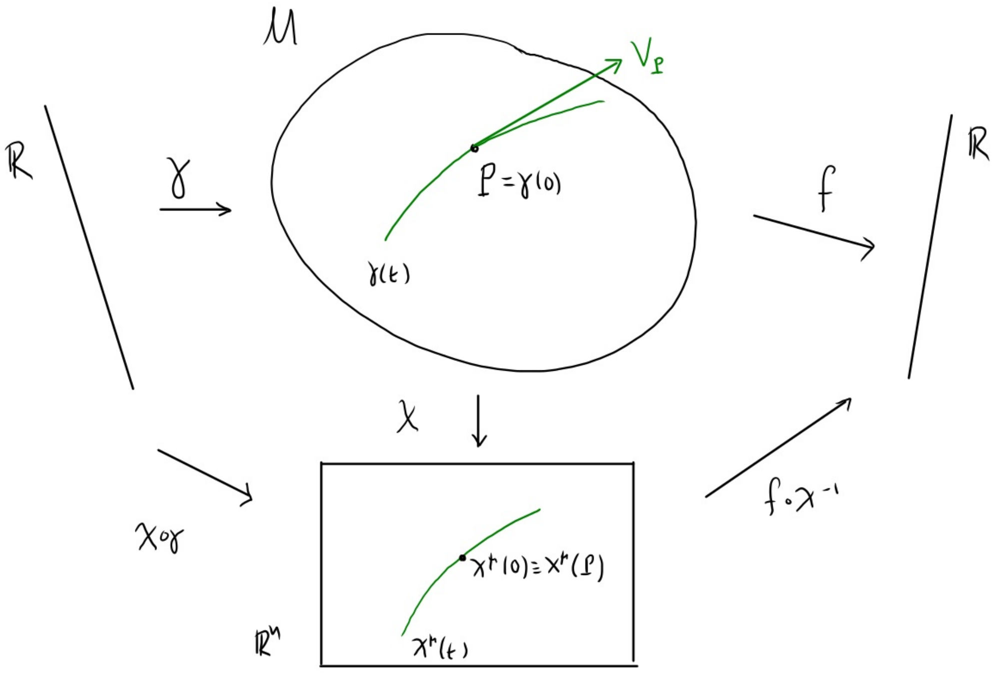

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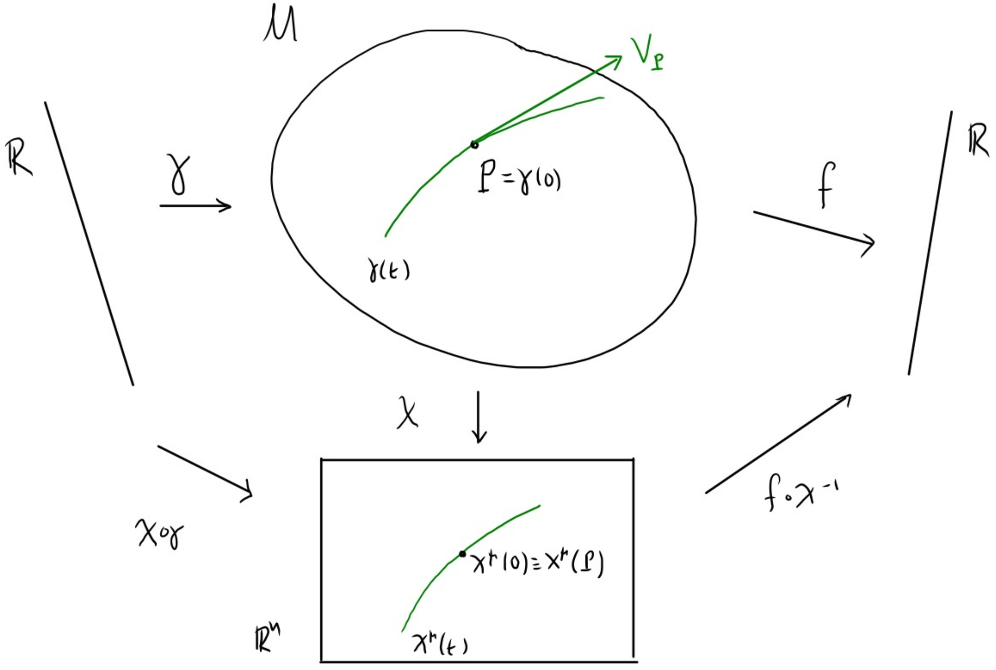

$$V_p(f) = \left.\frac{df}{dt}\right|_p$$

$$= \left.\frac{d}{dt} f\circ\gamma(t)\right|_0$$

$$= \left.\frac{d}{dt} \underbrace{f\circ\chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi\circ\gamma(t)}_{x^\mu(t)}\right|_0$$

$x^\mu(t)$ ⟶ Image of curve in coordinate system

$f(x^\nu)$ ⟶ function on coordinates

<u>Notation</u>: $f(x^\nu) = f(x^0, x^1, \dots, x^{n-1})$

$$x^\mu(t) = \left(x^0(t), x^1(t), \dots, x^{n-1}(t)\right) \in \mathbb{R}^n$$

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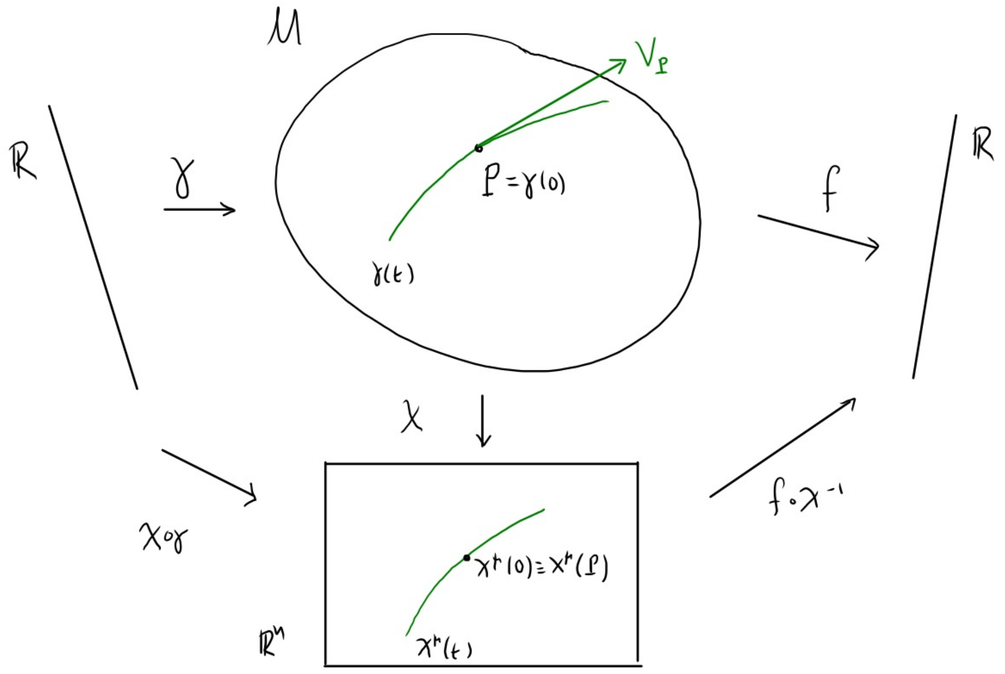
$$= \frac{d}{dt} \underbrace{f\circ\chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi\circ\gamma(t)}_{x^\mu(t)}\bigg|_0$$

$$= \frac{d}{dt} f\circ\chi^{-1}\left(x^\mu(t)\right)\bigg|_0$$



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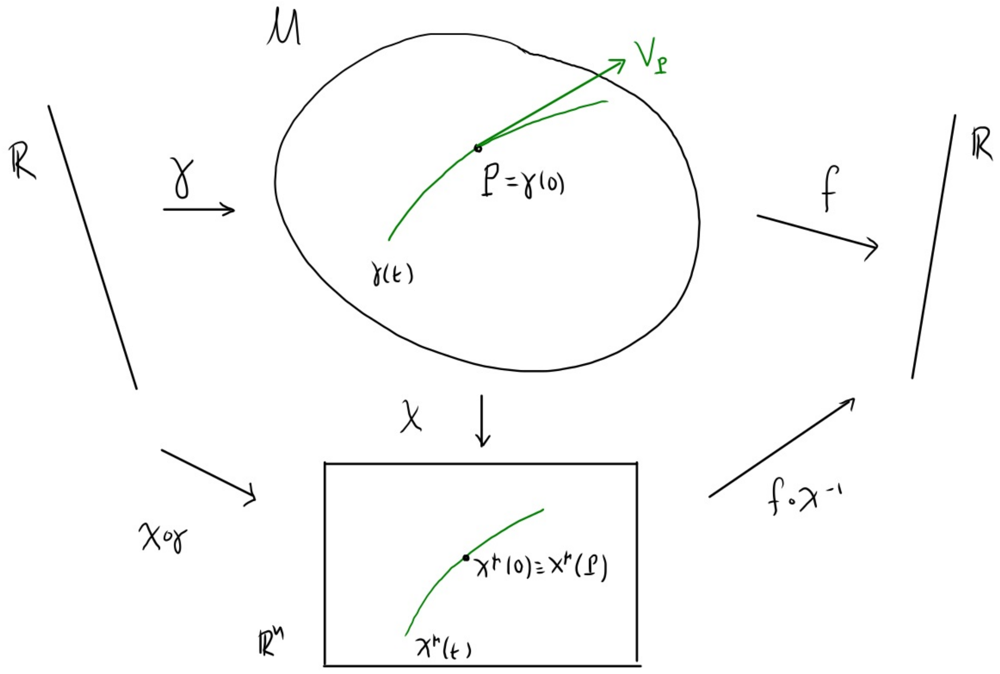


$$V_p(f) = \frac{df}{dt}\Big|_p$$

$$= \frac{d}{dt} f \circ \gamma(t) \Big|_0$$

$$= \frac{d}{dt} \underbrace{f \circ \chi^{-1}}_{f(x^\nu)} \circ \underbrace{\chi \circ \gamma(t)}_{x^\mu(t)} \Big|_0$$

$$= \frac{d}{dt} f \circ \chi^{-1}\left(x^\mu(t)\right) \Big|_0 \qquad = \frac{\partial f(x^\nu)}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

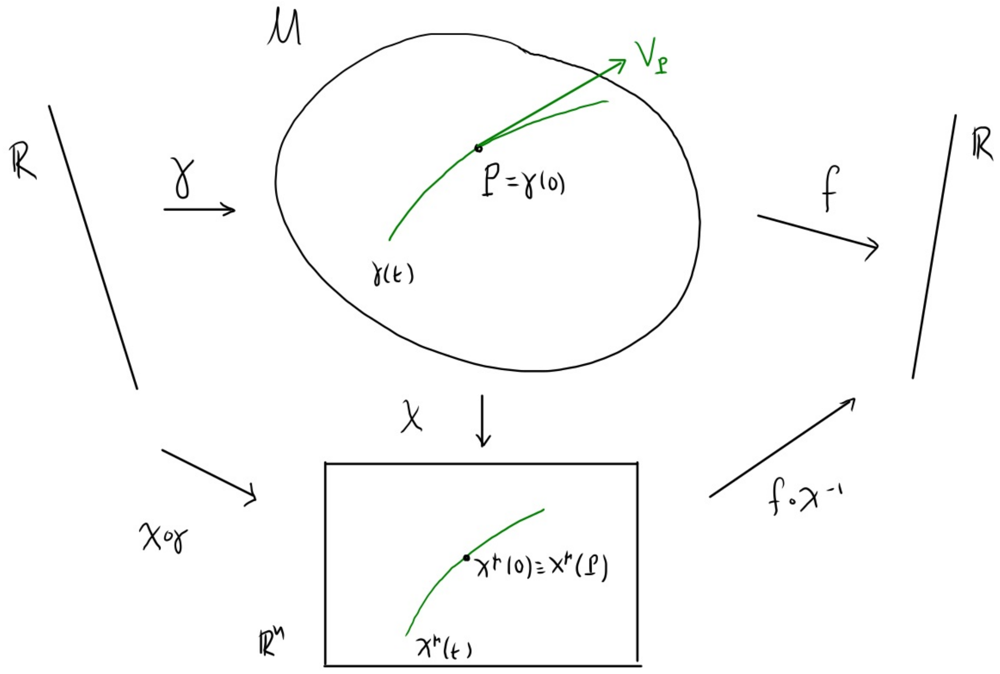
Summation over repeated indices

$$\sum_{\mu=0}^{n-1} \frac{\partial f}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

* Consider first $V_p(f)$:

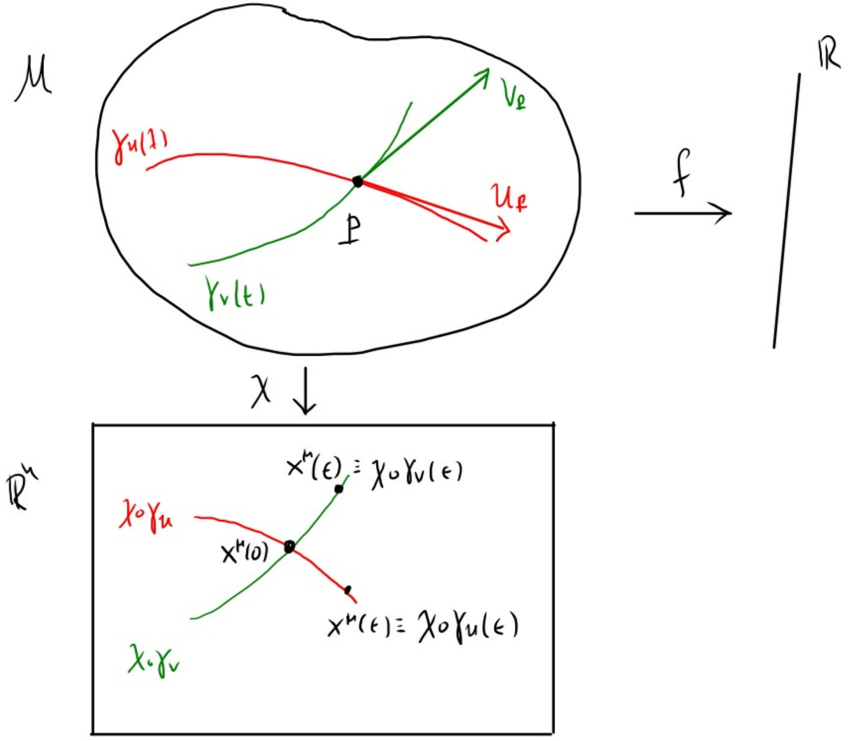
$$V_p(f) = \frac{\partial f(x^\nu)}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}$$

$$\equiv \frac{dx^\mu}{dt} \cdot \underbrace{\partial_\mu f}_{\text{Notation}}$$



$*$ Consider $\gamma_v(t)$ s.t. $\gamma_v(0)=p$

$\gamma_u(\lambda)$ s.t. $\gamma_u(0)=p$



and

$$V_p(f) = \frac{df}{dt}\Big|_p = \frac{dx^\mu}{dt}\,\frac{\partial f}{\partial x^\mu}\Big|_p$$

$$U_p(f) = \frac{df}{d\lambda}\Big|_p = \frac{dx^\mu}{d\lambda}\,\frac{\partial f}{\partial x^\mu}\Big|_p$$

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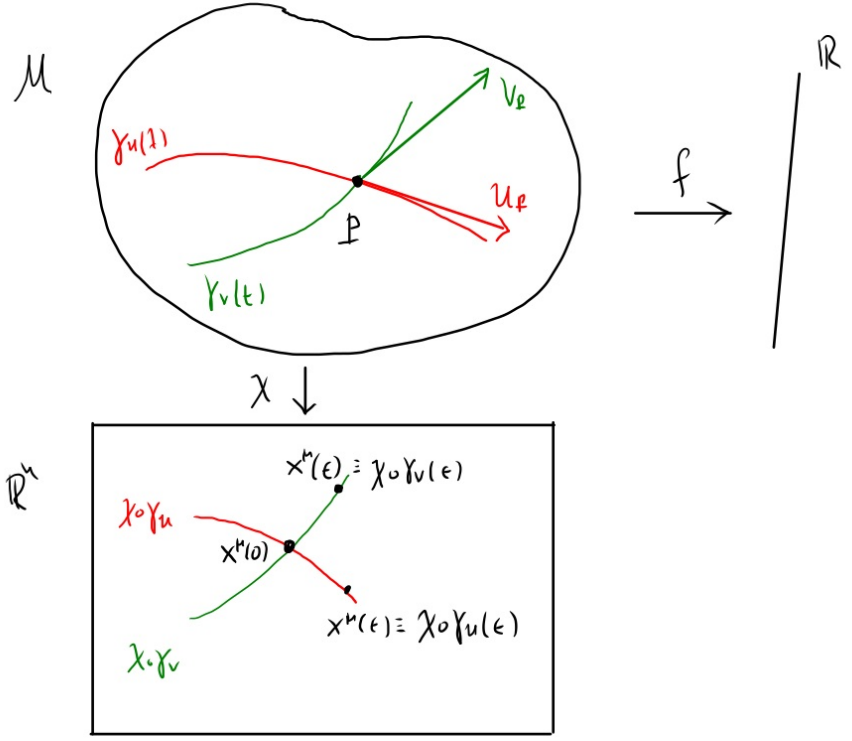

and

$$V_p(f) = \left.\frac{df}{dt}\right|_p = \frac{dx^\mu}{dt} \left.\frac{\partial f}{\partial x^\mu}\right|_p$$

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$*$ for a coordinate system with $\chi : p \mapsto x^\mu(p)$.

for a given μ $\quad p \mapsto x^\mu(p)$ is a real function on M

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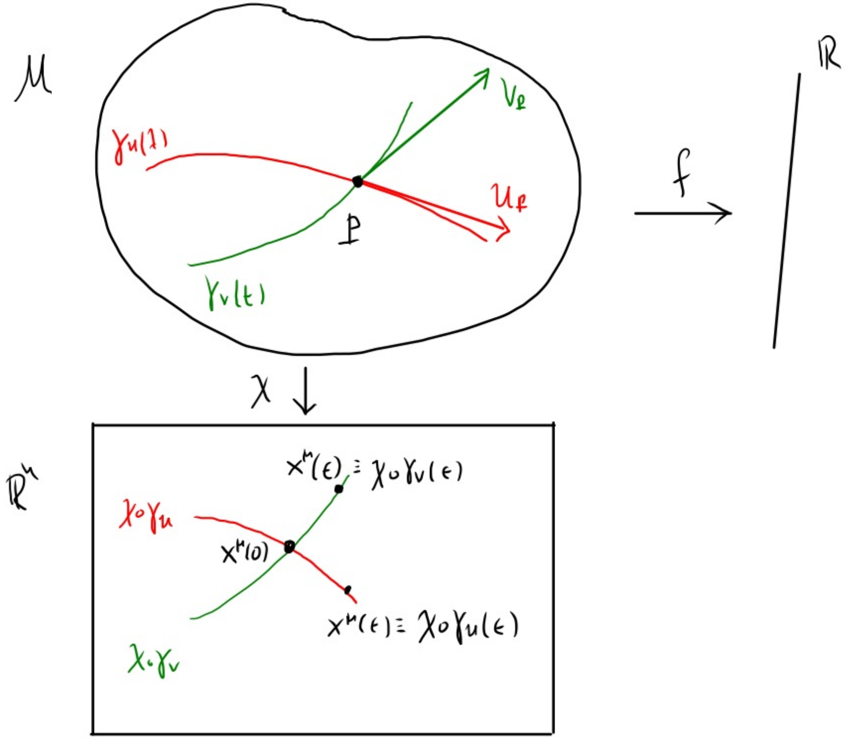

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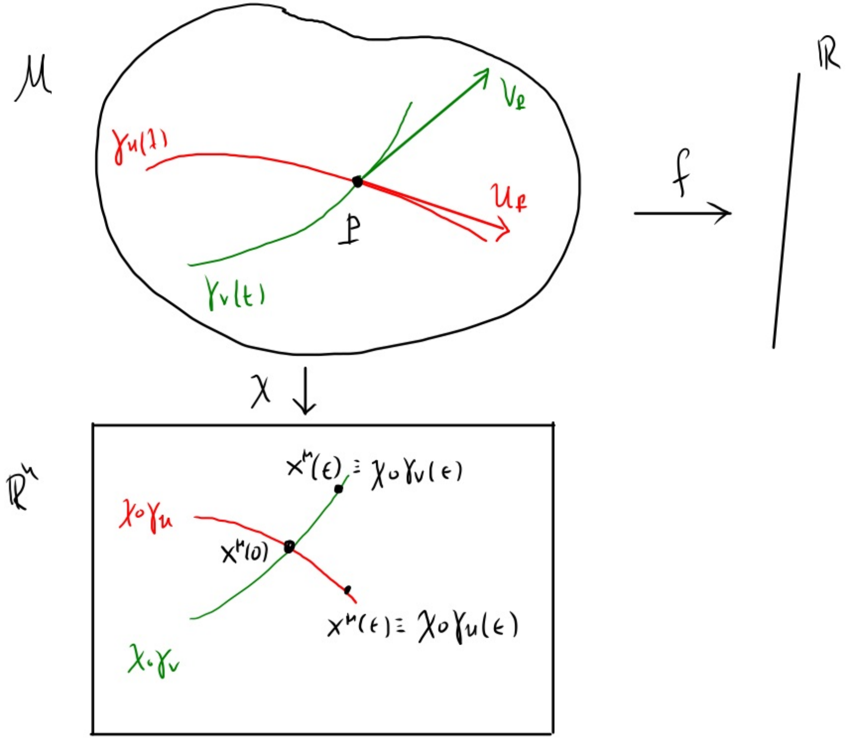
* for a coordinate system with $\chi : P \mapsto x^\mu(P)$.

for a given μ $P \mapsto x^\mu(P)$ is a real function on $\mathcal{U}$

such that: $$V_P(x^\mu) = \frac{dx^\nu}{dt} \cdot \frac{\partial x^\mu}{\partial x^\nu}\Big|_P = \frac{dx^\nu}{dt} \cdot \delta_\nu{}^\mu\Big|_P = \frac{dx^\mu}{dt}\Big|_0$$ (consistent with definition of V_P)

$*$ Consider $\gamma_v(t)$ s.t. $\gamma_v(0) = P$

$\gamma_u(\lambda)$ s.t. $\gamma_u(0) = P$



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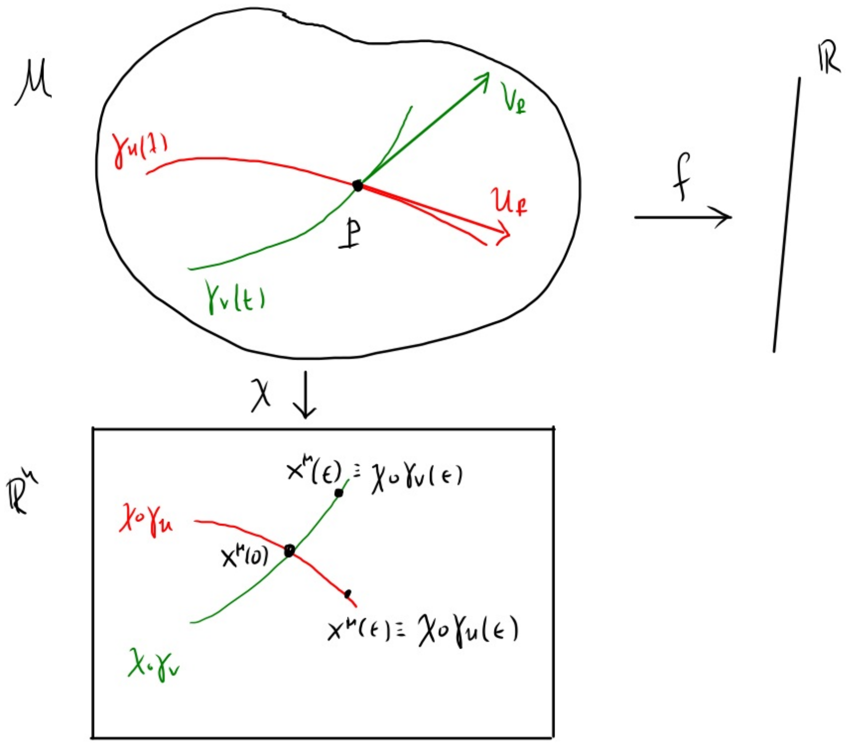

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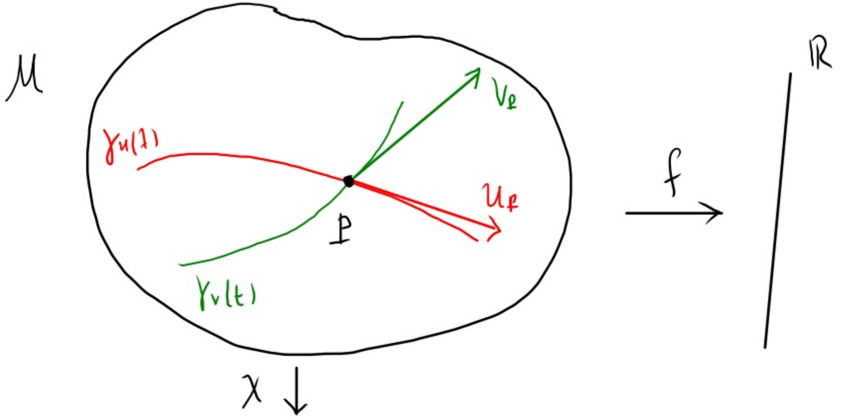

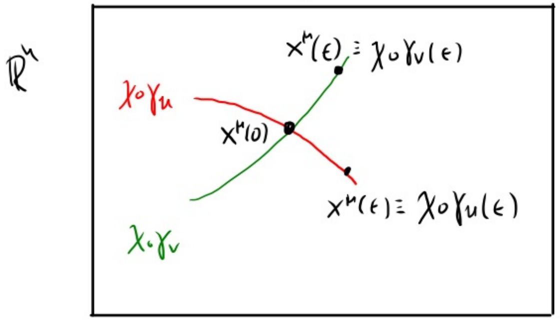

and

different points

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{dt}\right|_0 + \mathcal{O}_v(\epsilon^2)$$

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \mathcal{O}_u(\epsilon^2)$$

---

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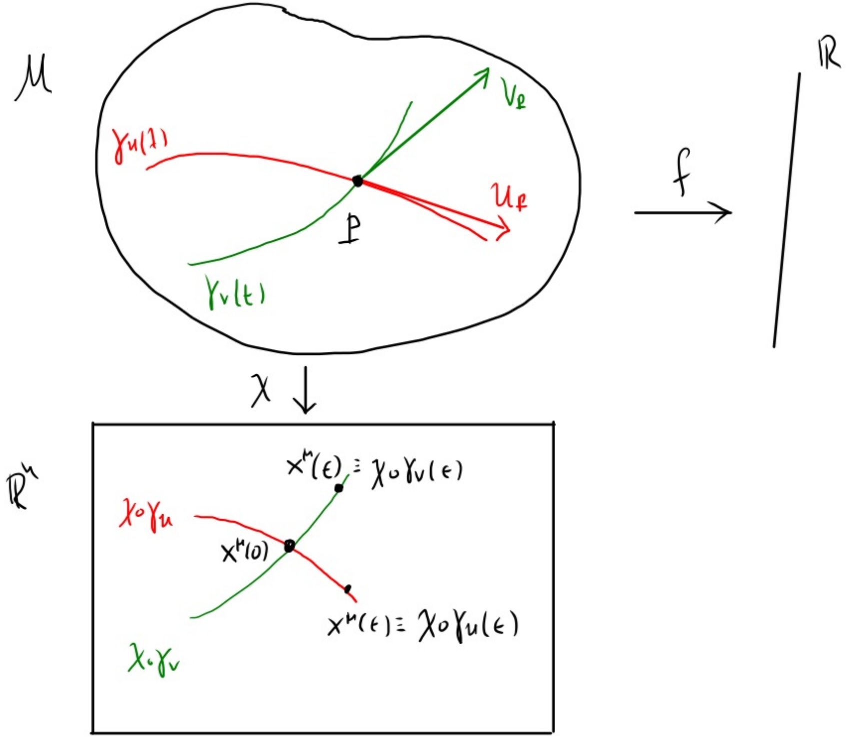


and

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{dt}\right|_0 + \Theta_v(\epsilon^2)$$

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left.\frac{dx^\mu}{d\lambda}\right|_0 + \Theta_u(\epsilon^2)$$

define <u>any</u> curve $\gamma_w(\epsilon)$, so that $x^\mu(\epsilon) \equiv \chi \circ \gamma_w(\epsilon)$ is given by:

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon \left[\alpha \left.\frac{dx^\mu}{dt}\right|_0 + \beta \left.\frac{dx^\mu}{d\lambda}\right|_0 \right] + \Theta_w(\epsilon^2)$$

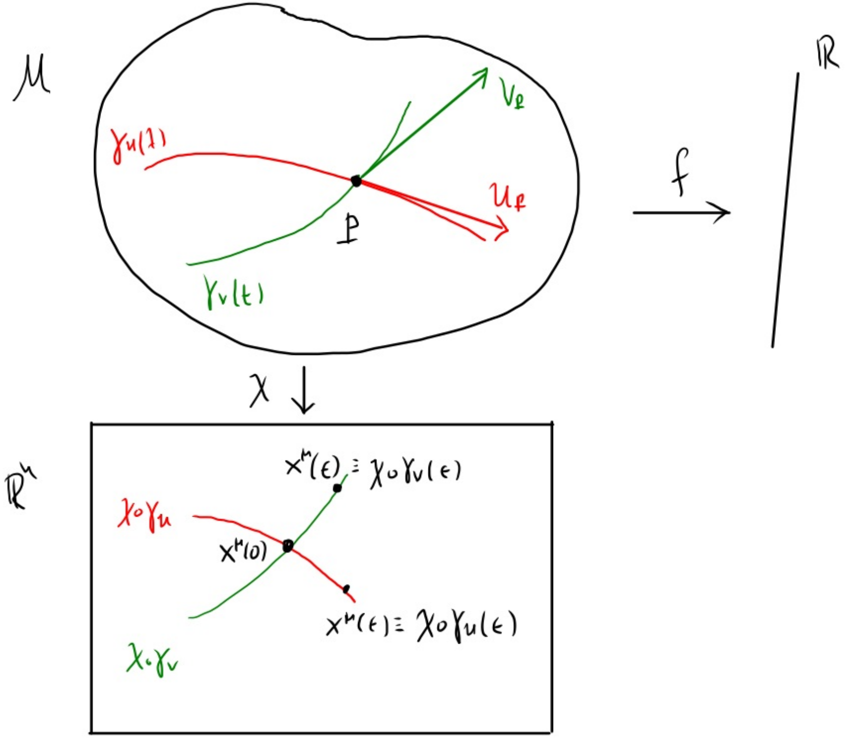
numbers

↳ any ϵ^2-infinitesimal you like!

$$\left| \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \Theta_w(\epsilon^2) \right| < \infty$$

* For $\gamma_w(\epsilon)$, the tangent vector is:

$$W_p(f) = \left.\frac{df}{d\epsilon}\right|_0$$



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numbers

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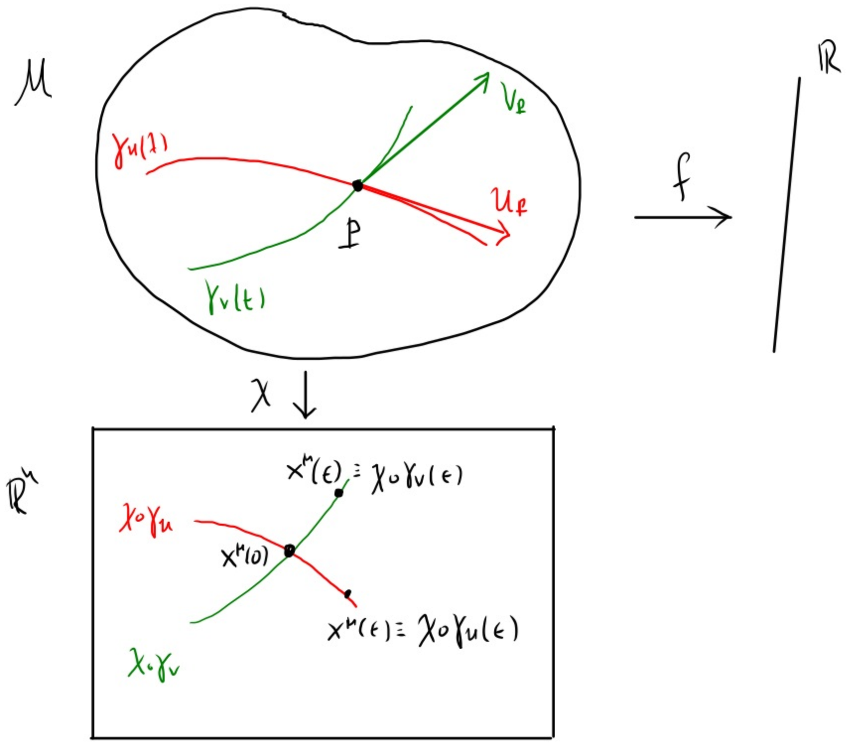
$$\left|\lim_{\epsilon \to 0} \frac{1}{\epsilon^2}\Theta_w(\epsilon^2)\right| < \infty$$

$\ast$ For $\gamma_w(\epsilon)$, the tangent vector is:

$$W_p(f) = \frac{df}{d\epsilon}\Big|_0 = \frac{d}{d\epsilon} f \circ \chi^{-1} \circ \chi \circ \gamma_w(\epsilon)\Big|_0$$



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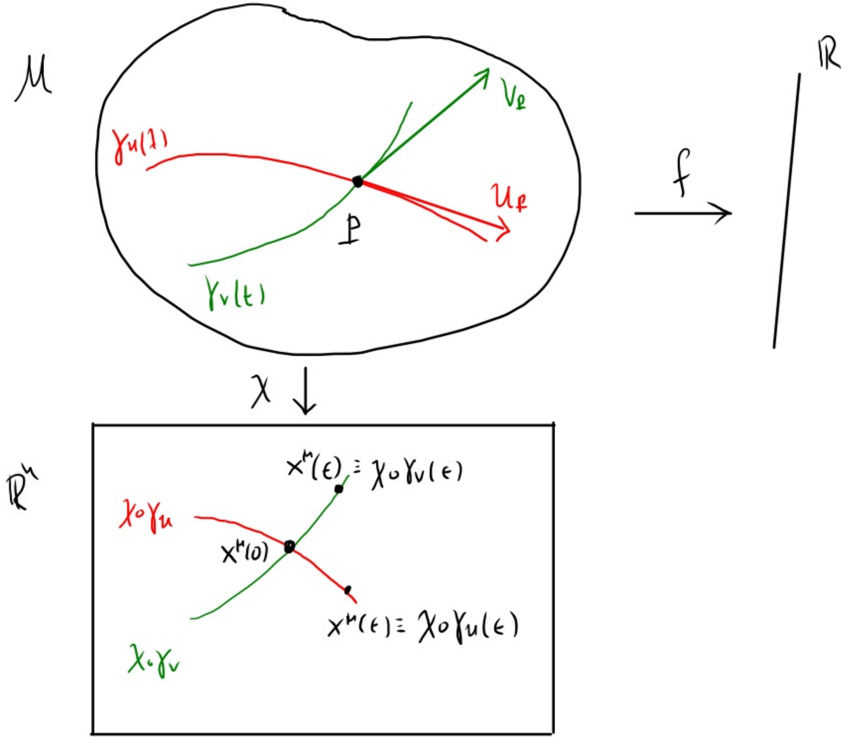

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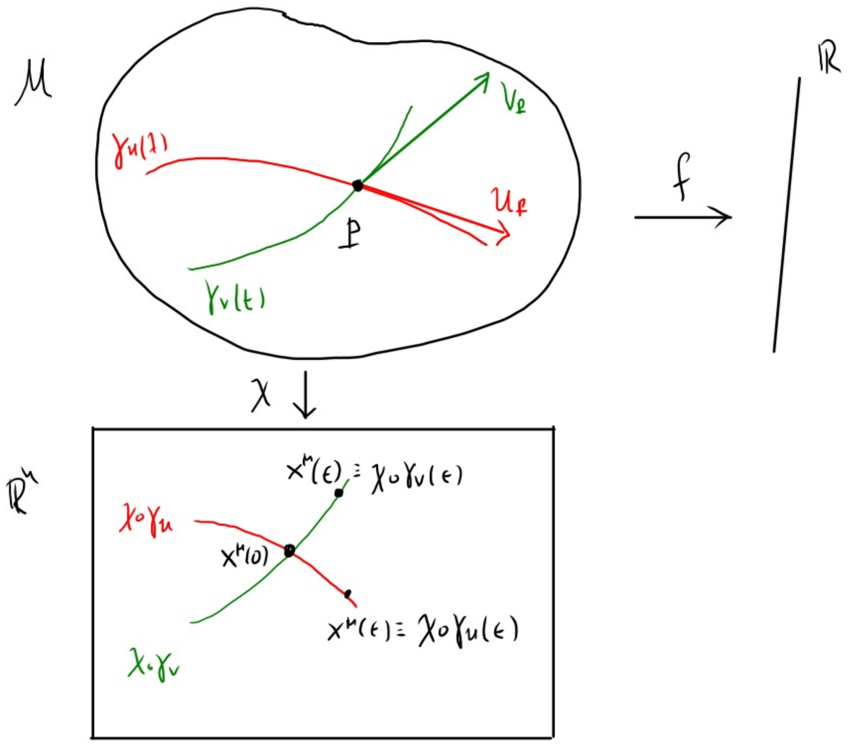

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define any curve $\gamma_w(\epsilon)$, so that $x^\mu(\epsilon) \equiv \chi\circ\gamma_w(\epsilon)$ is given by:

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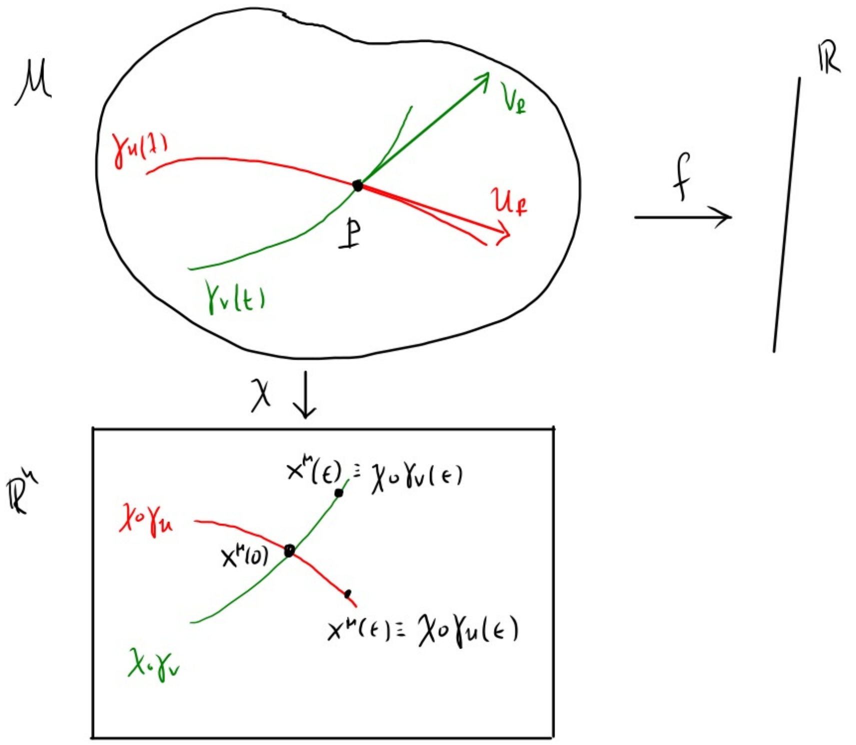

$$W_p(f) = \frac{df}{d\epsilon}\Big|_0 = \frac{d}{d\epsilon} f\circ\chi^{-1}\circ\chi\circ\gamma_w(\epsilon)\Big|_0$$

$$= \frac{d}{d\epsilon} f(x^\nu(\epsilon))\Big|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu}\,\frac{dx^\mu(\epsilon)}{d\epsilon}\Big|_0$$

---

define <u>any</u> curve $\gamma_w(\epsilon)$, so that $x^\mu(\epsilon) \equiv \chi\circ\gamma_w(\epsilon)$ is given by:

$$x^\mu(\epsilon) = x^\mu(0) + \epsilon\left[\alpha\,\frac{dx^\mu}{dt}\Big|_0 + \beta\,\frac{dx^\mu}{d\lambda}\Big|_0\right] + O_w(\epsilon^2)$$

numbers

↳ any ϵ^2-infinitesimal you like!

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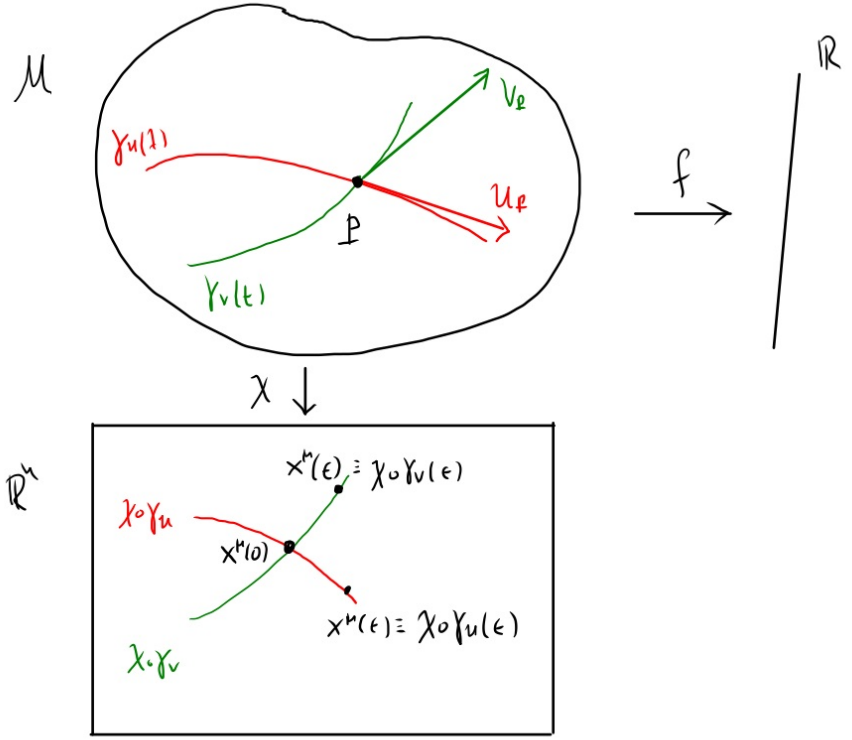
$$= \frac{d}{d\epsilon} f\left(x^\nu(\epsilon)\right)\Big|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu} \frac{dx^\mu(\epsilon)}{d\epsilon}\Big|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu}\left(\alpha \frac{dx^\mu}{dt}\Big|_0 + \beta \frac{dx^\mu}{d\lambda}\Big|_0 \right)$$



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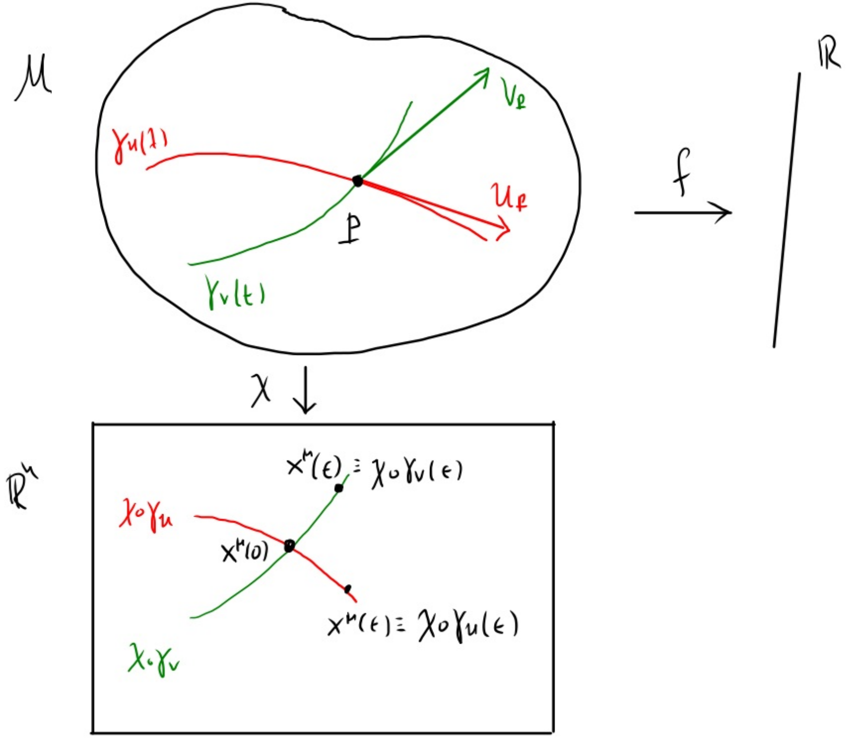


$$W_p(f) = \frac{df}{d\epsilon}\bigg|_0 = \frac{d}{d\epsilon} f \circ \chi^{-1} \circ \chi \circ \gamma_W(\epsilon)\bigg|_0$$

$$= \frac{d}{d\epsilon} f\left(x^\nu(\epsilon)\right)\bigg|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu} \frac{d x^\mu(\epsilon)}{d\epsilon}\bigg|_0$$

$$= \frac{\partial f(x^\nu)}{\partial x^\mu}\left(\alpha \frac{dx^\mu}{dt}\bigg|_0 + \beta \frac{dx^\mu}{d\lambda}\bigg|_0\right)$$

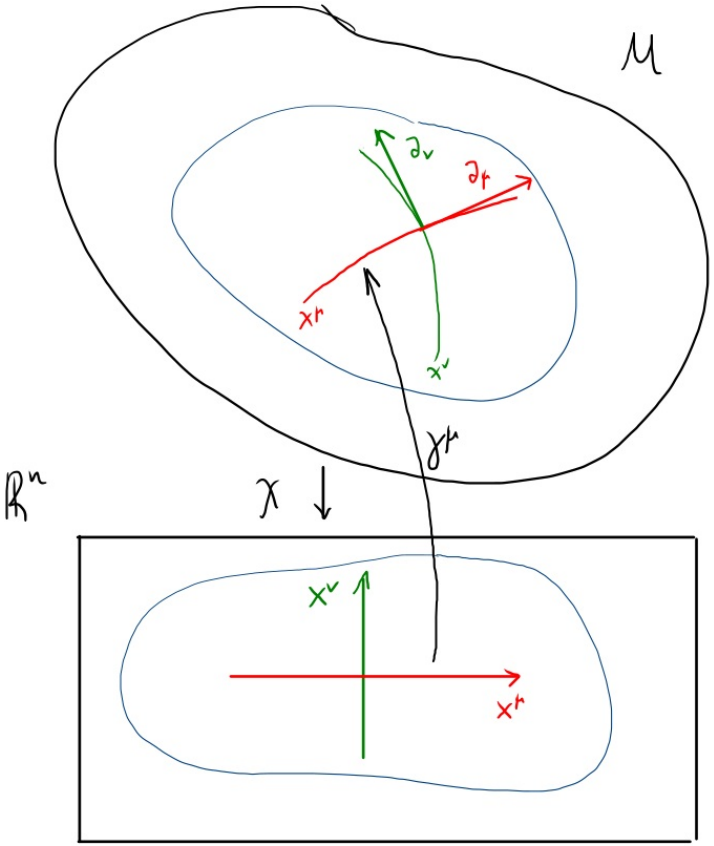
$$= \alpha V_p(f) + \beta U_p(f) \qquad \Rightarrow \qquad W_p = \alpha V_p + \beta U_p$$

* Coordinate basis:

Consider the curve:

$$\gamma^{\mu} : \mathbb{R} \longrightarrow \mathcal{M}$$
$$x^{\mu} \longmapsto \gamma^{\mu}(x^{\mu})$$

the parameter of the curve!



$$\gamma^{\mu}(x^{\mu}) = \chi^{-1}(c_0, c_1, \dots, c_{\mu-1}, x^{\mu}, c_{\mu+1}, \dots, c_{n-1})$$

$c_0, \dots, c_{n-1}$: constants

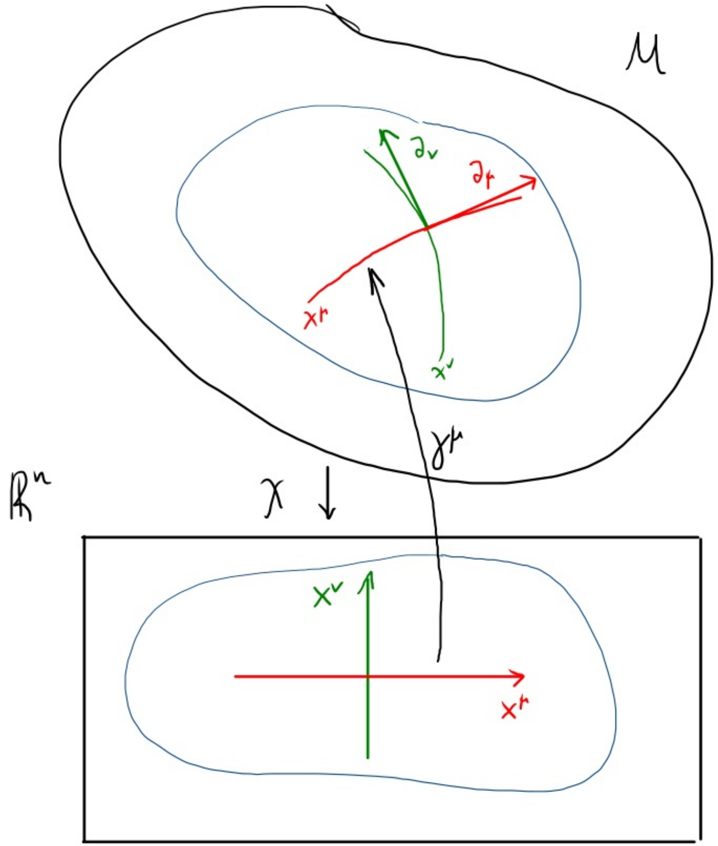
* Coordinate basis:

Consider the curve:

$$\gamma^\mu : \mathbb{R} \longrightarrow \mathcal{M}$$
$$x^\mu \longmapsto \gamma^\mu(x^\mu)$$

the parameter of the curve!

- In $\mathbb{R}^n$ we move parallel to the x^μ-axis (all other x^ν are held fixed)



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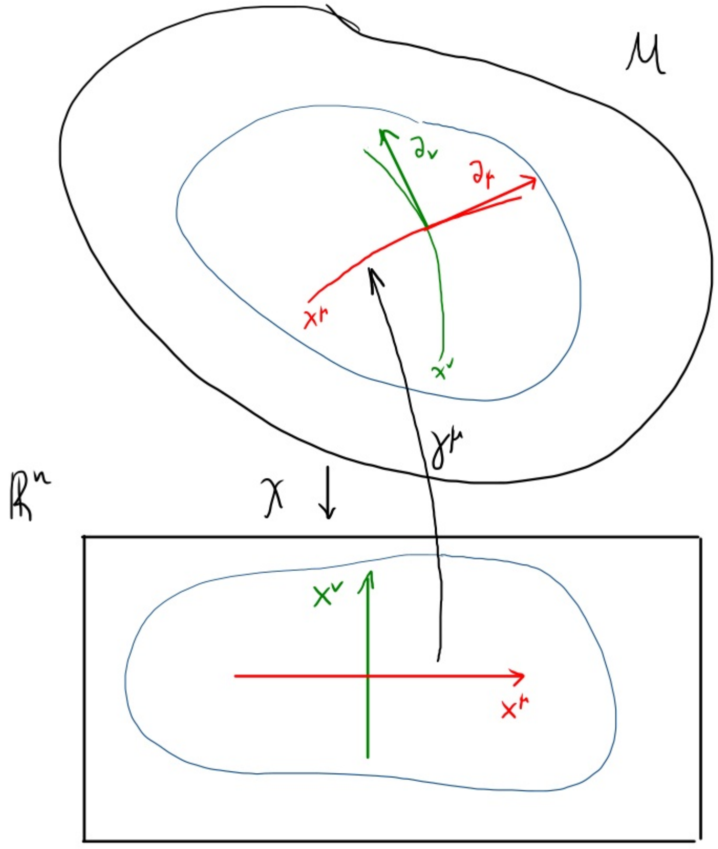


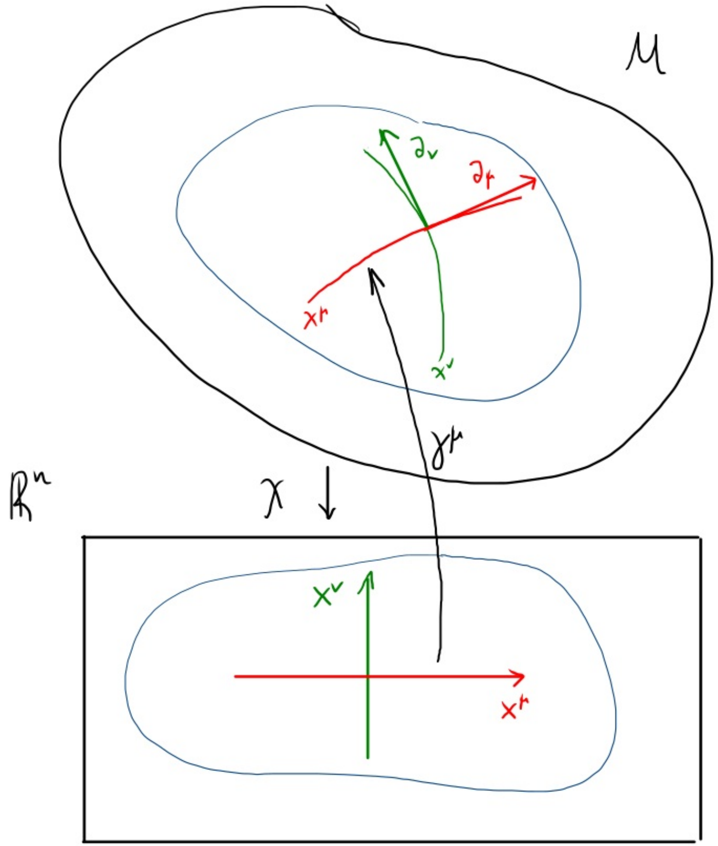
- In $\mathbb{R}^n$ we move parallel to the x^μ-axis (all other x^ν are held fixed)

- The tangent vector of γ^μ at P is written as ∂_μ

* Coordinate basis:

Then:

$$\frac{df}{dx^\mu}\Big|_p \equiv \frac{d}{dx^\mu} f \circ \gamma^\mu(x^\mu)\Big|_p$$



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* Coordinate basis:

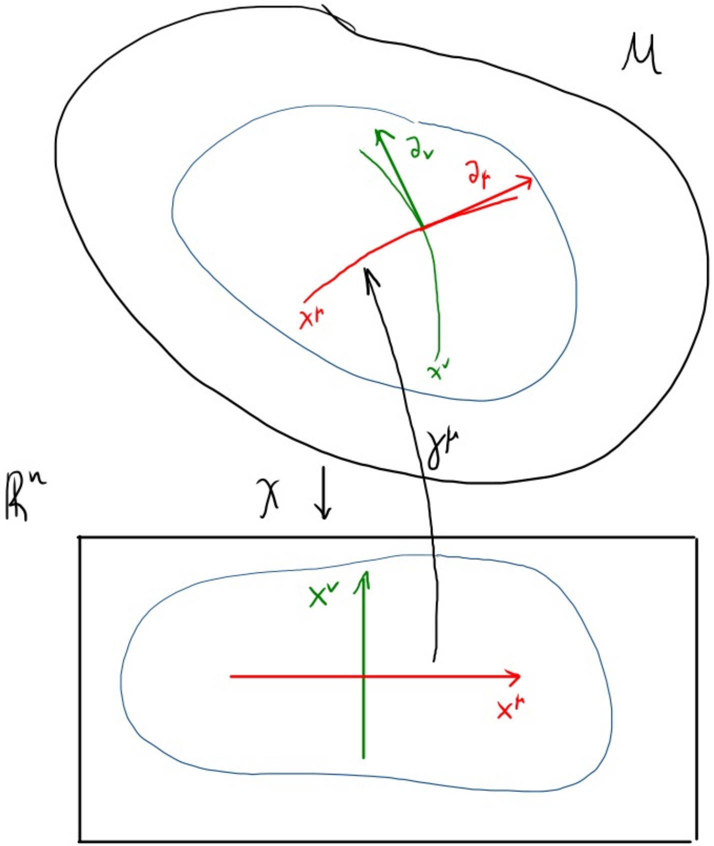
Then:

$$\left.\frac{df}{dx^\mu}\right|_p \equiv \left.\frac{d}{dx^\mu} f \circ \gamma^\mu(x^\mu)\right|_p$$

directional derivative along γ^μ

$$= \left.\frac{\partial}{\partial x^\mu} f \circ x^{-1}(x^\nu)\right|_p$$

definition of partial derivative: we vary x^μ, hold x^ν fixed



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Then:

$$\frac{df}{dx^\mu}\Big|_p \equiv \frac{d}{dx^\mu} f\circ\gamma^\mu(x^\mu)\Big|_p$$

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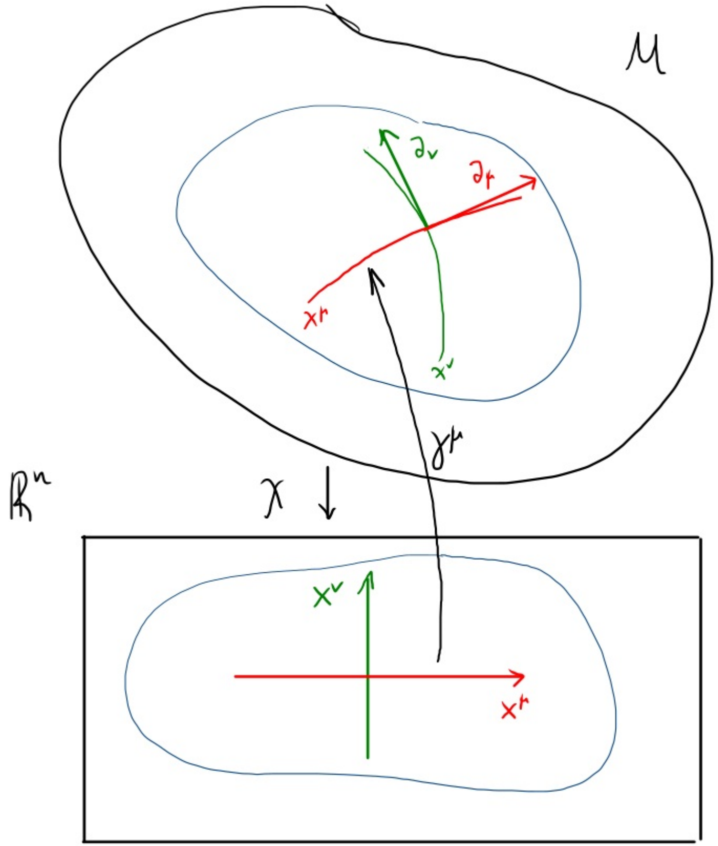


We define:

$$\partial_\mu\big|_p = \frac{d}{dx^\mu}\Big|_p \qquad \text{s.t.} \qquad \partial_\mu f\big|_p = \frac{\partial f\circ x^{-1}}{\partial x^\mu}\Big|_p$$

* Coordinate basis:

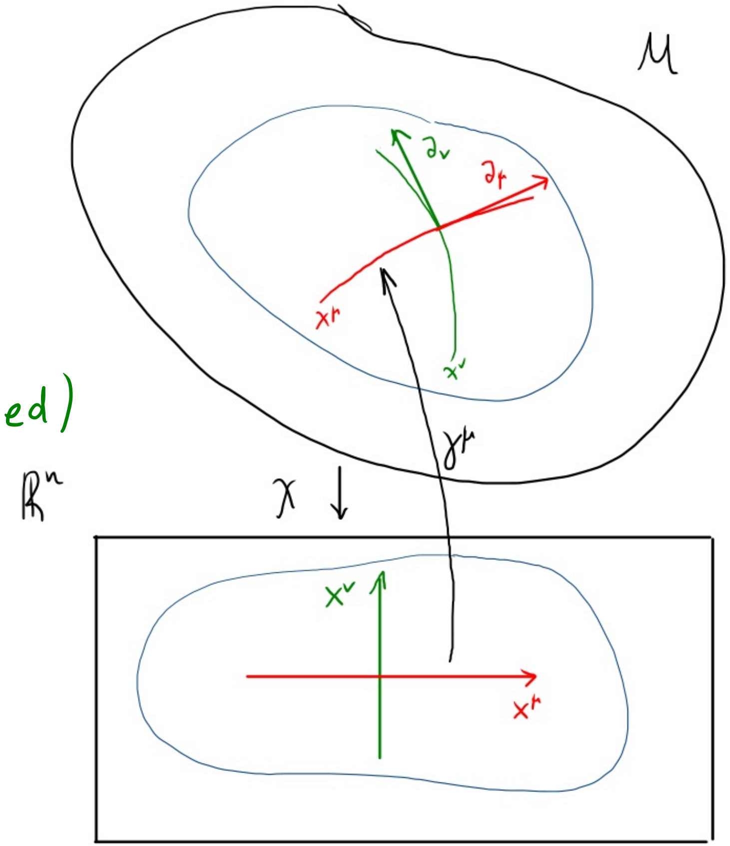
We have shown that: $\forall f$

$$V_p(f) = \frac{dx^\mu}{dt}\Big|_0 \frac{\partial f}{\partial x^\mu}\Big|_p$$

(summation over all μ implied)

or

$$V_p(f) = \frac{dx^\mu}{dt} \partial_\mu f \Big|_p$$



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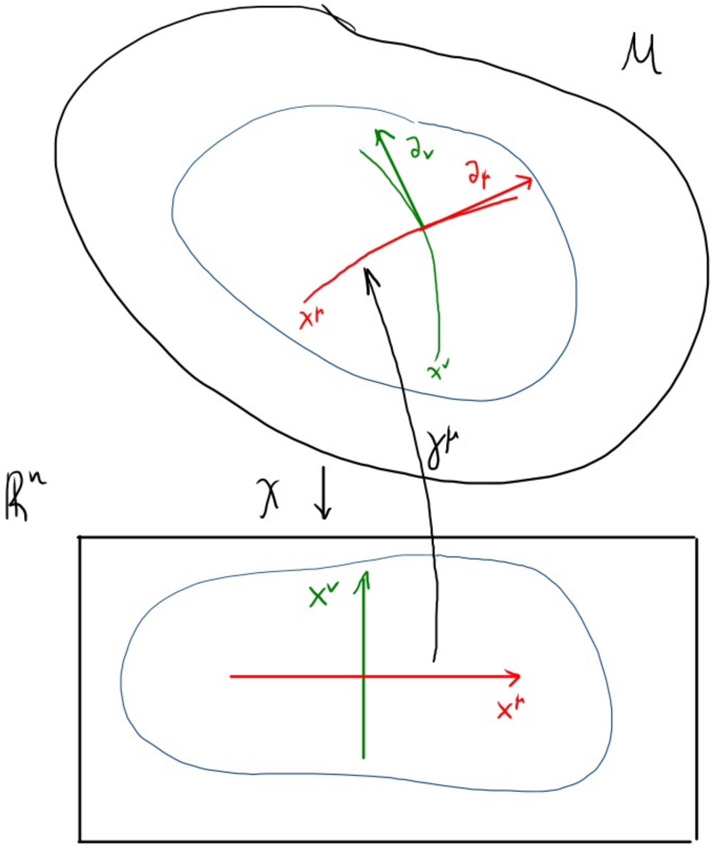
or

$$V_p(f) = \frac{dx^\mu}{dt} \partial_\mu f \Big|_p$$

$$\Rightarrow \quad V_p = \frac{dx^\mu}{dt} \partial_\mu \Big|_p$$



* Coordinate basis:

We have shown that: $\forall f$

$$V_p(f) = \left.\frac{dx^\mu}{dt}\right|_0 \left.\frac{\partial f}{\partial x^\mu}\right|_p$$

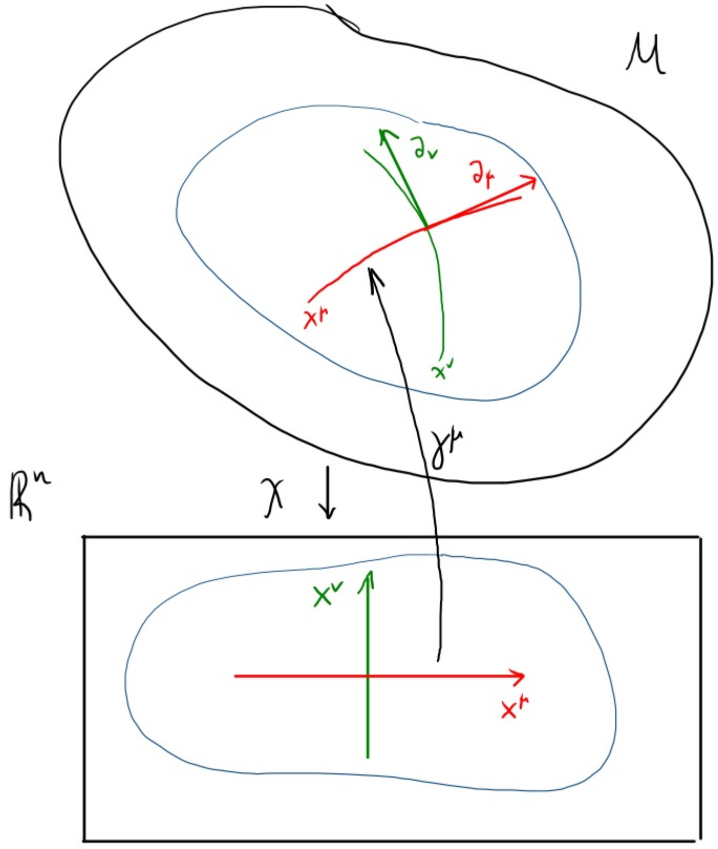
or

$$V_p(f) = \frac{dx^\mu}{dt}\, \partial_\mu f \Big|_p$$

$$\Rightarrow \quad \underbrace{V_p}_{\text{an operator acting on any } f} = \frac{dx^\mu}{dt}\, \partial_\mu \Big|_p$$

coordinate vectors for all μ

a linear combination of all ∂_μ

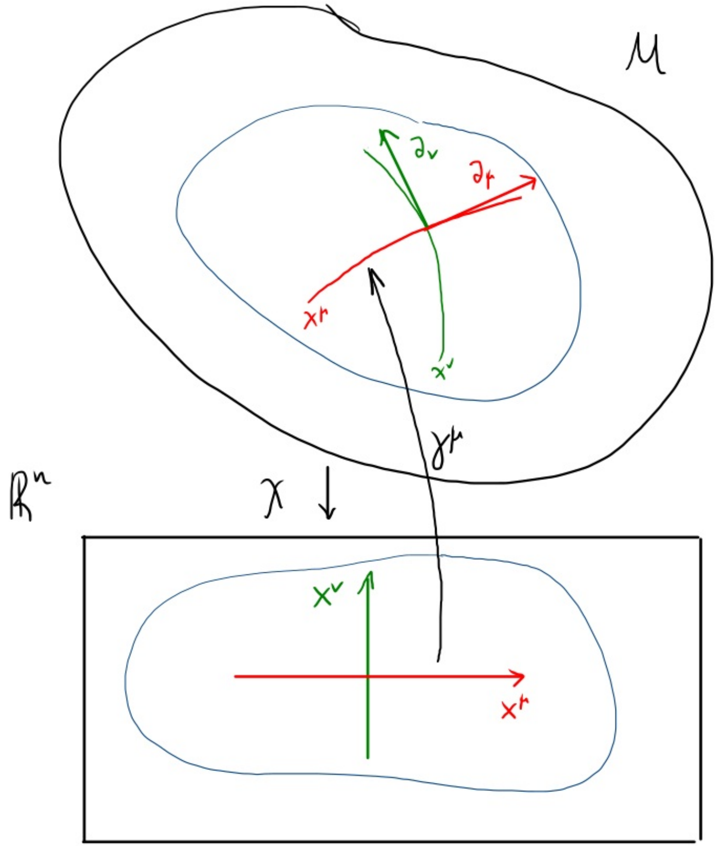


* Coordinate basis:

$$V = \underbrace{\frac{dx^\mu}{dt}}_{\text{components of } V \text{ in the } \{\partial_\mu\}\text{-basis}} \partial_\mu = V^\mu \partial_\mu$$

true for any f

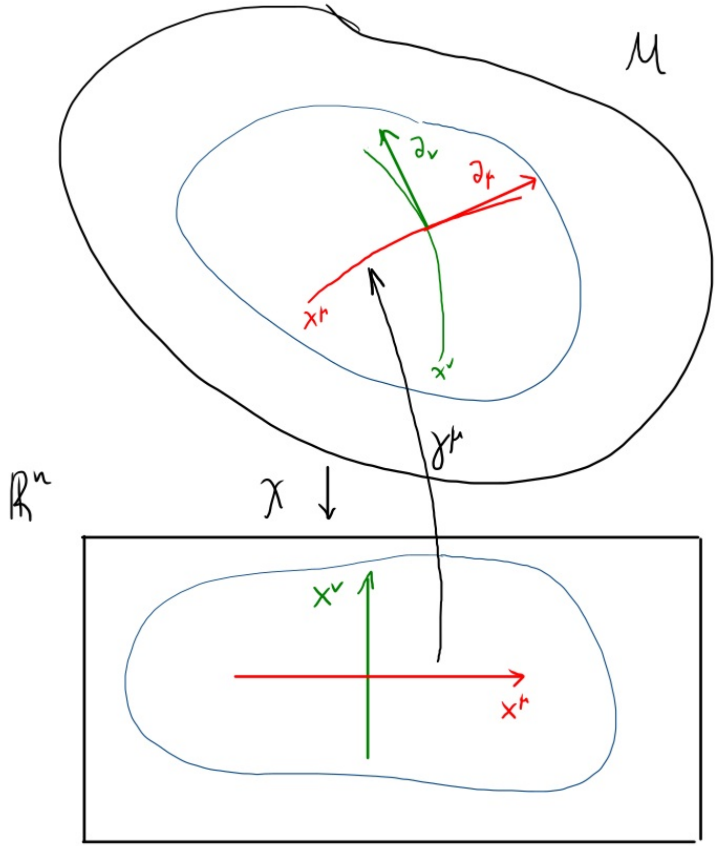
→ remove f from notation



* Coordinate basis:

$$V = \frac{dx^\mu}{dt}\partial_\mu = V^\mu \partial_\mu$$

$V^\mu = \dfrac{dx^\mu}{dt}$ components of V in the $\{\partial_\mu\}$ basis



* $\{\partial_\mu\}$ is a basis $\Rightarrow$ T_pM is n-dimensional

* Coordinate basis:

$$V = \frac{dx^\mu}{dt}\partial_\mu = V^\mu \partial_\mu$$

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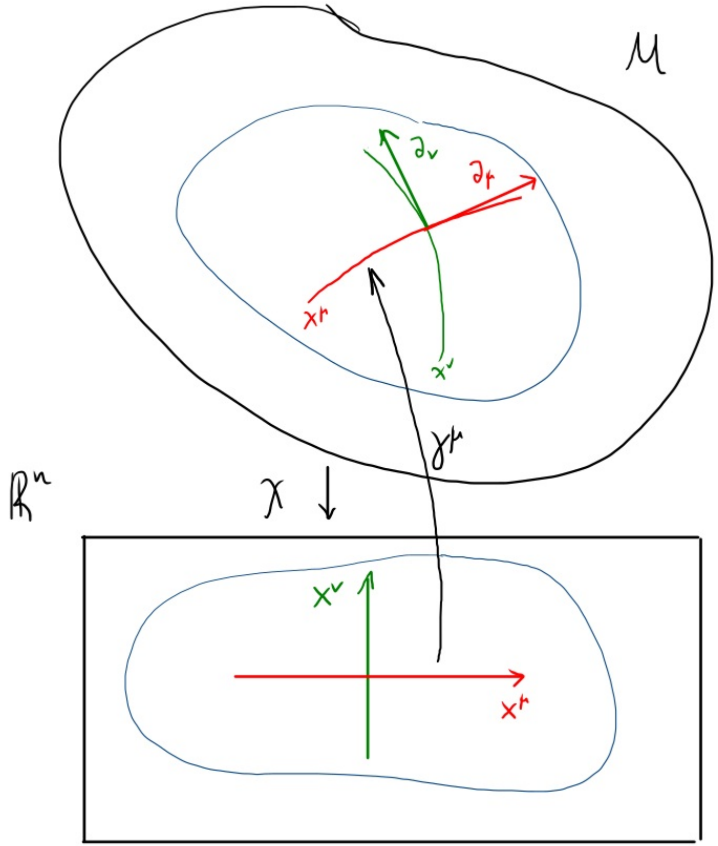


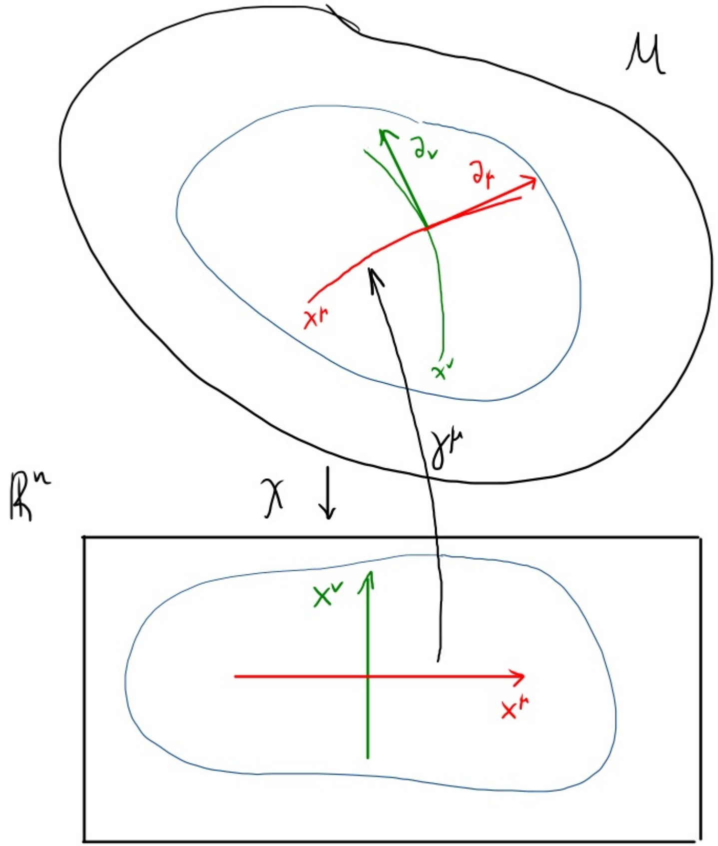
* $\{\partial_\mu\}$ is a basis $\Rightarrow$ T_pM is n-dimensional

* $\{\partial_\mu\}$ is derived from chosen coordinate system: a coordinate basis

* Coordinate basis:

$$V = \frac{dx^\mu}{dt} \partial_\mu = V^\mu \partial_\mu$$

$V^\mu = \frac{dx^\mu}{dt}$ components of V in the $\{\partial_\mu\}$ basis



* $\{\partial_\mu\}$ is a basis $\Rightarrow$ T_pM is n-dimensional

* $\{\partial_\mu\}$ is derived from chosen coordinate system: a coordinate basis

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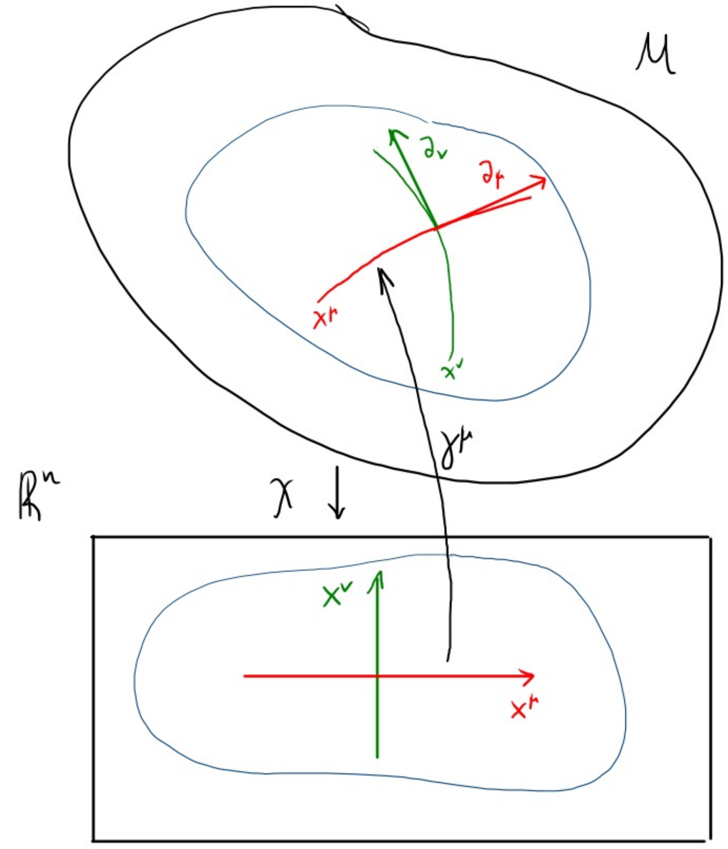

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* If there is a metric $\Rightarrow$ inner product in T_pM : $\{\partial_\mu\}$ may not be orthogonal and/or unit vectors

* Change a basis $\Rightarrow$ change of components (of the <u>same</u> vector)

- coordinate basis

Due to chain rule, for any f: $\partial_{\mu'} f(x^{\mu'}) = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial f(x^{\mu})}{\partial x^{\mu}}$

$$\Rightarrow \quad \partial_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \partial_{\mu}$$

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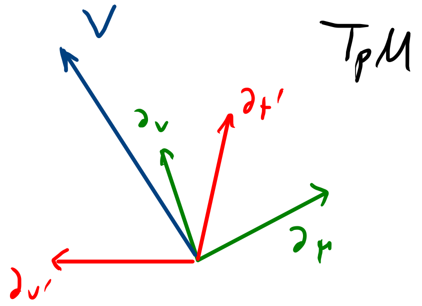
$$\left.\begin{array}{l} V = V^\mu \partial_\mu \\ V = V^{\mu'} \partial_{\mu'} \end{array}\right\} \begin{array}{l} \text{same vector} \\ \text{different basis} \end{array}$$



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Inverting the above system of linear equations, and using the fact that

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we obtain:

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu}$$

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- another basis: $\quad e_a = \Lambda_a{}^{\mu} \partial_{\mu}$

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- Example : Lorentz xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \quad \Rightarrow \quad \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu}$$

* Change a basis $\Rightarrow$ change of components (of the <u>same</u> vector)

- coordinate basis $V^{\mu'} = \dfrac{\partial x^{\mu'}}{\partial x^{\mu}} V^{\mu} \iff V^{\mu} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\mu'}$

- Example : Lorentz xfm

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^{\mu} \Rightarrow \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \Lambda^{\mu'}{}_{\mu}$$

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Vector fields

* Repeat the construction of vectors foreach $P \in M$

  Then, each point P has a $T_P M$ "hanging above" it

  ($\rightarrow$ a "fiber bundle")

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_ if $V(f)$ is smooth $\forall f$, then V is smooth

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- $V = V^\mu \partial_\mu$ with $V^\mu = V^\mu(P)$ a smooth fuction on the chart
  $V(x^\mu) = V^\mu$

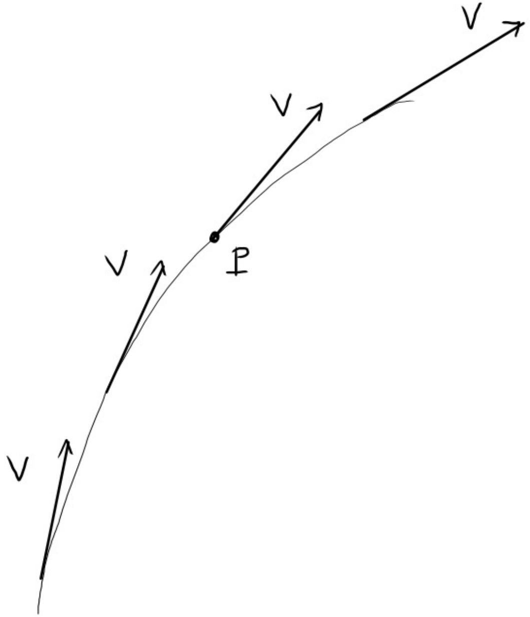
$\Rightarrow$ a smooth V has smooth components in a coord basis

Integral Curves

At each point P in a chart

$$\frac{dx^\mu}{dt} = V^\mu(x^\nu)$$

↳ value of component V^μ at point with coordinates $\{x^\nu\}$



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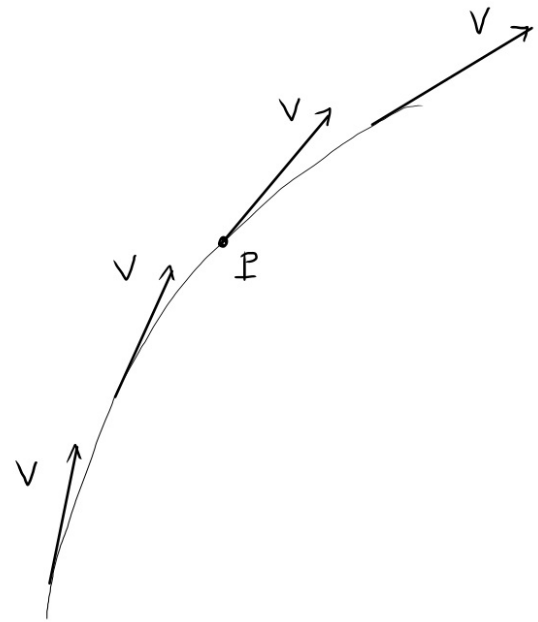

where t the parameter of a curve, for which V is tangent to it at all of its points

Integral Curves

At each point P in a chart

$$\frac{dx^\mu}{dt} = V^\mu(x^\nu) \qquad (1)$$

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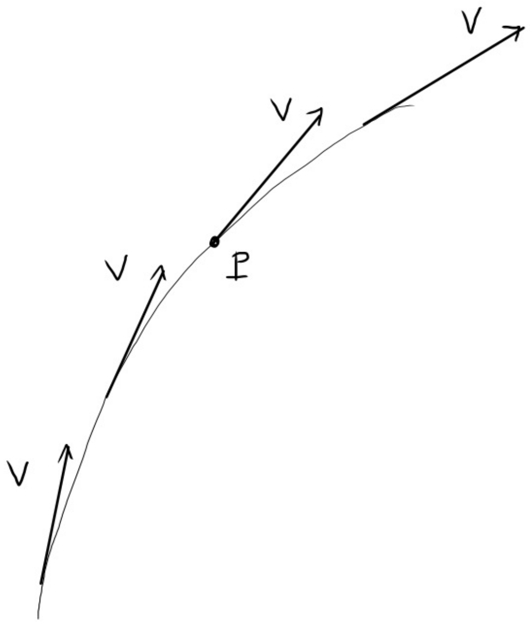

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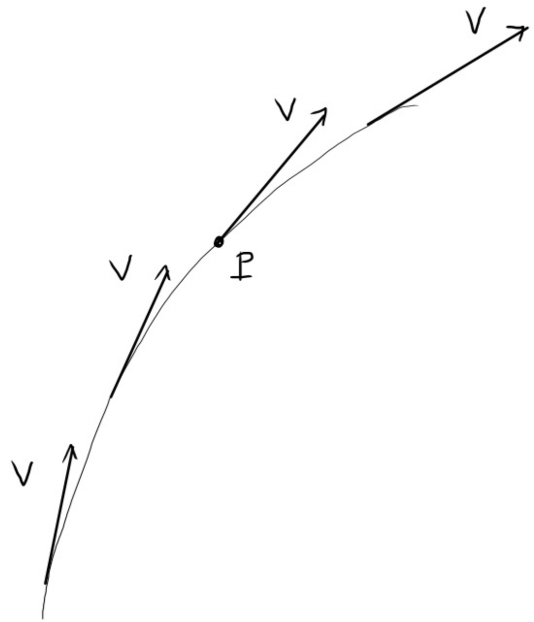

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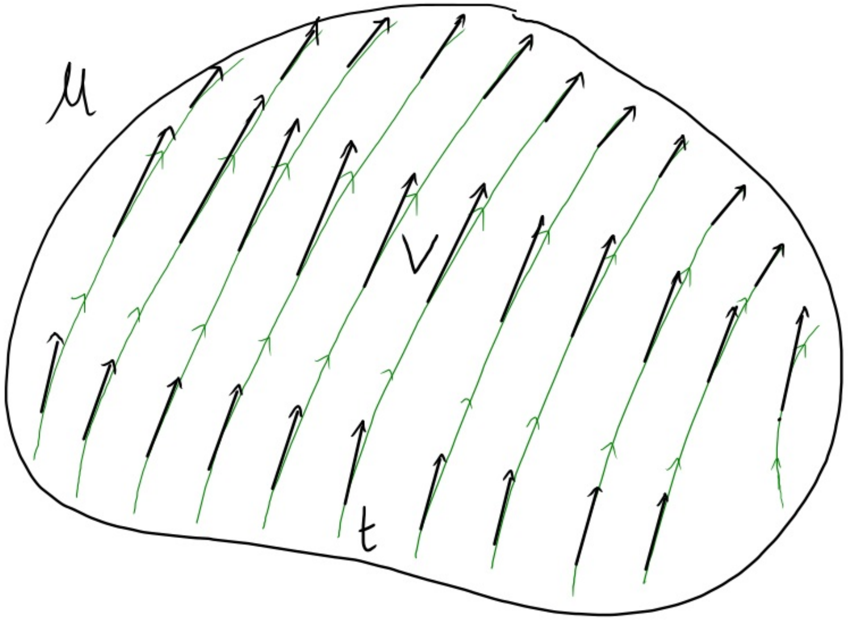
If $x^\mu(0)$ the coordinates of P, then

- (1) has a unique solution $x^\mu(t)$
- it is the <u>unique</u> integral curve of V going through P

* The integral curves of a nonvanishing vector field on $\mathcal{U} \subseteq \mathcal{M}$ "fill" $\mathcal{U}$



* The integral curves of a nonvanishing vector field on $U \subseteq M$ "fill" U

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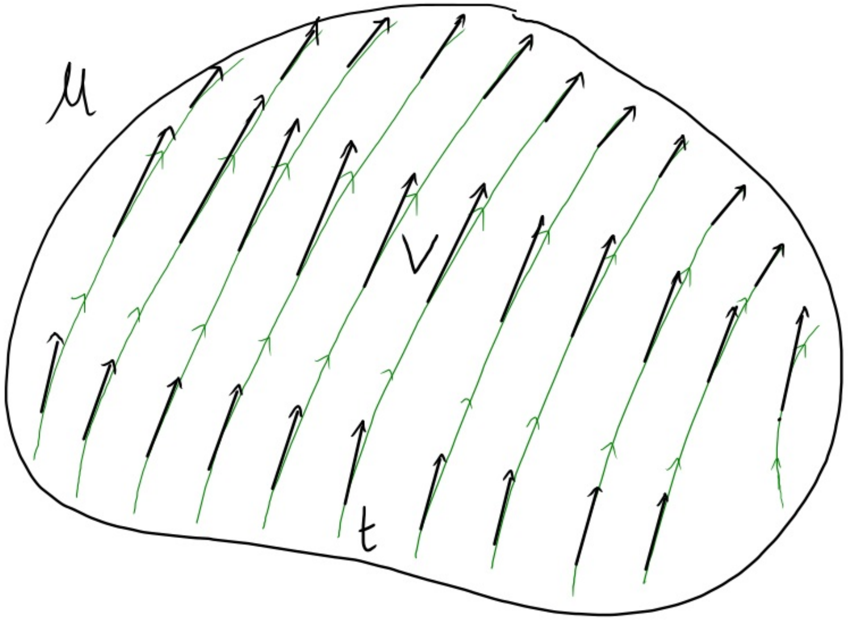

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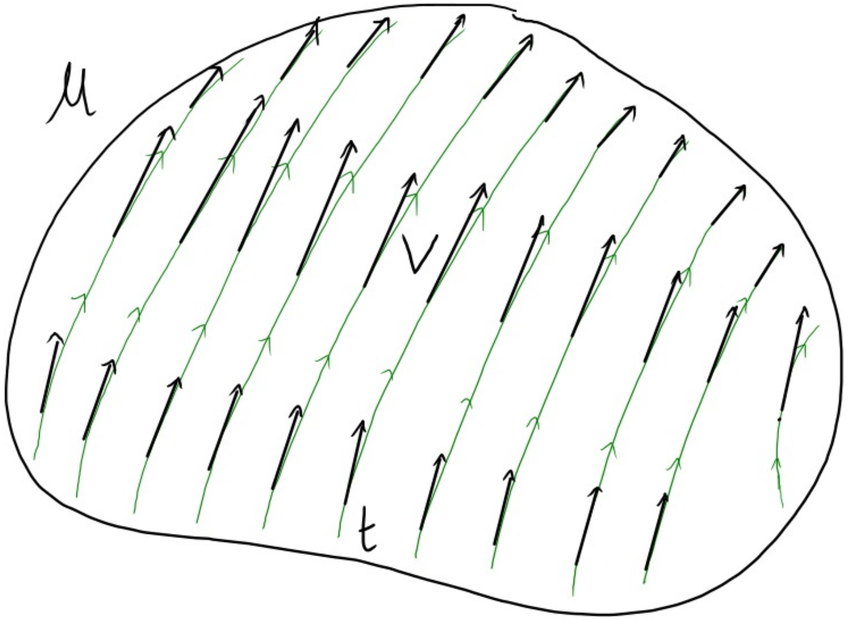

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- they never cross (one and only one)

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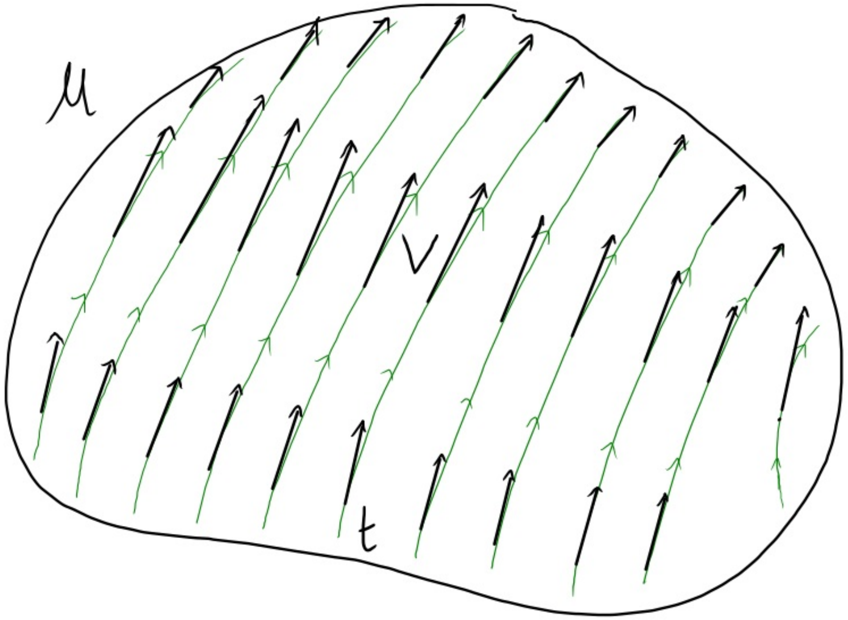

- they pass through each $P \in \mathcal{U}$
- they never cross (one and only one)

$\Rightarrow$ they form a "congruence"

Lie Bracket

If we have two vector fields V and W on M, then

$\forall f \in \mathcal{F}(M)$ $\quad V(f) = \frac{df}{dt}$, $\quad W(f) = \frac{df}{d\lambda}$ are smooth functions on M

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Careful, not an ordinary 2nd derivative:

- compute the directional derivative $\frac{df}{dt}$ along integral curves of V
- consider the result a function on $\mathcal{M}$
- compute the directional derivative $\frac{d}{d\lambda}\left(\frac{df}{dt}\right)$ of that function along integral curves of W

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Therefore $W(V(f)) = W\left(\frac{df}{dt}\right) = \frac{d}{d\lambda}\left(\frac{df}{dt}\right)$ a smooth function

- we write $V(W(f)) = VW(f)$

$\rightarrow$ VW is NOT a vector field (see that soon...)

* Define $[V, W] = VW - WV$

  - $[V, W](f) = VW(f) - WV(f)$

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Indeed, it is a derivation: $\forall \alpha, \beta \in \mathbb{R}$ $\forall f, g \in \mathcal{F}(M)$

(1) $[V,W](\alpha f + \beta g) = \alpha [V,W](f) + \beta [V,W](g)$

(2) $[V,W](f \cdot g) = [V,W](f) \cdot g + f \cdot [V,W](g)$

Proof of (2): $[V,W](fg) = [V,W](f)\cdot g + f\cdot [V,W](g)$

$$VW(fg) = V(W(fg))$$

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Proof of (2) : $[V,W](fg) = [V,W](f)\cdot g + f\cdot[V,W](g)$

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$$= V(W(f)) \cdot g + W(f) \cdot V(g) + V(f)\,W(g) + f\,V(W(g)) \qquad \text{V a derivation}$$

Proof of (2) : $[V,W](fg) = [V,W](f)\cdot g + f\cdot [V,W](g)$

$$
\begin{aligned}
VW(fg) &= V(W(fg)) \\
&= V(W(f)g + fW(g)) && \text{W a derivation} \\
&= V(W(f)\cdot g) + V(fW(g)) && \text{V is linear on fs} \\
&= \underbrace{V(W(f))\cdot g + W(f)\cdot V(g) + V(f)W(g) + fV(W(g))}_{\text{VW not a derivation!}} && \text{V a derivation}
\end{aligned}
$$

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$$WV(fg) = W(V(fg)) = W(V(f)g + fV(g)) = W(V(f)g) + W(fV(g))$$

$$= W(V(f))g + V(f)W(g) + W(f)V(g) + fW(V(g))$$

Proof of (2) : $[V,W](fg) = [V,W](f)\cdot g + f\cdot [V,W](g)$ (3)

$$VW(fg) = V(W(fg))$$
$$= V(W(f)g + fW(g)) \qquad W \text{ a derivation}$$
$$= V(W(f)\cdot g) + V(fW(g)) \qquad V \text{ is linear on } fs$$
$$(1) \quad = \underline{V(W(f))\cdot g} + \cancel{W(f)\cdot V(g)} + \cancel{V(f)W(g)} + \underline{f\,V(W(g))} \qquad V \text{ a derivation}$$

$$WV(fg) = W(V(fg)) = W(V(f)g + fV(g)) = W(V(f)g) + W(fV(g))$$
$$(2) \quad = \underline{W(V(f))g} + \cancel{V(f)W(g)} + \cancel{W(f)V(g)} + \underline{fW(V(g))}$$

$$(1)-(2) \Rightarrow \underbrace{(VW-WV)}_{[V,W]}(fg) = \underbrace{(VW-WV)}_{[V,W]}(f)\cdot g + f\underbrace{(VW-WV)}_{[V,W]}(g) \iff (3)$$

* In a coordinate basis

$$[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

(not true for noncoordinate bases)

* In a coordinate basis

$$[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Easy to remember:

$$V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

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$$[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Indeed, since $V(x^\mu) = V^\mu$ and $W(x^\mu) = W^\mu$

$V(f) = V^\mu \partial_\mu f$ $\quad$ $W(f) = W^\mu \partial_\mu f$ for every f :

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$$\Rightarrow [V,W]^\mu = [V,W](x^\mu) = VW(x^\mu) - WV(x^\mu) = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$$

Lie Derivative

(the practical way...)

The Lie bracket defines a derivative:

$$\mathcal{L}_V W = [V, W]$$

derivative of W w.r.t. V

Lie Derivative

The Lie bracket defines a derivative:

$$\mathcal{L}_V W = [V, W] \qquad \text{and} \qquad \mathcal{L}_V f = V(f)$$

a vector field

a function

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Indeed, for $\alpha, \beta \in \mathbb{R}$

- $\mathcal{L}_V(\alpha W + \beta U) = \alpha \mathcal{L}_V W + \beta \mathcal{L}_V U$

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- $\mathcal{L}_V (\alpha W + \beta U) = \alpha \mathcal{L}_V W + \beta \mathcal{L}_V U$
- $\mathcal{L}_V (f W) = (\mathcal{L}_V f) W + f \mathcal{L}_V W$

Indeed:

$$\mathcal{L}_V(\alpha W + \beta U) = [V, \alpha W + \beta U] = \alpha [V, W] + \beta [V, U] = \alpha \mathcal{L}_V W + \beta \mathcal{L}_V U$$

Indeed:

$$\mathcal{L}_V(fW) = [V, fW]$$

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$$\begin{aligned}[V, fW](g) &= V(fW(g)) - fW(V(g)) \\ &= \left[V(f)W(g) + fV(W(g))\right] - fW(V(g))\end{aligned}$$

Indeed:

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&= V(f)W(g) + f[VW(g) - WV(g)] \\
&= (V(f)W + f[V,W])(g) \\
&= (\mathcal{L}_V f \cdot W + f\mathcal{L}_V W)(g)
\end{aligned}$$

* The components $(\mathcal{L}_V W)^\mu = [V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$ transform as for any vector field:

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$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \quad , \quad W^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} W^\mu , \qquad \partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

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$$\partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}} = \underbrace{\frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}}_{\text{chain rule}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

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$= \delta_\nu{}^\sigma$ because $\left(\frac{\partial x^{\nu'}}{\partial x^\nu} \right)$ is the inverse of $\left(\frac{\partial x^\nu}{\partial x^{\nu'}} \right)$

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$$= V^\nu W^\mu \frac{\partial^2 x^{\mu'}}{\partial x^\nu \partial x^\mu} + \frac{\partial x^{\mu'}}{\partial x^\mu} V^\nu \partial_\nu W^\mu$$

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Similarly:

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$$(1) - (2) \Rightarrow \quad V^{\nu'} \partial_{\nu'} W^{\mu'} - W^{\nu'} \partial_{\nu'} V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \left(V^{\nu} \partial_{\nu} W^{\mu} - W^{\nu} \partial_{\nu} V^{\mu} \right)$$

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$$[V, W]^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} [V, W]^{\mu}$$

Learn more on Lie derivatives in a following video:

- geometric interpretation
- definition on any tensor field T

  $\mathcal{L}_v T$

- properties useful in calculations